On f-derivations of lattice implication algebras

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Abstract

In this paper, we introduced the notion of f-derivations, and considered the properties of f-derivations of lattice implication algebras. We give an equivalent condition to be isotone f-derivation in a lattice implication algebra. Also, we characterized the fixed set $Fix_d(L)$ and $Kerd$ by f-derivations. Moreover, we introduced the normal filter and obtained some properties of normal filters in lattice implication algebras.

Keywords: lattice implication algebra, f-derivation, normal filter, $Fix_d(L)$, $Kerd$.

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1. Introduction

In order to research a logical system whose propositional value is given in a lattice. Y. Xu [6] proposed the concept of lattice implication algebras, and some researchers have studied their properties and the corresponding logic systems. Also, in [7], Y. Xu and K. Y. Qin discussed the properties lattice H implication algebras, and gave some equivalent conditions about lattice H implication algebras. Y. Xu and K. Y. Qin [8] introduced the notion of filters in a lattice implication, and investigated their properties. In this paper, we introduced the notion of f-derivations, and considered

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the properties of f-derivations of lattice implication algebras. We give an equivalent condition to be isotone f-derivation in a lattice implication algebra. Also, we characterized the fixed set $Fix_d(L)$ and $Kerd$ by f-derivations. Moreover, we introduced the normal filter and obtained some properties of normal filters in lattice implication algebras.

2. Preliminary

A *lattice implication algebra* is an algebra $(L;\ \wedge,\ \vee,\ ',\ \to,\ 0,\ 1)$ of type $(2,2,1,2,0,0)$, where $(L;\wedge,\vee,0,1)$ is a bounded lattice, " $'$ " is an order-reversing involution and " $\to$ " is a binary operation, satisfying the following axioms:

(I1) $x \to (y \to z) = y \to (x \to z)$.

(I2) $x \to x = 1$.

(I3) $x \to y = y' \to x'$.

(I4) $x \to y = y \to x = 1 \Rightarrow x = y$.

(I5) $(x \to y) \to y = (y \to x) \to x$.

(L1) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$.

(L2) $(x \wedge y) \to z = (x \to z) \vee (y \to z)$.

for all $x, y, z \in L$. If L satisfies conditions (I1) – (I5), we say that L is a *quasi lattice implication algebra.* A lattice implication algebra L is called a *lattice H implication algebra* if it satisfies $x \vee y \vee ((x \wedge y) \to z) = 1$ for all $x, y, z \in L$.

In the sequel the binary operation " $\to$ " will be denoted by juxtaposition. We can define a partial ordering " $\leq$ " on a lattice implication algebra L by $x \leq y$ if and only if $x \to y = 1$.

In a lattice implication algebra L, the following hold (see [6]):

(u1) $0 \to x = 1$, $1 \to x = x$ and $x \to 1 = 1$.

(u2) $x \to y \leq (y \to z) \to (x \to z)$.

(u3) $x \leq y$ implies $y \to z \leq x \to z$ and $z \to x \leq z \to y$.

(u4) $x' = x \to 0$.

(u5) $x \vee y = (x \to y) \to y$.

(u6) $((y \to x) \to y')' = x \wedge y = ((x \to y) \to x')'$.

(u7) $x \le (x \to y) \to y$.

(u8) $(x \to y) \vee (y \to x) = 1$.

In a lattice H implication algebra L, the following hold:

(u9) $x \to (x \to y) = x \to y$.

(u10) $x \to (y \to z) = (x \to y) \to (x \to z)$.

A subset F of a lattice implication algebra L is called a *filter* of L it it satisfies:

(F1) $1 \in F$.

(F2) $x \in F$ and $x \to y \in F$ imply $y \in F$ for all $x, y \in L$.

Let L_1 and L_2 be lattice implication algebras. A map $f : L_1 \to L_2$ is called an *implication homomorphism* if $f(x \to y) = f(x) \to f(y)$ for all $x, y \in L_1$.

Definition 2.1 [5]. Let L be a lattice implication algebra. A map $d : L \to L$ is a *derivation* of L if

$$d(x \to y) = (x \to d(y)) \vee (d(x) \to y)$$

for all $x, y \in L$.

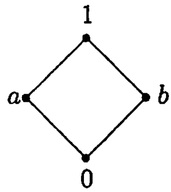
3. f-derivations of lattice implication algebras

In what follows, let L denote a lattice implication algebra unless otherwise specified.

Definition 3.1. Let L be a lattice implication algebra and let f be an implication endomorphism of L. A map $d : L \to L$ is a *f-derivation* of L if it satisfies the identity

$$d(x \to y) = (f(x) \to d(y)) \vee (d(x) \to f(y))$$

for all $x, y \in L$.



