Isolated Toughness and Fractional (g, f)-Factors of Graphs *

Sizhong Zhou †
School of Mathematics and Physics
Jiangsu University of Science and Technology
Mengxi Road 2, Zhenjiang, Jiangsu 212003
People's Republic of China
Ziming Duan
School of Science
China University of Mining and Technology
Xuzhou, Jiangsu, 221008
People's Republic of China
Bingyuan Pu
Department of Fundamental Education
Chengdu Textile College
Chengdu, Sichuan, 610023
People's Republic of China

Abstract

Let G be a graph, and let a and b be nonnegative integers such that $1 \leq a \leq b$, and let g and f be two nonnegative integer-valued functions defined on $V(G)$ such that $a \leq g(x) \leq f(x) \leq b$ for each $x \in V(G)$. A spanning subgraph F of G is called a fractional (g, f)-factor if $g(x) \leq d_G^h(x) \leq f(x)$ for all $x \in V(G)$, where $d_G^h(x) = \sum_{e \in E_x} h(e)$ is the fractional degree of $x \in V(F)$ with $E_x = \{e : e = xy \in E(G)\}$. The isolated toughness $I(G)$ of a graph G is defined as follows: If G is a complete graph, then $I(G) = +\infty$;

*This research was supported by Jiangsu Provincial Educational Department (07KJD110048).

†Corresponding author.E-mail address: zsz_cumt@163.com(S. Zhou)

else, $I(G) = \min\{\frac{|S|}{i(G-S)} : S \subseteq V(G), i(G-S) \geq 2\}$, where $i(G-S)$ denotes the number of isolated vertices in $G-S$. In this paper, we prove that G has a fractional (g, f)-factor if $\delta(G) \geq I(G) \geq \frac{b(b-1)}{a}+1$. This result is best possible in some sense.

Keywords: graph, isolated toughness, (g, f)-factor, fractional (g, f)-factor.

AMS Subject: 05C70

1 Introduction

We consider only finite undirected simple graph G with vertex set $V(G)$ and edge set $E(G)$. For $x \in V(G)$, we denote by $d_G(x)$ the degree of x in G, by $\delta(G)$ the minimum vertex degree of G and by $N_G(x)$ the set of vertices adjacent to x in G. For any $S \subseteq V(G)$, we define $N_G(S) = \cup_{x \in S} N_G(x)$. Let S and T be disjoint subsets of $V(G)$. We denote by $e_G(S, T)$ the number of edges joining S and T. For a subset $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced by S and by $G-S$ the subgraph obtained from G by deleting vertices in S together with the edges incident to vertices in S. A vertex set $S \subseteq V(G)$ is called independent if $G[S]$ has no edges. For any $S \subseteq V(G)$, we use $i(G-S)$ to denote the number of isolated vertices of $G-S$. The isolated toughness $I(G)$ of a graph G is defined by Ma and Liu [1] as follows.

$$I(G) = \min\{\frac{|S|}{i(G-S)} : S \subseteq V(G), i(G-S) \geq 2\},$$

if G is not complete; otherwise, $I(G) = +\infty$.

Let $g(x)$ and $f(x)$ be two nonnegative integer-valued functions defined on $V(G)$ such that $g(x) \leq f(x)$ for each $x \in V(G)$. Then a spanning subgraph F of G is called a (g, f)-factor if $g(x) \leq d_F(x) \leq f(x)$ for all $x \in V(G)$. If $g(x) = a$ and $f(x) = b$ for each $x \in V(G)$, then a (g, f)-factor of G is called an $[a, b]$-factor of G. If $g(x) = f(x) = k$ for each $x \in V(G)$, then a (g, f)-factor of G is called a k-factor of G.

Let $h(e) \in [0, 1]$ be a function defined on $E(G)$ and $d_G^h(x) = \sum_{e \in E_x} h(e)$, where $E_x = \{e : e = xy \in E(G)\}$. Then $d_G^h(x)$ is called the fractional degree of x in G. We call h an indictor function if $g(x) \leq d_G^h(x) \leq f(x)$ holds for each $x \in V(G)$. Let $E^h = \{e : e \in E(G), h(e) \neq 0\}$ and G_h be a spanning subgraph of G such that $E(G_h) = E^h$. We call G_h a fractional

(g, f)-factor. Similarly, we define the fractional $[a, b]$-factor and the fractional k-factor of G, where a, b and k are nonnegative integers. The other terminologies and notations not given in this paper can be found in [2,3].

Many authors have investigated (g, f)-factors [4-7] and $[a, b]$-factors[8-10]. The following results on fractional (g, f)-factors and fractional $[a, b]$-factors and fractional k-factors are known.

