Groups $PSL(n,q)$ and $3-(v,k,1)$ designs *

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Abstract

Let $T = PSL(n,q)$ be a projective linear simple group, where $n \geq 2$, q a prime power and $(n,q) \neq (2,2)$ and $(2,3)$. We classify all $3-(v,k,1)$ designs admitting an automorphism group G with $T \trianglelefteq G \leq Aut(T)$ and $v = \frac{q^n-1}{q-1}$.

2000 Mathematics Subject Classification: 05B05, 20B25
Keywords: $t-$designs, Group of automorphisms, Projective linear groups

1 Introduction

Let t, k, v and λ be integers such that $0 < t \leq k \leq v$ and $\lambda > 0$. Let X be a v-set and $P_k(X)$ denote the set of all k-subsets of X. A $t-(v,k,\lambda)$ design is a pair $\mathcal{D} = (X, \mathcal{B})$ in which $\mathcal{B}$ is a collection of elements of $P_k(X)$ (called blocks) such that every t-subset of X appears in exactly λ blocks. If $\mathcal{B}$ has no repeated blocks, then it is called simple. If every t-subset of X is in fact a block, i.e., the number of blocks is $b = \binom{v}{k}$, the binomial coefficient, then the design is called trivial. Here we are concerned only with simple and non-trivial designs. An automorphism of $\mathcal{D}$ is a permutation σ on X such that $\sigma(B) \in \mathcal{B}$ for each $B \in \mathcal{B}$. An automorphism group of $\mathcal{D}$ is a group whose elements are automorphisms of $\mathcal{D}$. Let G be a finite group acting on X. For $x \in X$, the orbit of x is $G(x) = \{gx | g \in G\}$ and the stabilizer of x is $G_x = \{g \in G | gx = x\}$. It is well known that $|G| = |G(x)||G_x|$. Sometimes we use also x^G to denote the orbit of x under G. Orbits of size $|G|$ are called regular and the others are called non-regular. If there is an $x \in X$ such that $G(x) = X$, then G is called transitive on X. The action of G on X induces a natural action on $P_k(X)$. If this latter action is transitive, then G is called k-homogeneous. The flags of $\mathcal{D}$ are the order pairs (x, B), where x is a point and B is a block containing x. A group is flag-transitive if it is transitive on the set of flags; this is equivalent to the assertion that the group is transitive on blocks (points) and the subgroup

*Supported by the NSFC (Grant No. 11271208 and 61272424) and the Research Foundation of Education Bureau of Hunan Province of China (No. 08c021).

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fixing a block (point) is transitive on the points of that block (the blocks through that point).

All $3-(v,k,1)$ designs which admit a flag-transitive automorphism group were classified by M. Huber in [9]. The most interesting examples which occur have an almost simple group with socle $PSL(2,q)$ as group of automorphisms. Moreover, In [8], M. Huber discussed the case where there is an almost simple groups with socle $PSL(n,q)$ acting on a $3-(v,4,1)$ design. As a continuation of his works, in this paper, we consider the case where there is an almost simple group with socle $PSL(n,q)$ acting on a $3-(v,k,1)$ design. We get the following theorem:

Theorem 1.1 *Let $\mathcal{D}=(X,\mathcal{B})$ be a nontrivial simple $3-(v,k,1)$ design and $G\leq \mathrm{Aut}(\mathcal{D})$. If $PSL(n,q)\trianglelefteq G\leq \mathrm{Aut}(PSL(n,q))$ and $v=\frac{q^n-1}{q-1}$ with $q=p^f$, where p is a prime and f a positive integer, then one of the following occurs:*