Example 3.2. Let $X = \{x, y\}$. Then

$$L = \mathcal{P}(X) = \{\emptyset, \{x\}, \{y\}, X\}.$$

Let $0 = \emptyset$, $a = \{x\}$, $b = \{y\}$, $1 = X$. Then $L = \{0, a, b, 1\}$ is a bounded lattice with above Hasse diagram.

We can make an implication $\rightarrow$ on L such as

$$a \rightarrow b = \{x\}^C \cup \{y\} = \{y\} \cup \{y\} = \{y\} = b.$$

Hence we have the operation table of the implication :

x	x'
0	1
a	b
b	a
1	0

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

If we define a map $f : L \rightarrow L$ by

$$f(x) = \begin{cases} 0 & \text{if } x = 0, a \\ 1 & \text{if } x = b, 1 \end{cases}$$

then this map f is an implication endomorphism, and define a map $d : L \rightarrow L$ by

$$d(x) = \begin{cases} b & \text{if } x = 0, a \\ 1 & \text{if } x = b, 1. \end{cases}$$

Then it is easy to check that d is a f-derivation of lattice implication algebra L. But d is not a derivation of L since $d(b \rightarrow 0) = d(a) = b$, but $(b \rightarrow d(0)) \vee (d(b) \rightarrow 0) = (b \rightarrow b) \vee (1 \rightarrow 0) = 1 \vee 0 = 1$, and so $d(b \rightarrow 0) \neq (b \rightarrow d(0)) \vee (d(b) \rightarrow 0)$.

Example 3.3. In Example 3.2, define a map $f : L \rightarrow L$ by

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ b & \text{if } x = a \\ a & \text{if } x = b \\ 1 & \text{if } x = 1. \end{cases}$$

Then it is easy to check that f is an implication endomorphism of lattice implication algebra L. Define a map $d : L \to L$ by

$$d(x) = \begin{cases} a & \text{if } x = 0, b \\ 1 & \text{if } x = a, 1. \end{cases}$$

Then it is easy to check that d is a f-derivation of lattice implication algebra L. But d is not a derivation of L since $d(a \to 0) = d(b) = a$, but $(a \to d(0)) \vee (d(a) \to 0) = (a \to a) \vee (1 \to 0) = 1 \vee 0 = 1$, and so $d(a \to 0) \neq (a \to d(0)) \vee (d(a) \to 0)$.

Also, define a map $f : L \to L$ by

$$f(x) = \begin{cases} 0 & \text{if } x = 0, b \\ 1 & \text{if } x = a, 1. \end{cases}$$

Then it is easy to check that f is an implication endomorphism of lattice implication algebra L. Define a map $d : L \to L$ by

$$d(x) = \begin{cases} 1 & \text{if } x = a, 1 \\ a & \text{if } x = 0, b. \end{cases}$$

Then it is easy to check that d is a f-derivation of lattice implication algebra L. But d is not a derivation of L since $d(a \to 0) = d(b) = a$, but $(a \to d(0)) \vee (d(a) \to 0) = (a \to a) \vee (1 \to 0) = 1 \vee 0 = 1$, and so $d(a \to 0) \neq (a \to d(0)) \vee (d(a) \to 0)$.

Proposition 3.4. Let d be a f-derivation of L. Then we have $d(1) = 1$.

Proof. Let d be a f-derivation of L. From (u8), we have

$$\begin{aligned} d(1) &= d(1 \to 1) = (f(1) \to d(1)) \vee (d(1) \to f(1)) \\ &= (1 \to d(1)) \vee (d(1) \to 1) = d(1) \vee 1 = 1 \end{aligned}$$

since $f(1) = 1$.

Proposition 3.5. Let d be a f-derivation of a lattice implication algebra L. Then the following properties hold for all $x, y \in L$.

(i) $d(x) = d(x) \vee f(x)$.

(ii) $f(x) \leq d(x)$.

(iii) $f(x) \vee f(y) \leq d(x) \vee d(y)$.

Proof. (i) Let $x \in L$. Then we have

$$\begin{aligned} d(x) &= d(1 \to x) = (f(1) \to d(x)) \vee ((d(1) \to f(x)) \\ &= (1 \to d(x)) \vee 1 \to f(x)) \\ &= d(x) \vee f(x). \end{aligned}$$

(ii) From (i), we have

$$\begin{aligned} f(x) \to d(x) &= f(x) \to (d(x) \vee f(x)) = f(x) \to ((d(x) \to f(x)) \to f(x)) \\ &= (d(x) \to f(x)) \to (f(x) \to f(x)) = (d(x) \to f(x)) \to 1 = 1, \end{aligned}$$

which implies $f(x) \leq d(x)$.