Theorem 1 [11] *Let G be a graph, and let g and f be two non-negative integer-valued functions defined on $V(G)$ such that $g(x) \leq f(x) \leq d_G(x)$ for each $x \in V(G)$. If $(f(x) - k)d_G(y) \geq (d_G(x) - k)g(y)$ for each $x, y \in V(G)$ with $x \neq y$, then G has a fractional (g, f)-factor containing any k edges of G. Where k is a non-negative integer.*

Theorem 2 [12] *Let $k \geq 2$ be an integer, and let G be a graph of order n such that $n \geq 4k - 6$. Then*
(1) If kn is even, and $bind(G) > \frac{(2k-1)(n-1)}{k(n-2)+3}$, then G has a fractional k-factor; and
(2) If kn is odd, and $bind(G) > \frac{(2k-1)(n-1)}{k(n-2)+2}$, then G has a fractional k-factor.

Theorem 3 [13] *Suppose that G is a graph with $\delta(G) \geq k$ and $I(G) \geq k$, where k is a positive integer. Then G has a fractional k-factor.*

Theorem 4 [13] *Let G be a graph and $a < b$ be positive integers. If the minimum degree of G and the isolated toughness of G satisfying $\delta(G) \geq I(G) \geq a - 1 + \frac{a}{b}$, then G has a fractional $[a, b]$-factor.*

2 The Proof of Main Theorem

In this paper, we give an isolated toughness condition for a graph to have a fractional (g, f)-factor. Our theorem is a more general form of Theorem 3 and Theorem 4 in a certain sense.

Theorem 5 *Let G be a graph, and let a and b be integers such that $1 \leq a \leq b$, and let g and f be two nonnegative integer-valued functions defined on $V(G)$ such that $a \leq g(x) \leq f(x) \leq b$ for each $x \in V(G)$. If $\delta(G) \geq I(G) \geq \frac{b(b-1)}{a} + 1$, then G has a fractional (g, f)-factor.*

The proof of Theorem 5 depends on the following theorems.

Anstee [14] obtained the necessary and sufficient condition for a graph to have fractional (g, f)-factor by algorithm. Liu [15] proved it by graphical methods.

Theorem 6 [14,15] *Let G be a graph, and let g and f be two nonnegative integer-valued functions defined on $V(G)$ such that $g(x) \le f(x)$ for each $x \in V(G)$. Then G has a fractional (g, f)-factor if and only if for any $S \subseteq V(G)$,*

$$g(T) - d_{G-S}(T) \le f(S),$$

where $T = \{x : x \in V(G) - S, d_{G-S}(x) \le g(x)\}$.

Theorem 7 [16] *Let H be a graph and $a \ge 1$ be an integer, and let $T_1, \cdots, T_{a-1}$ be a partition of $V(H)$ such that $d_H(x) \le j$ for $\forall x \in T_j$ (T_j may be empty sets), $j = 1, \cdots, a-1$. Then there exist an independent set I and a covered set C of H such that*

$$\sum_{j=1}^{a-1}(a-j)c_j \le (a-1)\sum_{j=1}^{a-1}(a-j)i_j,$$

where $i_j = |I \cap T_j|$, $c_j = |C \cap T_j|$, $j = 1, \cdots, a-1$.

The proof of Theorem 5. Since $\delta(G) \ge \frac{b(b-1)}{a} + 1 \ge b \ge g(x)$, if G is a complete graph, then G has a (g, f)-factor, and then G has a fractional (g, f)-factor. In the following we suppose that G is not a complete graph.

Suppose that G satisfies the assumption of theorem, but it has not a fractional (g, f)-factor. Then, by Theorem 6, there exists a subset S of $V(G)$ such that

$$g(T) - d_{G-S}(T) > f(S), \tag{1}$$

where $T = \{x : x \in V(G) - S,\ d_{G-S}(x) \le g(x)\}$.

We choose subsets S and T such that $|T|$ is minimum and S and T satisfy (1).

Here, we prove the following claims.

Claim 1. $d_{G-S}(x) \le g(x) - 1 \le b - 1$ for each $x \in T$.

Proof. Suppose that there exists a vertex $x \in T$ such that $d_{G-S}(x) \ge g(x)$. Then the subsets S and $T - \{x\}$ satisfy (1), which contradicts the choice of T.
Completing the proof of Claim 1.

Claim 2. $|S| \ge 1$.

Proof If $|S| = 0$, then we have

$$g(T) - d_{G-S}(T) \leq f(S).$$

This contradicts (1). This completes the proof of Claim 2.