1. *If G is block-transitive, then G is flag-transitive, and*
   - (a) *$\mathcal{D}$ is isomorphic to a $3-(p^f+1,p^m+1,1)$ design whose points are the elements of the projective line $GF(p^f)\cup\{\infty\}$ and whose blocks are the images of $GF(p^m)\cup\{\infty\}$ under $PGL(2,p^f)$ (resp. $PSL(2,p^f)$ with f/m odd), and $PSL(2,p^f)\leq G\leq P\Gamma L(2,p^f)$, where $m|f$,*
   - (b) *$\mathcal{D}$ is isomorphic to a $3-(q+1,4,1)$ design whose points are the elements of the projective line $GF(q)\cup\{\infty\}$ with $q\equiv 7 \pmod{12}$ and whose blocks are the images of $\{0,1,\varepsilon,\infty\}$ under $PSL(2,q)$, where ε is a primitive sixth root of unity in $GF(q)$, and $PSL(2,q)\leq G\leq P\Sigma L(2,q)$,*
2. *If G is not block-transitive, then $p^f\equiv 1 \pmod 4$, and $\mathcal{D}$ is isomorphic to a $3-(p^f+1,p^m+1,1)$ design with $2m|f$, and $PSL(2,p^f)\leq G\leq P\Sigma L(2,p^f)$, $\mathcal{B}=\Gamma\cup\Gamma'$, where $\Gamma=\{GF(p^m)\cup\{\infty\}\}^{PSL(2,p^f)}$ and $\Gamma'=B^{PSL(2,q)}$, where B is a k-subset of $GF(p^f)\cup\{\infty\}$ with $P_3(B)\cap\{0,1,\infty\}^G=\emptyset$.*

The second section contains some preliminary results about $PSL(2,q)$ and $t-$designs. In the third section we give the proof of the main theorem.

2 Preliminary Results

Let $\mathcal{D}=(X,\mathcal{B})$ be a $t-(v,k,\lambda)$ design, and $G\leq Aut(\mathcal{D})$. Let b denote the number of blocks of $\mathcal{D}$, and r the number of blocks that is incident with a fixed point of $\mathcal{D}$. Now, we introduce the following results which play an important role in the proof of the main theorem.

Lemma 2.1 ([6]) *Let $\mathcal{D}=(X,\mathcal{B})$ be a $t-(v,k,\lambda)$ design. Then the following holds:*

1. $bk = vr$;

2. $r = \frac{\lambda(v-1)\ldots(v-t+1)}{(k-1)\ldots(k-t+1)}$;

3. $b = \frac{\lambda v\ldots(v-t+1)}{k\ldots(k-t+1)}$.

Lemma 2.2 ([2]) If $\mathcal{D} = (X, \mathcal{B})$ is a non-trivial Steiner t-design, then $v - t + 1 \geq (k-t+2)(k-t+1)$ for $t > 2$. If equality holds, then $(t,k,v) = (3,4,8), (3,6,22), (3,12,112), (4,7,23)$ or $(5,8,24)$.

From now on, we let $X := GF(q) \cup \{\infty\}$ with $q = p^f$, where p is a prime and f a positive integer. The set of all mappings of the form: $x \mapsto \frac{ax+b}{cx+d}$ with $a,b,c,d \in GF(q)$, $ad-bc$ being a square in $GF(q)$ constitutes the projective special linear group, $PSL(2,q)$ on X. Let π be a mapping of the form: $x \mapsto x^p$, $x \in X$. Then we use the notation $P\Sigma L(2,q)$ to denote $PSL(2,q) \rtimes \langle\pi\rangle$. The action of $P\Sigma L(2,q)$ induces a natural action on $P_k(X)$. It is well known that this action is transitive on $P_3(X)$ if and only if $q \equiv 3 \pmod 4$. But in the case where $q \equiv 1 \pmod 4$, $P\Sigma L(2,q)$ is not transitive on $P_3(X)$. In this case, we start with a simple observation which is quite straightforward to verify: any 3-subset of X belongs to $\Delta_1 = G(\{0,\infty,1\})$ or $\Delta_2 = G(\{0,\infty,\theta\})$, where $G = P\Sigma L(2,q)$ and θ is a primitive root of unity in $GF(q)$.

The following lemma and perspective come from [5].

Lemma 2.3 *Let k be a positive integer and consider the action of $G := PSL(2,q)$ on $P_k(X)$. If Γ is an orbit for this action then so is $\theta\Gamma$, where θ is a primitive root of unity in $GF(q)$. Moreover, $\theta^2\Gamma = \Gamma$.*

Remark 2.4 *There is a more general perspective by Lemma 2.3 as follows. Fix $k > 4$ and consider the induced action of $PSL(2,q)$ on $P_k(X)$. The set $P_k(X)$ is partitioned into orbits Γ_i, $i = 1, \ldots, m$ for some m, so that we have*

$$P_k(X) = \bigcup_{i=1}^{m} \Gamma_i.$$

Let $\mathcal{F} = \{\Gamma_1, \Gamma_2, \ldots, \Gamma_m\}$. Then the group $GF(q)^*$ acts on $\mathcal{F}$ as

$$\gamma \cdot \Gamma = \gamma\Gamma, \; for \gamma \in GF(q)^*, \Gamma \in \mathcal{F}.$$

