Cubic Graphs with Minimum Number of Spanning Trees

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Abstract

In this paper we prove that there exists one type of connected cubic graph, which minimizes the number of spanning trees over all other connected cubic graphs of the same order n, $n \geq 14$.

Keywords: Spanning trees, Enumeration, Cubic graphs.

1. Introduction

The problem of identifying connected graphs that maximize/minimize the number of spanning trees over other connected graphs for a given number of vertices and edges has been extensively studied in the literature [1-3,5-7]. For regular graphs Kelmans and Chelnokov [6] showed that complete graph K_{2n} with removed n nonadjacent edges maximizes the number of spanning trees. In addition, Cheng [3] proved that a complete multipartite graph maximizes the number of spanning trees. Subsequently, Boesch [2] conjectured that regular graphs of maximum girth maximize the number of spanning trees.

If the above conjecture holds, then for cubic graphs the attention shifts to finding out if there exists one type of connected cubic graph that maximizes the number of spanning trees over the set of all connected cubic graphs of the same order. We know that the Petersen-graph, Möbius Ladder and $K_{3,3}$ maximize the number of spanning trees among connected cubic graphs of order 10, 8 and 6 respectively. However, the above conjecture would also imply the identification of 3-cages, which by itself is a challenging problem.

In this paper we consider simple undirected cubic graphs and focus our attention on the opposite (i.e., minimization) type of problem. That is, we will characterize a certain type of connected cubic graph, and prove that such a graph minimizes the number of spanning trees over the set of all connected cubic graphs of the same order. In fact, Valdés characterized the 2-connected cubic graphs for which the number of spanning trees is minimum [8]. So, in this work we extend Valdés's result to all connected cubic graphs, which cannot be derived from his work in a straightforward way. Note, it's not even obvious that there exists one type of 1-connected cubic graph for which the number of spanning trees is minimum in respect to 1-connected cubic graphs.

Let G_n denote a connected cubic graph of order n with $V(G_n)$ vertices and $E(G_n)$ edges. Let $t(G_n)$ denote the number of spanning trees in G_n. We say that G_n minimizes the number of spanning trees over all connected cubic graphs of the same order if for any connected cubic graph H_n, $t(H_n) \geq t(G_n)$. Let $K_{4,-1}$ denote a complete graph of order 4 with a single edge removed, and let $K_{5,-3}$ denote a complete graph of order 5 with three (two adjacent and one nonadjacent) edges removed. In addition, we denote a subgraph H of G by $H(G)$.

For $n \equiv 2 \pmod 4$ we define Q_n as a cubic graph that consists of two $K_{5,-3}$ components and k $K_{4,-1}$ components connected together with edges, where $k \geq 0$. Clearly, there is only one unique type of such a graph and it is illustrated in Figure 1. Q_n can be constructed for $n \geq 10$. Let $n=4k-2$, for $k \geq 3$. The number of spanning trees for $K_{5,-3}$ equals 24, and for $K_{4,-1}$ equals 8. Hence, the number of spanning trees in Q_n equals $t(Q_n) = 3^2 8^{k-1}$.

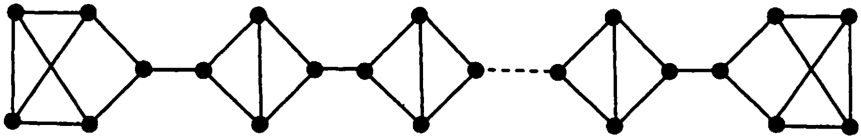

Figure 1 - Q_n for $n \equiv 2 \pmod 4$

For $n \equiv 0 \pmod 4$ we define Q_n as a cubic graph that consists of three $K_{5,-3}$ components and k $K_{4,-1}$ components connected together with edges, where $k \geq 0$. In this case Q_n can be obtained from Q_{n-6} as follows. Let $x_i x_j$ be an edge in Q_{n-6} that does not belong to a cycle. Then, replace $x_i x_j$ with $x_a x_b x_c$ and connect the vertex of degree 2 of $K_{5,-3}$ with the edge to x_b. Q_n can be constructed for $n \geq 16$. If $n=4k$ and $k \geq 4$ then the number of spanning trees in Q_n equals $t(Q_n) = 3^3 8^{k-1}$. Note, here Q_n is not unique because it depends on an edge being chosen for replacement in Q_{n-6}.