(iii) Since $f(x) \leq d(x)$, we have $d(x) \to f(y) \leq f(x) \to f(y)$, which implies

$$\begin{aligned} f(x) \vee f(y) &= (f(x) \to f(y)) \to f(y) \leq (d(x) \to f(y)) \to f(y) \\ &= (f(y) \to d(x)) \to d(x). \end{aligned}$$

Similarly, from $f(y) \leq d(y)$, we have

$$(f(y) \to d(x)) \to d(x) \leq (d(y) \to d(x)) \to d(x) = d(x) \vee d(y).$$

Hence we obtain $f(x) \vee f(y) \leq d(x) \vee d(y)$.

Theorem 3.6. Let L be a lattice implication algebra and let d be a f-derivation on L. Then we have $d(x \to y) = f(x) \to d(y)$ for all $x, y \in L$.

Proof. Let d be a f-derivation on lattice implication algebra L and $x, y \in L$. Then we have $d(x) \to f(y) \leq d(x) \to d(y)$ and $d(x) \to d(y) \leq f(x) \to d(y)$ from Proposition 3.5 (ii) and (u3). Hence we get

$$\begin{aligned} d(x \to y) &= (f(x) \to d(y)) \vee (d(x) \to f(y)) \\ &= ((f(x) \to d(y)) \to (d(x) \to f(y))) \to (d(x) \to f(y)) \\ &= ((d(x) \to f(y)) \to (f(x) \to d(y))) \to (f(x) \to d(y)) \\ &= 1 \to (f(x) \to d(y)) = f(x) \to d(y) \end{aligned}$$

from (I5).

Proposition 3.7. Let d be a f-derivation of lattice implication algebra L. If it satisfies $d(x \to y) = d(x) \to f(y)$ for all $x, y \in L$, we have $d(x) = f(x)$.

Proof. Let d be a f-derivation of L. If it satisfies $d(x \to y) = d(x) \to f(y)$ for all $x, y \in L$, we have

$$\begin{aligned} d(x) &= d(1 \to x) = d(1) \to f(x) \\ &= 1 \to f(x) = f(x). \end{aligned}$$

This completes the proof.

Proposition 3.8. Let d be a f-derivation of lattice implication algebra L. If it satisfies $f(x) \to d(y) = d(x) \to f(y)$ for all $x, y \in L$, then $d = f$.

Proof. Let d be a f-derivation of L. If it satisfies $f(x) \to d(y) = d(x) \to f(y)$ for all $x, y \in L$, we have

$$\begin{aligned} d(x) &= d(1 \to x) = f(1) \to d(x) \\ &= d(1) \to f(x) = 1 \to f(x) \\ &= f(x) \end{aligned}$$

from Theorem 3.6. This completes the proof.

Definition 3.9. Let L be a lattice implication algebra and d be a f-derivation on L. If $x \leq y$ implies $d(x) \leq d(y)$ for all $x, y \in L$, then d is called an *isotone f-derivation* of L.

Proposition 3.10. Let d be a f-derivation of a lattice implication algebra L. Then the following conditions are equivalent:

(i) d is an isotone f-derivation.

(ii) $d(x) \vee d(y) \leq d(x \vee y)$ for all $x, y \in L$.

Proof. (i)$\Rightarrow$ (ii): Suppose that d is an isotone f-derivation. We know that $x \leq x \vee y$ and $y \leq x \vee y$. Since d is isotone, $d(x) \leq d(x \vee y)$ and $d(y) \leq d(x \vee y)$. Hence we obtain $d(x) \vee d(y) \leq d(x \vee y)$.

(ii)$\Rightarrow$ (i): Suppose that $d(x) \vee d(y) \leq d(x \vee y)$ and $x \leq y$. Then we have $d(x) \leq d(x) \vee d(y) \leq d(x \vee y) = d(y)$.

Let d be a f-derivation of L. Define a set $Fix_d(L)$ by

$$Fix_d(L) := \{x \in L \mid d(x) = f(x)\}$$

for all $x \in L$. Clearly, $1 \in Fix_d(L)$.

Proposition 3.11. Let L be a lattice implication algebra and let d be a f-derivation on L. Then we have the following properties:

(i) If $x \in L$ and $y \in Fix_d(L)$, we have $x \to y \in Fix_d(L)$,

(ii) If $x \in L$ and $y \in Fix_d(L)$, $x \vee y \in Fix_d(L)$.

Proof. (i) Let $x \in L$ and $y \in Fix_d(L)$. Then we have $d(y) = f(y)$. Hence we get

$$\begin{aligned} d(x \to y) &= f(x) \to d(y) \\ &= f(x) \to f(y) \\ &= f(x \to y) \end{aligned}$$

from Theorem 3.6. This completes the proof.

(ii) Let $x, y \in Fix_d(L)$. Then we get

$$\begin{aligned} d(x \vee y) &= d((x \to y) \to y) \\ &= f(x \to y) \to d(y) \\ &= f(x \to y) \to f(y) \\ &= f((x \to y) \to y) \\ &= f(x \vee y) \end{aligned}$$

from Theorem 3.6. This completes the proof.