Thus we can write

$$\mathcal{F} = \bigcup_i \mathcal{F}_i,$$

where each $\mathcal{F}_i$ is an orbit of $\mathcal{F}$ under $GF(q)^*$. Furthermore, since the map $x \mapsto \alpha^2 x$ is an element of $PSL(2,q)$, it follows that $|\mathcal{F}_i| = 1$ or 2. Hence each orbit for the action described above gives us a simple 3-design with $PSL(2,q)$ acting as a group of automorphisms.

The results contained in the following lemma are established in [7] and [11].

Lemma 2.5 *Let $q = p^f$ with p a prime and f a positive integer, and $d = (2, q-1)$. Then the subgroups of $PSL(2,q)$ are as follows:*

1. *Cyclic subgroups of order z where $z|\frac{q\pm1}{d}$;*
2. *Dihedral subgroups of order $2z$ where $z|\frac{q\pm1}{d}$;*
3. *A_4 when $p > 2$ or $p = 2$ and f even;*
4. *S_4 when $q^2 - 1 \equiv 0 \pmod{16}$;*
5. *A_5 when $p = 5$ or $q^2 - 1 \equiv 0 \pmod 5$;*
6. *The semidirect product of the elementary abelian subgroup of order p^m and the cyclic group of order z where $m \le f$, $z|\frac{q-1}{d}$, $z|(p^m - 1)$ and z may be 1;*
7. *Subgroups $PSL(2,p^m)$ with $m|f$, and subgroups $PGL(2,p^m)$ with $2m|f$.*

We state that the sizes of orbits from the action of subgroups of $PSL(2,q)$ on the projective line X as follows. When $q \equiv 3 \pmod 4$, P. J. Cameron, H. R. Maimani, G. R. Omidi and B. Tayfeh-Rezaie has determined the sizes of orbits in [3]. In [10], M. Huber have determined the sizes of orbits with all cases. The main techniques involve Cauchy-Frobenius-Burnside Lemma and the fact that if $H_1 \le H_2 \le G$ then the orbit of H_2 is a union of orbits from the action of H_1. In the following lemmas we suppose that H is a subgroup of $PSL(2,q)$ and N_l denotes the number of orbits of size l.

Lemma 2.6 ([10]) *Let H be the cyclic group of order $c > 1$ and $d = (2, q-1)$,*

1. *if $c|(q+1)/d$, then $N_c = (q+1)/c$,*
2. *if $c|(q-1)/d$, then $N_1 = 2$ and $N_c = (q-1)/c$,*
3. *if $c = p$, then $N_1 = 1$ and $N_c = q/c$.*

Lemma 2.7 ([10]) *Let H be the dihedral group of order $2c$ with $c|(q\pm1)/d$ and $d = (2, q-1)$, where $c > 1$. Then*

1. *for $q \equiv 1 \pmod 4$*
   - (a) *if $c|(q+1)/2$, then $N_c = 2$ and $N_{2c} = (q-2c+1)/(2c)$,*
   - (b) *if $c|(q-1)/2$, then $N_2 = 1$, $N_c = 2$ and $N_{2c} = (q-2c-1)/(2c)$, unless $c = 2$, in which case $N_2 = 3$ and $N_4 = (q-5)/4$.*
2. *for $q \equiv 3 \pmod 4$*
   - (a) *if $c|(q+1)/2$, then $N_{2c} = (q+1)/(2c)$,*
   - (b) *if $c|(q-1)/2$, then $N_2 = 1$ and $N_{2c} = (q-1)/(2c)$;*

3. for $q \equiv 0 \pmod 2$

(a) if $c|(q+1)$*, then* $N_c = 1$ *and* $N_{2c} = (q-c+1)/(2c)$*,*
(b) if $c|(q-1)$*, then* $N_2 = 1$*,* $N_c = 1$ *and* $N_{2c} = (q-c-1)/(2c)$*.*

Lemma 2.8 ([10]) *Let H be isomorphic to* A_5*, and* $q \equiv 1 \pmod 4$*. Then*