In the next section we prove that Q_n minimizes the number of spanning trees over all connected cubic graphs of order n, $n \geq 14$.

2. Connected Cubic Graph with Minimum Number of Spanning Trees.

We first prove two simple Lemmas.

Lemma 2.1 *Q_n minimizes the number of spanning trees over all cubic graphs of order $n = 14$ or 16.*

Proof: We executed a computer program based on Depth-first search that generated all cubic graphs for $n=14, 16$. For every generated cubic graph G_n we verified using Matrix-Tree Theorem [4] that $t(Q_{14}) \leq t(G_{14})$, $t(Q_{16}) \leq t(G_{16})$. □

Lemma 2.2 *For given Q_n, $n \geq 14$, the relation $t(Q_{n+2})/ t(Q_n) \leq 3$ is satisfied.*

Proof: Let k be an integer and $k \geq 4$. If $n = 4k-2$ then $t(Q_{n+2})/ t(Q_n) = (3^3 8^{k-1})/ (3^2 8^{k-1}) = 3$. If $n = 4k$ then $t(Q_{n+2})/ t(Q_n) = (3^2 8^k)/ (3^3 8^{k-1}) = 8/3$. □

Basic strategy in proving next main theorem is based on showing the following contradiction. If there exists G_n such that $G_n \neq Q_n$, $t(G_n) < t(Q_n)$ and n is minimized then there exists $G_{n-k} \neq Q_{n-k}$ such that $t(G_{n-k}) < t(Q_{n-k})$, where $k \geq 2$. We now present our main result.

Thorem 2.3 *Q_n minimizes the number of spanning trees over all connected cubic graphs of order n, where n ≥14.*

Proof: Suppose that there exists G_n such that $G_n \neq Q_n$ and $t(G_n) < t(Q_n)$. By Lemma 2.1 G_n exists if $n \geq 18$. Without loss of generality assume that G_n represents a graph of minimum order n $(n \geq 18)$, which satisfies $t(G_n) < t(Q_n)$. Consider first a subgraph $G^1(G_n)$ with a triangle that is not a subgraph of $K_{5,-3}(G_n)$ (Figure 2). Suppose $G^1(G_n)$ exists.

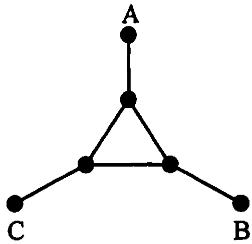

Figure 2 – Subgraph $G^1(G_n)$.

Suppose vertices A,B,C are pairwise distinct in G_n. Thus we can transform $G^1(G_n) \rightarrow H^1(G_{n-2})$ as follows:

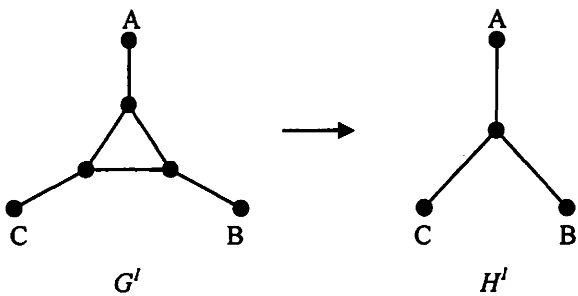

Figure 3 - Transformation $G^1(G_n) \rightarrow H^1(G_{n-2})$

where H^1 replaces G^1 in G_n inducing simple cubic graph G_{n-2}. In this case we have $3t(G_{n-2}) \leq t(G_n) < t(Qn)$ and by Lemma 2.2 $t(Q_n) \leq 3t(Q_{n-2})$. This implies $t(G_{n-2}) < t(Q_{n-2})$, which is a contradiction.