Proposition 3.12. Let d be a f-derivation of a lattice implication algebra L. If $x \leq y$ and $x \in Fix_d(L)$, we have $y \in Fix_d(L)$.

Proof. Let $x \leq y$ and $x \in Fix_d(L)$. Then we have $x \to y = 1, f(x) \leq f(y)$ and $d(x) = f(x)$. Thus we get

$$\begin{aligned} d(y) &= d((1 \to y) = d((x \to y) \to y) \\ &= d((y \to x) \to x) = f(y \to x) \to d(x) \\ &= f(y \to x) \to f(x) = (f(y) \to f(x)) \to f(x) \\ &= (f(x) \to f(y)) \to f(y) = f(x) \vee f(y) = f(y), \end{aligned}$$

from Theorem 3.6. Hence $y \in Fix_d(L)$.

Definition 3.13. Let L be a lattice implication algebra and let d be a f-derivation. Define a $Kerd$ by

$$Kerd = \{x \in L \mid d(x) = 1\}.$$

Proposition 3.14. Let d be a f-derivation of a lattice implication algebra L. If d is an endomorphism of L, $Kerd$ is a filter of L.

Proof. Clearly, $1 \in Kerd$. Let $x, x \to y \in Kerd$. Then $d(x) = 1$ and $d(x \to y) = 1$. Hence we have

$$1 = d(x \to y) = d(x) \to d(y) = 1 \to d(y) = d(y),$$

which implies $y \in Kerd$.

Proposition 3.15. Let L be a lattice implication algebra and let d be a f-derivation. If $y \in Kerd$ and for all $x \in L$, then $x \vee y \in Kerd$.

Proof. Let d be a f-derivation and $y \in Kerd$. Then we get $d(y) = 1$, and so

$$\begin{aligned} d(x \vee y) &= d((x \to y) \to y) = f(x \to y) \to d(y) \\ &= f(x \to y) \to 1 = 1 \end{aligned}$$

from Theorem 3.6. Hence we have $x \vee y \in Kerd$. This completes the proof.

Proposition 3.16. Let L be a lattice implication algebra and let d be a f-derivation. If $x \leq y$ and $x \in Kerd$, we have $y \in Kerd$.

Proof. Let $x \leq y$ and $x \in Kerd$. Then we get $x \to y = 1$ and $d(x) = 1$, and so

$$\begin{aligned} d(y) &= d(1 \to y) = d((x \to y) \to y) = d((y \to x) \to x) \\ &= f(y \to x) \to d(x) = f(y \to x) \to 1 \\ &= 1 \end{aligned}$$

from Theorem 3.6. Hence we have $y \in Kerd$.

Proposition 3.17. Let L be a lattice implication algebra and let d be a f-derivation. If $y \in Kerd$, we have $x \to y \in Kerd$ for all $x \in L$.

Proof. Let $y \in Kerd$. Then $d(y) = 1$. Thus we have

$$\begin{aligned} d(x \to y) &= f(x) \to d(y) \\ &= f(x) \to 1 \\ &= 1, \end{aligned}$$

from Theorem 3.6. Hence $x \to y \in Kerd$.

Definition 3.18. Let L be a lattice implication algebra. A non-empty set F of L is called a *normal filter* if it satisfies the following conditions:
(i) $1 \in F$,
(ii) $x \in L$ and $y \in F$ imply $x \to y \in F$.

Example 3.19. In Example 3.2, let $F = \{1, a\}$. Then F is a normal filter of a lattice implication algebra L.

Proposition 3.20. Let L be a lattice implication algebra and let d be a f-derivation. Then $Fix_d(L)$ is a normal filter of L.

Proof. Clearly, $1 \in Fix_d(L)$. Let $x \in L$ and $y \in Fix_d(L)$. Then we have $d(y) = f(y)$, and so

$$\begin{aligned} d(x \to y) &= f(x) \to d(y) \\ &= f(x) \to f(y) \\ &= f(x \to y), \end{aligned}$$

which implies $x \to y \in Fix_d(L)$ from Theorem 3.6. This completes the proof.

Proposition 3.21. Let L be a lattice implication algebra and let d be a f-derivation. Then $Kerd$ is a normal filter of L.

Proof. Clearly, $1 \in Kerd$. Let $x \in L$ and $y \in Kerd$. Then we have $d(y) = 1$, and so

$$\begin{aligned} d(x \to y) &= f(x) \to d(y) \\ &= f(x) \to 1 \\ &= 1, \end{aligned}$$

which implies $x \to y \in Kerd$ from Theorem 3.6. Hence $Kerd$ is a normal filter of L.

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