Let $T_j = \{x : x \in T,\ d_{G-S}(x) = j\}$, and $|T_j| = t_j$, $j = 0, 1, \cdots, b-1$. Set $H = G[T_1 \cup T_2 \cup \cdots \cup T_{b-1}]$, we have $d_H(x) \leq j$ for $\forall x \in T_j$. According to Theorem 7, there exist an independent set I and a covered set C of H such that

$$\sum_{j=1}^{b-1}(b-j)c_j \leq (b-1)\sum_{j=1}^{b-1}(b-j)i_j, \tag{2}$$

where $i_j = |I \cap T_j|$, $c_j = |C \cap T_j|$, $j = 1, \cdots, b-1$.

Let's assume that I be a maximal independent set of H. Put $W = G - S - T$, $U = S \cup C \cup (N_G(I) \cap V(W))$. Then

$$|U| \leq |S| + \sum_{j=1}^{b-1} j i_j \tag{3}$$

and

$$i(G-U) \geq t_0 + \sum_{j=1}^{b-1} i_j, \tag{4}$$

where t_0 is the number of isolated vertices in T.

Here, we prove the following claim.

Claim 3. $|U| \geq i(G-U)I(G)$.

Proof The proof splits into two cases.

Case 1. $i(G-U) \geq 2$.

According to the definition of $I(G)$, we obtain

$$|U| \geq i(G-U)I(G).$$

Case 2. $i(G-U) \leq 1$.

In view of (4), we get that

$$1 \geq i(G-U) \geq t_0 + \sum_{j=1}^{b-1} i_j.$$

Thus, there exists at most an integer j_0 such that $i_{j_0} = 1$ and $1 \leq j_0 \leq b-1$. Therefore, H is a complete graph. Let v_0 be a vertex of H. Then, we have

$$\begin{aligned} |U| &= |S \cup C \cup (N_G(I) \cap V(W))| \\ &\geq |S| + d_{G-S}(v_0) \geq d_G(v_0) \geq \delta(G) \\ &\geq I(G) \geq i(G-U)I(G). \end{aligned}$$

Completing the proof of Claim 3.

According to (3), (4) and Claim 3, we have

$$|S| + \sum_{j=1}^{b-1} j i_j \geq I(G)(t_0 + \sum_{j=1}^{b-1} i_j). \tag{5}$$

According to (1) and $a \leq g(x) \leq f(x) \leq b$ for each $x \in V(G)$, we obtain

$$b|T| - d_{G-S}(T) \geq g(T) - d_{G-S}(T) > f(S) \geq a|S|. \tag{6}$$

By (5) and (6), we get that

$$\begin{aligned} bt_0 + \sum_{j=1}^{b-1}(b-j)i_j + \sum_{j=1}^{b-1}(b-j)c_j &> a|S| \\ &\geq a(I(G)t_0 + I(G)\sum_{j=1}^{b-1} i_j - \sum_{j=1}^{b-1} j i_j) \\ &= \sum_{j=1}^{b-1}(aI(G) - aj)i_j + at_0 I(G), \end{aligned}$$

that is,

$$bt_0 + \sum_{j=1}^{b-1}(b-j)i_j + \sum_{j=1}^{b-1}(b-j)c_j > \sum_{j=1}^{b-1}(aI(G) - aj)i_j + at_0 I(G). \tag{7}$$

Since $I(G) \geq \frac{b(b-1)}{a} + 1$, we have $aI(G) \geq a(\frac{b(b-1)}{a} + 1) \geq b$. Then by (7), we obtain

$$\sum_{j=1}^{b-1}(b-j)i_j + \sum_{j=1}^{b-1}(b-j)c_j > \sum_{j=1}^{b-1}(aI(G) - aj)i_j. \tag{8}$$

In view of (2) and (8), we obtain

$$\begin{aligned} b\sum_{j=1}^{b-1}(b-j)i_j &= (b-1)\sum_{j=1}^{b-1}(b-j)i_j + \sum_{j=1}^{b-1}(b-j)i_j \\ &\geq \sum_{j=1}^{b-1}(b-j)i_j + \sum_{j=1}^{b-1}(b-j)c_j \\ &> \sum_{j=1}^{b-1}(aI(G) - aj)i_j. \end{aligned}$$

i.e.,

$$\sum_{j=1}^{b-1}(b(b-j)-aI(G)+aj)i_j > 0. \tag{9}$$

Let $\Phi(j) = b(b-j) - aI(G) + aj$. By $b \geq a$, we have $\Phi^{'}(j) \leq 0$. Moreover, by $I(G) \geq \frac{b(b-1)}{a} + 1$, we get $\Phi(1) = b(b-1) - aI(G) + a \leq 0$. Thus, we have

$$\Phi(j) \leq 0, \qquad j = 1, \cdots, b-1.$$

Thus, we get that

$$\sum_{j=1}^{b-1}(b(b-j)-aI(G)+aj)i_j \leq 0,$$

which contradicts (9).