The above case implies that A,B,C cannot be pairwise distinct in $G^1(G_n)$. Suppose vertices A,B,C are not pairwise distinct in $G^1(G_n)$. So, without loss of generality assume $B=C$. Because $G^1(G_n)$ is not a subgraph of $K_{5,-3}$ then we have $A \neq B = C$. Consider the following two cases.

Case 1: Vertex A not adjacent to vertex B in G_n.
Then we can transform $G^1(G_n) \rightarrow H^2(G_{n-4})$ as follows:

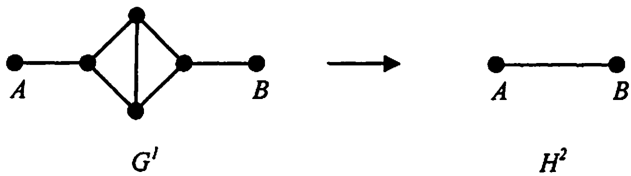

Figure 4 – Transformation $G^1(G_n) \rightarrow H^1(G_{n-4})$

where H^2 replaces G^1 in G_n inducing simple cubic G_{n-4}. Then, $8t(G_{n-4}) \leq t(G_n) < t(Qn)$ and $8t(Q_{n-4}) = t(Q_n)$. This implies $t(G_{n-4}) < t(Q_{n-4})$, which is a contradiction.

Case 2: Vertex A adjacent to vertex B in G_n.
Then $G^1(G_n)$ implies $G^2(G_n)$ in Figure 5, and we can transform $G^2(G_n) \rightarrow H^3(G_{n-2})$ as follows:

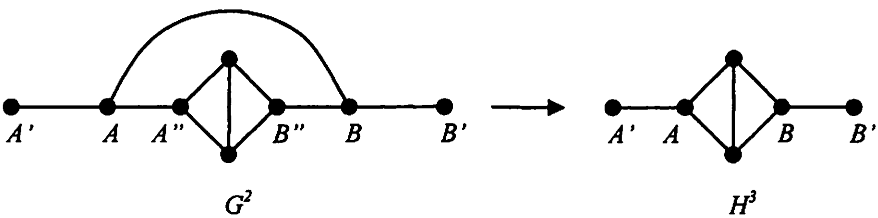

Figure 5 – Transformation $G^2(G_n) \rightarrow H^3(G_{n-2})$

where H^3 replaces G^2 in G_n inducing simple cubic G_{n-2}.

If i'th spanning tree in graph G induces a spanning tree in subgraph R of G then such a spanning tree in G we denote by $T_i(G,R)$. Otherwise, i'th spanning tree in G we denote by $\bar{T}_i(G,R)$. Let W_1 be a subgraphs of G_n induced by vertex set $\{A, A', A'', B, B', B''\}$. Similarly, let W_2 be a subgraph of G_{n-2} induced by $\{A, A', B, B'\}$. For every spanning tree of G_{n-2} we will identify globally unique spanning trees of G_n with identical edges, except for the edges implied by W_1 and W_2. So, for every spanning tree $T_1\ (G_{n-2}, H^3(G_{n-2}))$ in G_{n-2} there are three distinct spanning trees in G_n, which can be identified as follows:

1. $T_{i_1}(G_n, G^2(G_n))$ contains edges $A'A$, $A''A$, $B'B$, $B''B$,
2. $T_{i_2}(G_n, G^2(G_n))$ contains edges $A'A$, AB, $B'B$, $B''B$,
3. $T_{i_3}(G_n, G^2(G_n))$ contains edges $A'A$, $A''A$, AB, $B'B$.

Each $T_{i_j}(G_n, G^2(G_n))$ also contains edges $E(T_i(G_{n-2}, W_2)) - E(W_2)$, for $3 \geq j \geq 1$. In addition, for every spanning tree $\mathcal{T}_i\ (G_{n-2}, H^3(G_{n-2}))$ in G_{n-2} there are three sub-cases with three globally distinct spanning trees each in G_n. They can be identified as follows.