1. *if* $q = 5^e$*,* $e \equiv 1 \pmod 2$*, then* $N_6 = 1$ *and* $N_{60} = (q-5)/60$*,*
2. *if* $q = 5^e$*,* $e \equiv 0 \pmod 2$*, then* $N_6 = 1, N_{20} = 1$ *and* $N_{60} = (q-25)/60$*,*
3. *if* $15|(q+1)$*, then* $N_{30} = 1$ *and* $N_{60} = (q-29)/60$*,*
4. *if* $3|(q+1)$ *and* $5|(q-1)$*, then* $N_{12} = 1, N_{30} = 1$ *and* $N_{60} = (q-41)/60$*,*
5. *if* $3|(q-1)$ *and* $5|(q+1)$*, then* $N_{20} = 1, N_{30} = 1$ *and* $N_{60} = (q-49)/60$*,*
6. *if* $15|(q-1)$*, then* $N_{12} = 1, N_{20} = 1, N_{30} = 1$ *and* $N_{60} = (q-61)/60$*,*
7. *if* $3|q$ *and* $5|(q+1)$*, then* $N_{10} = 1$ *and* $N_{60} = (q-9)/60$*,*
8. *if* $3|q$ *and* $5|(q-1)$*, then* $N_{10} = 1, N_{12} = 1$ *and* $N_{60} = (q-21)/60$*.*

Lemma 2.9 ([10]) *Let H be the elementary Abelian group of order* p^m *with* $1 \le m \le f$*, then* $N_1 = 1$*, and other orbits are regular.*

Lemma 2.10 ([10]) *Let H be a semidirect product of the elementary Abelian group of order* p^m *and the cyclic group of order* $z > 1$ *where* $1 \le m \le f$*,* $z|(p^m-1)$ *and* $z|(q-1)/d$ *with* $d = (2, q-1)$*, then* $N_1 = 1$*,* $N_{p^m} = 1$ *and other orbits are regular.*

Lemma 2.11 *Let H be* $PSL(2, p^m)$ *where* $m|f$*, then*
(i) if f/m is odd, then $N_{p^m+1} = 1$ and other orbits are regular;
(ii) if f/m is even, then $N_{p^m+1} = N_{p^m(p^m-1)} = 1$ and other orbits are regular.
Moreover, the orbit of size $p^m + 1$ is $GF(p^m) \cup \{\infty\}$.

Proof. (i) and (ii) come from [10]. Since $N_{p^m+1} = 1$ and $PSL(2, p^m)$ acts transitively on $GF(p^m) \cup \{\infty\}$, it follows that the orbit of size $p^m + 1$ is $GF(p^m) \cup \{\infty\}$.

Lemma 2.12 *Let H be* $PGL(2, p^m)$ *where* $2m|f$*, then* $N_{p^m+1} = N_{p^m(p^m-1)} = 1$ *and other orbits are regular. Moreover, the orbit of size* p^m+1 *is* $GF(p^m) \cup \{\infty\}$*.*

Proof. The proof is similar to that of Lemma 2.11.

The following lemma is useful in the proof of Theorem 1.1.

Lemma 2.13 *Let S_n be a symmetric group on the set X with $|X| = n$, and H a subgroup of S_n. If $|H| = n(n-1)(n-2)$ or $n(n-1)(n-2)/2$ with $3 \leq n \leq 5$, then H is transitive on X. If $n = 6$, then $H \cong S_5$ or A_5, where A_n denotes the alternating group on X.*

Proof. When $n = 3$ and 4, the lemma is obvious. If $n = 5$ and $|H| = n(n-1)(n-2)$, then $H \cong A_5$. If H is a subgroup of S_5 of order 30, then clearly H is not a subgroup of A_5 since A_5 is simple. So H must contain an odd permutation and it can easily be shown that $|H \cap A_5| = |H|/2 = 15$ which is impossible. Thus $H \cong A_5$, and so the conclusion holds. If $n = 6$ and $|H| = n(n-1)(n-2) = 120$, then H must contain odd permutations since A_6 has no any subgroups of order 120 by [4], and so $H \cong S_5$. Note that the subgroups of order 60 of S_6 are isomorphic to A_5, and so if $|H| = n(n-1)(n-2)/2 = 60$ then $H \cong A_5$. Thus the conclusion holds. □

3 The Proof of Main Theorem

Let $\mathcal{D}$ be a non-trivial simple $3-(v,k,1)$ design, and $T \trianglelefteq G \leq Aut(T)$ with a projective linear simple group $T = PSL(n,q)$, where $n \geq 2$, q a prime power and $(n,q) \neq (2,2), (2,3)$. We consider the natural action of G on the projective space $PG(n-1,q)$, $v = \frac{q^n-1}{q-1}$.