From the argument above, we deduce the contradiction. Hence, G has a fractional (g,f)-factor.
Completing the proof of Theorem 5.

Remark Let $a = b$. Then, we have $g(x) = f(x) = a$ and $I(G) \geq a$. In the following, let us show that the condition $I(G) \geq a$ in Theorem 5 can not be replaced by $I(G) \geq a - \varepsilon$, where ε is any positive real number. We construct a graph H from K_{na^2}, K_{a-1} and $(na+1)K_1$ as follows. Let $V(K_{na^2}) = \{x_1, x_2, \cdots, x_{na+1}, \cdots, x_{na^2}\}$, $V((na+1)K_1) = \{y_1, y_2, \cdots, y_{na+1}\}$ and $V(K_{a-1}) = \{z_1, z_2, \cdots, z_{a-1}\}$, where n is any positive integer and let $E(H) = E(K_{na^2}) \cup E(K_{a-1}) \cup (\bigcup_{i=1}^{na+1} x_i y_i) \cup \{uv : u \in V(K_{a-1}), v \in (na+1)K_1\}$. Set $S = V(K_{a-1} \cup V(K_{na^2}))$. Obviously, $I(H) = \frac{na^2+a-1}{na+1} = a - \frac{1}{na+1}$ and where n is large enough and for any given positive real number ε, $I(H) = a - \frac{1}{na+1} \geq a - \varepsilon$. Let $S = V(K_{a-1})$ and $T = V((na+1)K_1)$. Then, we obtain

$$a|T| - d_{H-S}(T) = a|T| - |T| = (a-1)|T| = (a-1)(na+1) > a(a-1) = a|S|.$$

By Theorem 6, there are not any fractional a-factors in H. In the above sense, the result in Theorem 5 is best possible.

Acknowledgments. The authors would like to express their gratitude to the referees for their very helpful and detailed comments in improving this paper.

References

[1] Yinghong Ma and Guizhen Liu, Isolated toughness and existence of fractional factors in graphs, Acta. Appl. Math. Sinica, 26(2003), 133-140.

[2] J. A. Bondy, U. S. R. Murty, Graph Theory with Applications. London, The Macmillan Press, 1976.

[3] Edward R. Schinerman, D.H.Ullman. Fractional Graph Theory. John Wiley and Son, Inc. New York, 1997.

[4] Guizhen Liu, $(g < f)$-factors of graphs, Acta Math. Sci. (in China), 14(1994), 285-290.

[5] Sizhong Zhou, Some sufficient conditions for graphs to have (g, f)-factors, Bulletin of the Australian Mathematical Society, 75(2007), 447-452.

[6] Y. Egawa and M. Kano, Sufficient conditions for graphs to have (g, f)-factors, Discrete Mathematics, 151(1996), 87-90.

[7] Sizhong Zhou and Xiuqian Xue, Complete-factors and (g, f)-covered graphs, Australasian Journal of Combinatorics, 37(2007), 265-269.

[8] H. Matsuda, Fan-type results for the existence of $[a, b]$-factors, Discrete Mathematics, 306(2006), 688-693.

[9] Sizhong Zhou and Jiashang Jiang, Notes on the Binding Numbers for (a, b, k)-Critical Graphs, Bulletin of the Australian Mathematical Society, to appear.

[10] M. Kano, A sufficient condition for a graph to have $[a, b]$-factors, Graphs and Combinatorics, 6(1990), 245-251.

[11] Sizhong Zhou and Changming Shang, Some sufficient conditions with fractional (g, f)-factors in graphs, Chinese Journal of Engineering Mathematics, 24(2)(2007), 31-35.

[12] Sizhong Zhou and Ziming Duan, Binding numbers and fractional k-factors of graphs, Ars Combinatoria, to appear.

[13] Yinghong Ma and Guizhen Liu, Fractional factors and isolated toughness of graphs, Mathematica Applicata, 19(1)(2006), 188-194.

[14] Anstee R P., An algorithmic proof Tutte's f-factor theorem, J. Algorithms, 6(1985), 112-131.

[15] Guizhen Liu and Lanju Zhang, Fractional (g, f)-factors of graphs, Acta Mathematica Scientia, 21B(4)(2001), 541-545.

[16] P. Katerinis, Toughness of graphs and the existence of factors, Discrete Mathematics, 80(1990), 81-92.