Case 2.1: $\mathcal{T}_i\ (G_{n-2}, H^3(G_{n-2}))$ includes edges $A'A$, $B'B$.
Then corresponding unique $\mathcal{T}_i\ (G_n, G^2(G_n))$ is identified by

1. $\mathcal{T}_{i_1}(G_n, G^2(G_n))$ contains edges $A'A$, $A''A$, $B'B$, $B''B$,
2. $\mathcal{T}_{i_2}(G_n, G^2(G_n))$ contains edges $A'A$, AB, $B'B$, $B''B$,
3. $\mathcal{T}_{i_3}(G_n, G^2(G_n))$ contains edges $A'A$, $A''A$, AB, $B'B$.

In addition, each $T_{i_j}(G_n, G^2(G_n))$ contains edges $E(T_i(G_{n-2}, W_2)) - \{A'A, B'B\}$, for $3 \geq j \geq 1$.

Case 2.2: $\mathcal{T}_i\ (G_{n-2}, H^3(G_{n-2}))$ includes edge $A'A$ and doesn't include edge $B'B$.
Then corresponding unique $\mathcal{T}_i\ (G_n, G^2(G_n))$ is identified by

1. $\mathcal{T}_{i_1}(G_n, G^2(G_n))$ contains edges $A'A$, $A''A$, $B'B$,
2. $\mathcal{T}_{i_2}(G_n, G^2(G_n))$ contains edges $A'A$, $A''A$, $B''B$,
3. $\mathcal{T}_{i_3}(G_n, G^2(G_n))$ contains edges $A'A$, $A''A$, AB.

In addition, each $T_{i_j}(G_n, G^2(G_n))$ contains edges $E(T_i(G_{n-2}, W_2)) - \{A'A\}$, for $3 \geq j \geq 1$.

Case 2.3: $\mathcal{T}_i\ (G_{n-2}, H^3(G_{n-2}))$ includes edge $B'B$ and doesn't include edge $A'A$.
Then corresponding unique $\mathcal{T}_i\ (G_n, G^2(G_n))$ is identified by

1. $\mathcal{T}_{i_1}(G_n, G^2(G_n))$ contains edges $B'B$, $B''B$, $A'A$,
2. $\mathcal{T}_{i_2}(G_n, G^2(G_n))$ contains edges $B'B$, $B''B$, $A''A$,
3. $\mathcal{T}_{i_3}(G_n, G^2(G_n))$ contains edges $B'B$, $B''B$, AB.

In addition, each $T_{i_j}(G_n, G^2(G_n))$ contains edges $E(T_i(G_{n-2}, W_2)) - \{B'B\}$, for $3 \geq j \geq 1$.

So, by Lemma 2.2 it follows that $t(G_{n-2}) \leq (1/3)t(G_n) < (1/3)t(Q_n) \leq t(Q_{n-2})$, which is a contradiction. Consequently, $G^1(G_n)$ cannot exist.

If a cycle of G_n has an edge in common with $K_{5,-3}(G_n)$ then it must be completely included in $K_{5,-3}(G_n)$. Otherwise, a cycle of G_n is excluded from $K_{5,-3}(G_n)$. Let $C_k = v_1v_2...v_kv_1$ be a shortest cycle in G_n, which is not included in $K_{5,-3}(G_n)$. Let $G^3(G_n)$ be a subgraph of G_n, (i.e., $G^3(G_n) \subseteq G_n$), such that $V(G_n) = \{v_1, v_2, ..., v_k, u_1, u_2, ..., u_k\}$, $v_1v_k \in E(G_n)$, $v_iv_{i+1} \in E(G_n)$, $v_ju_j \in E(G_n)$, for $k > i \geq 1$, $k \geq j \geq 1$. For analysis assume $v_1 = A'$, $v_2 = B'$, $v_3 = C'$, $v_4 = D'$, and $u_1 = A$, $u_2 = B$, $u_3 = C$, $u_4 = D$. Suppose $G^3(G_n)$ exists. Then we have two subsequent cases to consider.