Lemma 3.1 *Let $\mathcal{D} = (X, \mathcal{B})$ be a non-trivial simple $3-(v,k,1)$ design, and $T = PSL(2,q)$, $v = q+1$. If $T \trianglelefteq G \leq Aut(T)$ is 3-homogeneous on X and $G \leq Aut(\mathcal{D})$, then G is flag-transitive, and*

1. *$\mathcal{D}$ is isomorphic to a $3-(p^f+1, p^m+1, 1)$ design whose points are the elements of the projective line $GF(p^f) \cup \{\infty\}$ and whose blocks are the images of $GF(p^m) \cup \{\infty\}$ under $PGL(2,p^f)$ (resp. $PSL(2,p^f)$ with f/m odd), and $PSL(2,p^f) \leq G \leq P\Gamma L(2,p^f)$, where $m|f$,*

2. *$\mathcal{D}$ is isomorphic to a $3-(q+1,4,1)$ design whose points are the elements of the projective line $GF(q) \cup \{\infty\}$ with $q \equiv 7 \pmod{12}$ and whose blocks are the images of $\{0, 1, \varepsilon, \infty\}$ under $PSL(2,q)$, where ε is a primitive sixth root of unity in $GF(q)$, and and $PSL(2,p^f) \leq G \leq P\Sigma L(2,p^f)$.*

Proof. If G is 3-homogeneous, then in particular G is block-transitive by the definition of design. If $q \equiv 0$ or $3 \pmod 4$, then $PSL(2,q)$ and $PGL(2,q)$ are all 3-homogeneous on X. If $q \equiv 1 \pmod 4$, then $PGL(2,q)$ is 3-homogeneous but $P\Sigma L(2,q)$ is not that on X. Thus if $q \equiv 0$ or $1 \pmod 4$, then we may assume that $G = PGL(2,q)$, and if $q \equiv 3 \pmod 4$, then we may assume that $G = PSL(2,q)$. Since $G = PGL(2,q)$ is 3-transitive on X, the Lemma is true by [12]. Thus we assume that $q \equiv 3 \pmod 4$ and $G = PSL(2,q)$. By Lemma 2.1 we have

$$\frac{(q+1)(q-1)q}{k\cdot(k-1)\cdot(k-2)}=\frac{|G|}{|G_B|}. \tag{1}$$

Therefore, $|G_B|=|PSL(2,q)_B|=k(k-1)(k-2)/2$. Note that the representation $G_B\to Sym(B)\cong S_k$ is faithful. Thus we may suppose that $G_B\leq Sym(B)$. If $k=4$ or 5, then, by Lemma 2.13, G_B is transitive on B, and so G is flag-transitive since G is block-transitive. The assertion comes from [9]. If $k=6$, then by Lemma 2.13, $G_B\cong A_5$. If A_5 acts transitively on B, then G is flag transitive, and the assertion comes from [9]. Assume that A_5 is not transitive on B. By [4], A_5 has no subgroups of orders 30, 20 and 15, and so it has no orbits of sizes 2, 3 and 4. This deduces that A_5 has an orbit of size 1 on B, say α. Thus $A_5\leq G_\alpha$, and so $G_{\alpha,B}$ contains a subgroup A_4. Note that A_4 has no any subgroup of order 6. Thus A_4 has no orbits of size 2. Moreover, A_4 is not transitive on $B\setminus\alpha$ since $5\nmid|A_4|$. Therefore, A_4 has an orbit of size 1, say $\beta\in B\setminus\alpha$, which implies that A_4 fixes two points of X. On the other hand, the stabilizer of two point in G is cyclic by [11], a contradiction as A_4 is not cyclic. Now let $k\geq 7$, and so $|G_B|\geq 105$. It follows that G_B is isomorphic to a subgroup of type 1, 2, 6 or 7 in Lemma 2.5. Set $s=k(k-1)(k-2)/2$. Suppose that G_B is a subgroup of type 1, that is, a cyclic subgroup of order s. Note that B is a union of some orbits of G_B on X. Hence by Lemma 2.6, we have $k\geq|G_B|=k(k-1)(k-2)/2$, which is absurd. If G_B is a subgroup of type 2, that is, a dihedral subgroup of order s, then by Lemma 2.7 we have $k\geq s/2=k(k-1)(k-2)/4$, which is also absurd since $k\geq 7$. If G_B is a subgroup of type 6, then by Lemmas 2.9 and 2.10 we have $k\geq p^m$, and so