Case 3: Cycle C_k is of length $k=4$.
We can transform $G^3(G_n) \rightarrow H^4(G_{n-2})$ as follows:

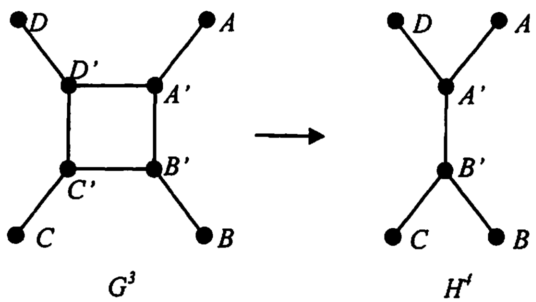

Figure 6 - Transformation $G^3(G_n) \rightarrow H^4(G_{n-2})$

where H^4 replaces G^3 in G_n inducing G_{n-2}. Furthermore, because $G^1(G_n)$ cannot exist, vertices A,D are distinct and so are vertices B,C. This in turn assures that induced cubic graph G_{n-2} is simple. Let W_1 be a subgraphs of G_n induced by $\{A', B', C', D'\}$. Similarly, let W_2 be a subgraph of G_{n-2} induced by $\{A', B'\}$. For every spanning tree of G_{n-2} we will identify globally unique spanning trees of G_n with identical edges, except for the edges implied by W_1 and W_2. In particular, for any spanning tree $T_i(G_{n-2}, W_2)$ in G_{n-2} there are four distinct spanning trees of type $T_i(G_n, W_1)$ in G_n, which can be identified as follows:

1. $T_{i_1}(G_n, W_1)$ contains edges $A'B', B'C', C'D' \in E(W_1)$,
2. $T_{i_2}(G_n, W_1)$ contains edges $B'C', C'D', D'A' \in E(W_1)$,
3. $T_{i_3}(G_n, W_1)$ contains edges $C'D', D'A', A'B' \in E(W_1)$,
4. $T_{i_4}(G_n, W_1)$ contains edges $D'A', A'B', B'C' \in E(W_1)$.

In addition each $T_{i_j}(G_n, W_1)$ contains edges $E(T_i(G_{n-2}, W_2)) - E(W_2)$, for $4 \geq j \geq 1$. Spanning trees are assured in above four scenarios because otherwise a cycle in $T_i(G_{n-2}, W_2)$ is implied.

Consider now $\mathcal{T}_i(G_{n-2}, W_2)$. Clearly, $\mathcal{T}_i(G_{n-2}, W_2)$ does not include edge $A'B'$. So, by definition of spanning tree there exists exactly one corresponding path $P_{A'B'}(G_{n-2})$ from vertex A' to vertex B' in G_{n-2} that includes exactly two edges from $H^4(G_{n-2})$. So, we have four cases to consider.

Case 3.1: $P_{A'B'}(G_{n-2}) = A'A \ldots BB'$.
Corresponding path in G_n is defined by $P_{A'B'}(G_n) = A'A \ldots CB'$. Then corresponding unique spanning trees in G_n are identified as follows:

1. $\mathcal{T}_{i_1}(G_n, G^3(G_n))$ contains edges $A'D', B'C' \in E(W_1)$,
2. $\mathcal{T}_{i_2}(G_n, G^3(G_n))$ contains edges $B'C', C'D' \in E(W_1)$,
3. $\mathcal{T}_{i_3}(G_n, G^3(G_n))$ contains edges $A'D', C'D' \in E(W_1)$.

In addition each $\mathcal{T}_{i_j}(G_n, W_1)$ contains edges $E(\mathcal{T}_i(G_{n-2}, W_2))$, for $4 \geq j \geq 1$. Spanning trees are assured in above three scenarios because in each case every combination of two edges does not create a cycle together with $P_{A'B'}(G_n)$.

Case 3.2: $P_{A'B'}(G_{n-2}) = A'A \ldots CB'$.
Corresponding path in G_n is defined by $P_{A'B'}(G_n) = A'A \ldots CC'$. Then corresponding unique spanning trees in G_n are identified as follows:

1. $\mathcal{T}_{i_1}(G_n, G^3(G_n))$ contains edges $A'D', B'C' \in E(W_1)$,
2. $\mathcal{T}_{i_2}(G_n, G^3(G_n))$ contains edges $B'C', C'D' \in E(W_1)$,
3. $\mathcal{T}_{i_3}(G_n, G^3(G_n))$ contains edges $A'B', A'D' \in E(W_1)$,
4. $\mathcal{T}_{i_4}(G_n, G^3(G_n))$ contains edges $A'B', C'D' \in E(W_1)$.

In addition each $\mathcal{T}_{i_j}(G_n, W_1)$ contains edges $E(\mathcal{T}_i(G_{n-2}, W_2))$, for $4 \geq j \geq 1$. Spanning trees are assured in above four scenarios because in each case every combination of two edges does not create a cycle together with $P_{A'B'}(G_n)$.

Case 3.3: $P_{A'B'}(G_{n-2}) = A'D \ldots BB'$.
Corresponding path in G_n is defined by $D'D \ldots BB'$. Then corresponding unique spanning trees in G_n are identified as follows:

1. $\mathcal{T}_{i_1}(G_n, G^3(G_n))$ contains edges $A'D', B'C' \in E(W_1)$,
2. $\mathcal{T}_{i_2}(G_n, G^3(G_n))$ contains edges $A'B', B'C' \in E(W_1)$,

3. $\mathcal{T}_{i_3}(G_n, G^3(G_n))$ contains edges $A'D'$, $C'D' \in E(W_1)$,
4. $\mathcal{T}_{i_4}(G_n, G^3(G_n))$ contains edges $A'B'$, $C'D' \in E(W_1)$.

In addition each $\mathcal{T}_{i_j}(G_n, W_1)$ contains edges $E(\mathcal{T}_i(G_{n-2}, W_2))$, for $4 \geq j \geq 1$. Spanning trees are assured in above four scenarios because in each case every combination of two edges does not create a cycle together with $P_{A'B'}(G_n)$.

Case 3.4: $P_{A'B'}(G_{n-2}) = A'D \dots CB'$.
Corresponding path in G_n is defined by $D'D \dots CC'$. Then corresponding unique spanning trees in G_n are identified as follows:

1. $\mathcal{T}_{i_1}(G_n, G^3(G_n))$ contains edges $A'D'$, $B'C' \in E(W_1)$,
2. $\mathcal{T}_{i_2}(G_n, G^3(G_n))$ contains edges $A'B'$, $B'C' \in E(W_1)$,
3. $\mathcal{T}_{i_3}(G_n, G^3(G_n))$ contains edges $A'B'$, $A'D' \in E(W_1)$.

In addition each $\mathcal{T}_{i_j}(G_n, W_1)$ contains edges $E(\mathcal{T}_i(G_{n-2}, W_2))$, for $4 \geq j \geq 1$. Spanning trees are assured in above three scenarios because in each case every combination of two edges does not create a cycle together with $P_{A'B'}(G_n)$.

So, by Lemma 2.2 it follows that $t(G_{n-2}) \leq (1/3)t(G_n) < (1/3)t(Q_n) \leq t(Q_{n-2})$, which is a contradiction. It means that G_n does not contain C_k, where $k=4$.

Case 4: Cycle C_k is of length at least $k=5$.
Then we can transform $G^3(G_n) \rightarrow H^5(G_{n-2})$ as illustrated in Figure 7.