$$p^m(p^m-1)\geq p^m z=|G_B|\geq s\geq p^m(p^m-1)(p^m-2)/2.$$

It follows that $p^m\leq 4$ and $|G_B|\leq 12$. This implies that $k\leq 4$, contrary to $k\geq 7$. If G_B is a subgroup of type 7, then

$$k(k-1)(k-2)/2\leq|G_B|\leq p^m(p^m-1)(p^m+1),$$

and so $k<p^m(p^m-1)$ (recall here $k\geq 7$). Hence, by Lemmas 2.11 and 2.12, we have $k=p^m+1$ and $B=GF(p^m)\cup\{\infty\}$. If $G_B\cong PGL(2,p^m)$ then $\mathcal{B}=B^{PGL(2,p^f)}$, and if $G_B\cong PSL(2,p^m)$ with f/m odd, then $\mathcal{B}=B^{PSL(2,p^f)}$. It follows that G_B is transitive on B, and so G is flag-transitive. Therefore, by [9], we get a $3-(p^f+1,p^m+1,1)$ design whose points are the elements of the projective line $GF(p^f)\cup\{\infty\}$ and whose blocks are the images of $GF(p^m)\cup\{\infty\}$ under $PGL(2,p^f)$ (respectively, $PSL(2,p^f)$ with f/m odd), where $m|f$. □

Lemma 3.2 *Let $\mathcal{D}=(X,\mathcal{B})$ be a non-trivial simple $3-(v,k,1)$ design, and $T=PSL(2,q)$ and $v=q+1$ with $q=p^f$. If $T\trianglelefteq G\leq Aut(T)$ is not 3-homogeneous on X and $G\leq Aut(\mathcal{D})$, then $p^f\equiv 1 \pmod 4$, and*

1. *if G is block-transitive, then G is flag-transitive, and $\mathcal{D}$ is isomorphic to a $3-(p^f+1, p^m+1, 1)$ design whose points are the elements of the projective line $GF(p^f) \cup \{\infty\}$ and whose blocks are the images of $GF(p^m) \cup \{\infty\}$ under $PSL(2,p^f)$, and $PSL(2,p^f) \leq G \leq P\Sigma L(2,p^f)$, where $m|f$; otherwise*

2. *$\mathcal{D}$ is isomorphic to a $3-(p^f+1, p^m+1, 1)$ design with $2m|f$, and $PSL(2,p^f) \leq G \leq P\Sigma L(2,p^f)$, $\mathcal{B} = \Gamma \cup \Gamma'$, where $\Gamma = \{GF(p^m) \cup \{\infty\}\}^{PSL(2,p^f)}$ and $\Gamma' = B^{PSL(2,q)}$, where B is a k-subset of $GF(p^f) \cup \{\infty\}$ with $P_3(B) \cap \{0,1,\infty\}^G = \emptyset$.*

Proof. By hypothesis, we have the case $q \equiv 1 \pmod 4$ and $PSL(2,q) \leq G \leq P\Sigma L(2,q)$. Since G and $PSL(2,q)$ are both 2-transitive on X, we may restrict ourselves to the case $G = PSL(2,q)$. In this case, The 3-subsets of X under $PSL(2,q)$ are split in exactly two orbits Δ_1 and Δ_2 of equal length.

If G is transitive on $\mathcal{B}$, then there exists a k-subset $B \in P_k(X)$ such that $\mathcal{B} = B^G$. By Theorem 1 of [13], we have $|P_3(B) \cap \Delta_1| = |P_3(B) \cap \Delta_2| = k(k-1(k-2)/12$. Note that $\frac{|G|}{|G_B|} = |B^G| = b = \frac{(q+1)q(q-1)}{k(k-1)(k-2)}$. Thus $|G_B| = k(k-1)(k-2)/2$. By the proof of Lemma 3.1, we get (1) of Lemma 3.2.