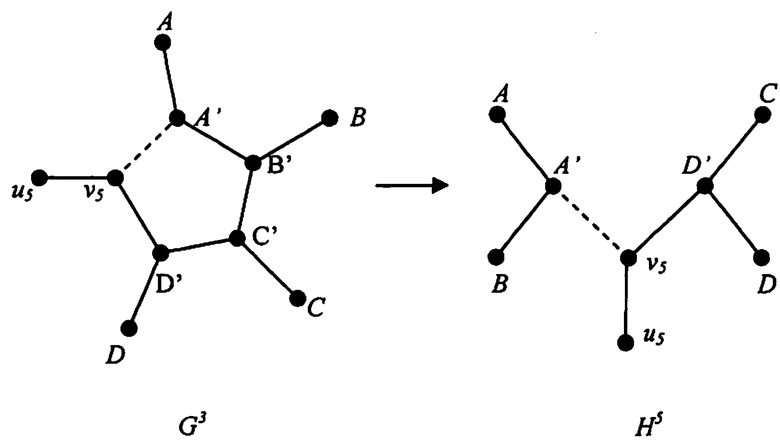

Figure 7 - Transformation $G^3(G_n) \rightarrow H^5(G_{n-2})$

where H^5 replaces G^3 in G_n inducing G_{n-2}. Dashed lines in Figure 7 represent path $v_5v_6 \dots v_kv_1$, where $k \geq 5$. Furthermore, because $G^1(G_n)$ cannot exist, vertices A,B are distinct and so are vertices C,D. This in turn assures that induced cubic graph G_{n-2} is simple.

Let W_1 be a subgraphs of G_n induced by $\{A', B, B', C, C', D'\}$, and let W_2 be a subgraph of G_{n-2} induced by $\{A', B, C, D'\}$. For every spanning tree of G_{n-2} we will identify globally unique spanning trees of G_n with identical edges, except for the edges implied by W_1 and W_2. Edge $A'D'$ (or BC) does not exist in G_n because it would imply a cycle $A'B'C'D'A'$ (or $BCC'B'B$) of length 4 – a contradiction. Edge $A'C$ (or $D'B$) does not exist in G_n because it would imply a cycle $A'B'C'CA'$ (or $D'C'B'BD'$) of length 4 – a contradiction. So, based on the transformation illustrated in Figure 7, W_2 consists of edges $E(W_2) = \{A'B, CD'\}$. Hence, there are four cases to consider.

Case 4.1: $\mathcal{T}_i(G_{n-2}, W_2)$ contains neither $A'B$ nor CD'.
Then corresponding unique spanning trees in G_n are identified as follows:

1. $\mathcal{T}_{i_1}(G_n, W_1)$ contains edges $A'B', B'C' \in E(W_1)$,
2. $\mathcal{T}_{i_2}(G_n, W_1)$ contains edges $BB', B'C' \in E(W_1)$,
3. $\mathcal{T}_{i_3}(G_n, W_1)$ contains edges $B'C', CC' \in E(W_1)$,
4. $\mathcal{T}_{i_4}(G_n, W_1)$ contains edges $B'C', C'D' \in E(W_1)$.

In addition each $\mathcal{T}_{i_j}(G_n, W_1)$ contains edges $E(\mathcal{T}_i(G_{n-2}, W_2))$, for $4 \geq j \geq 1$.

Case 4.2: $\mathcal{T}_i(G_{n-2}, W_2)$ contains $A'B$ and does not contain CD'.
Then corresponding unique spanning trees in G_n are identified as follows:

1. $\mathcal{T}_{i_1}(G_n, W_1)$ contains edges $A'B', BB', B'C' \in E(W_1)$,
2. $\mathcal{T}_{i_2}(G_n, W_1)$ contains edges $A'B', BB', CC' \in E(W_1)$,
3. $\mathcal{T}_{i_3}(G_n, W_1)$ contains edges $A'B', BB', C'D' \in E(W_1)$.

In addition each $\mathcal{T}_{i_j}(G_n, W_1)$ contains edges $E(\mathcal{T}_i(G_{n-2}, W_2)) - \{A'B\}$, for $3 \geq j \geq 1$.

Case 4.3: $\mathcal{T}_i(G_{n-2}, W_2)$ contains CD' and does not contain $A'B$.
Then corresponding unique spanning trees in G_n are identified as follows:

1. $\mathcal{T}_{i_1}(G_n, W_1)$ contains edges $CC', C'D', C'B' \in E(W_1)$,
2. $\mathcal{T}_{i_2}(G_n, W_1)$ contains edges $CC', C'D', BB' \in E(W_1)$,
3. $\mathcal{T}_{i_3}(G_n, W_1)$ contains edges $CC', C'D', A'B' \in E(W_1)$.

In addition each $T_{i_j}(G_n, W_1)$ contains edges $E(T_i(G_{n-2}, W_2)) - \{CD'\}$, for $3 \geq j \geq 1$.

Case 4.4: $T_i(G_{n-2}, W_2)$ contains $A'B$ and CD'.
Corresponding unique spanning trees in G_n are identified as follows.

1. $T_{i_1}(G_n, W_1)$ contains edges $A'B', BB', CC', C'D' \in E(W_1)$.

If we remove either CC' or $C'D'$ from $T_{i_1}(G_n, W_1)$ then an induced graph I_n must consist of two components, which are trees. In addition, vertices C, D' must belong to two different components in I_n. This means that either vertices C, A' or vertices D', A' belong to two different components in either case. If vertices C, A' belong to two different components then we obtain

2. $T_{i_2}(G_n, W_1)$ contains edges $A'B', BB', B'C', CC' \in E(W_1)$.

Otherwise, when vertices D', A' belong to two different components we obtain

2. $T_{i_2}(G_n, W_1)$ contains edges $A'B', BB', B'C', C'D' \in E(W_1)$.

If we remove either $A'B'$ or BB' from $T_{i_1}(G_n, W_1)$ then an induced graph I_n must also consist of two components, which are trees. In addition, vertices A', B must belong to two different components in I_n. This means that either vertices A', D' or vertices B, D' must belong to two different components in either case. If vertices A', D' belong to two different components then we obtain

3. $T_{i_3}(G_n, W_1)$ contains edges $A'B', B'C', CC', C'D' \in E(W_1)$.

Otherwise, when vertices B, D' belong to two different components we obtain

3. $T_{i_3}(G_n, W_1)$ contains edges $BB', B'C', CC', C'D' \in E(W_1)$.

In addition each $T_{i_j}(G_n, W_1)$ contains edges $E(T_i(G_{n-2}, W_2)) - \{A'B, CD'\}$, for $3 \geq j \geq 1$.

So, by Lemma 2.2 it follows that $t(G_{n-2}) \leq (1/3)t(G_n) < (1/3)t(Q_n) \leq t(Q_{n-2})$, which is a contradiction. It means that G_n does not contain C_k, where $k \geq 5$.

Hence, based on Cases 1,2,3,4 every edge which belongs to a cycle in G_n must also belong to a subgraph $K_{5,-3}$ of G_n. This implies that G_n must contain at least four $K_{5,-3}$ subgraphs. Otherwise, $G_n = Q_n$ that implies a contradiction. Consequently, G_n must contain G^4 (Figure 8), where AC and BC do not belong to any cycle. But then we can transform $G^4(G_n) \rightarrow H^6(G_{n-2})$ as illustrated in Figure 8.

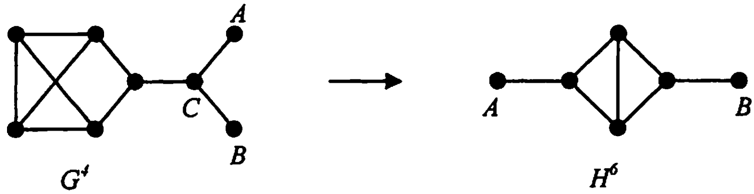

Figure 8 - Transformation $G^4(G_n) \rightarrow H^6(G_{n-2})$

Since $t(G^4(G_n)) = 24$ and $t(H^6(G_{n-2})) = 8$, it follows from our assumption that $t(G_{n-2}) = (1/3)t(G_n) < (1/3)t(Q_n) \leq t(Q_{n-2})$ - a contradiction. □

Finally, generalization of the result obtained for cubic graphs in this paper to all regular graphs (with possible strict inequality) would be a more challenging goal.

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