Assume that G is not transitive on $\mathcal{B}$, and $B_1^G, \ldots, B_\rho^G$ are ρ distinct orbits of G on the set of blocks with $\rho > 1$. Thus for any two distinct blocks B_i and B_j above, we know that one is empty set and the other one nonempty set in $\Delta_l \cap P_3(B_i)$ and $\Delta_l \cap P_3(B_j)$, where $l = 1$ and 2. Otherwise, both $P_3(B_i)$ and $P_3(B_j)$ are contained in the same orbit of 3-subsets, and so there is an element $g \in G$ such that $S_i^g = S_j$, where $S_i \in P_3(B_i)$ and $S_j \in P_3(B_j)$. By the definition of design, we have $B_i^g = B_j$, which conflicts with $B_i^G \neq B_j^G$. Hence $\rho = 2$ and $\mathcal{B} = B_1^G \cup B_2^G$. Let $u_{ij} = |\Delta_i \cap P_3(B_j)|$ where $i,j = 1, 2$. Then $u_{11} = u_{22} = \binom{k}{3}$ and $u_{12} = u_{21} = 0$ or $u_{11} = u_{22} = 0$ and $u_{12} = u_{21} = \binom{k}{3}$. Moreover, by Theorem 1 of [13], we have

$$\frac{u_{i1} \cdot |B_1^G|}{|\Delta_i|} + \frac{u_{i2} \cdot |B_2^G|}{|\Delta_i|} = 1, \text{ where } i = 1,\ 2.$$

Note that $|\Delta_1| = |\Delta_2| = (q+1)q(q-1)/12$. Therefore, we have $|B_1^G| = |B_2^G|$. Set $B = B_1$. It follows that

$$|\mathcal{B}| = 2|B^G| = \frac{2|PSL(2,q)|}{|G_B|} = \frac{q(q+1)(q-1)}{|G_B|}.$$

On the other hand, $|\mathcal{B}| = \frac{(q+1)q(q-1)}{k \cdot (k-1) \cdot (k-2)}$ by Lemma 2.1. Comparing the two expressions, we get $|G_B| = k(k-1)(k-2)$. If $k = 4$, then the design is example 3 in [8] (There exists an error in [8], $\{0,1,a,\infty\}$ is replaced by $\{0,-a,a,\infty\}$ there). If $k = 5$, then by Lemma 2.13 $G_B \cong A_5$. Since B is a union of some orbits of G_B acting on X, we have $|B| = k \geq 6$ by Lemma 2.8, a contradiction. Hence $k \geq 6$ and $|G_B| \geq 6 \cdot 5 \cdot 4 = 120$. Considering the orders of subgroups, we have G_B is isomorphic to a subgroup of type 1, 2, 6 or 7 in Lemma 2.5.

Therefore, by using the same method as in Lemma 3.1, we have $k = p^m + 1$ and $B = GF(p^m) \cup \{\infty\}$. Therefore, $G_B = PGL(2, p^m)$ with $2m|f$. If we set $B_2 = B$ above, then we can get $B_2 = GF(p^m) \cup \{\infty\}$ and $G_{B_2} = PGL(2, p^m)$. Let $\Gamma = (GF(p^m) \cup \{\infty\})^G$. If $x \mapsto ax$ is an automorphism of $\mathcal{D}$ for some non-square element $a \in GF(q) \backslash GF(p^m)$, then $\mathcal{B}$=$\Gamma \cup a\Gamma$. Otherwise, $\mathcal{B}$=$\Gamma \cup \Gamma'$, where $\Gamma' = B_2^G$ with $P_3(B_2) \cap \Delta_1 = \emptyset$, Γ and Γ' belong to different $\mathcal{F}_i$ in Remark 2.4. Note that here $PGL(2, p^m) \leq PSL(2, q)$ and $PGL(2, p^m)$ acts 3-homogeneous on $B_1 = GF(p^m) \cup \{\infty\}$. Thus $P_3(B_1) \subset \Delta_1$. By Theorem 1 of [13], we get a $3 - (p^f + 1, p^m + 1, 1)$ design. □

The result contained in the following lemma is established in [9].

Lemma 3.3 ([9]) *Let $T = PSL(n, q)$ with $n \geq 3$, $v = \frac{q^n - 1}{q - 1}$. Then $T \trianglelefteq G \leq Aut(T)$ cannot act as a group of automorphisms on any $3 - (v, k, 1)$ design.*

Proof of the Theorem 1.1. The result is obtained by putting together Lemmas 3.1, 3.2 and 3.3. □

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