On the lower and upper bounds for the Euclidean norm of a complex matrix and its Applications

Ayse Dilek Gungor
Selcuk University
Faculty of Arts and Science
Department of Mathematics
42031, Konya, TURKEY
drdilekgungor@gmail.com

June 26, 2008

Abstract

In this study, we obtained lower and upper bounds for the Euclidean norm of a complex matrix A of order $n \times n$. In addition, we found lower and upper bounds for the spectral norms and Euclidean norms of Hilbert matrix, its Hadamard square root, Cauchy-Toeplitz and Cauchy-Hankel matrices in the forms $H = (1/(i+j-1))_{i,\ j=1}^{n}$, $H^{\circ 1/2} = (1/(i+j-1)^{1/2})_{i,\ j=1}^{n}$, $T_n = [1/(g+(i-j)h)]_{i,\ j=1}^{n}$ and $H_n = [1/(g+(i+j)h)]_{i,j=1}^{n}$, respectively.

Keywords: Hilbert matrix; Cauchy-Toeplitz matrix; Cauchy-Hankel matrix; Norm; Lower and Upper bounds.

1 Introduction and Preliminaries

Let $A = (a_{ij})$ be an $n \times n$ symmetric matrix with all positive entries. Then the Hadamard inverse of A is defined by $A^{\circ(-1)} = (1/a_{ij})_{i,j=1}^{n}$, and the Hadamard square root by $A^{\circ 1/2} = (a_{ij}^{1/2})_{i,j=1}^{n}$ [6].

The matrix

$$H = (1/(i+j-1))_{i,j=1}^{n} \tag{1.1}$$

is well known as Hilbert matrix. Hence the Hadamard square root of Hilbert matrix is denoted by

$$H^{\circ 1/2} = (1/(i+j-1)^{1/2})_{i,j=1}^{n}. \tag{1.2}$$

Let $C = [1/(x_i - y_j)]_{i,j=1}^{n} (x_i \neq y_j)$ be a Cauchy matrix and $T_n = [t_{j-i}]_{i,j=0}^{n-1}$ be a Toeplitz matrix. In generally Cauchy-Toeplitz matrix is being defined as

$$T_n = \left[\frac{1}{g + (i-j)h} \right]_{i,j=1}^{n} \tag{1.3}$$

where $h \neq 0$, g and h are any numbers and g/h is not integer. Toeplitz matrices are precisely those matrices that one constant along all diagonals parallel to the main diagonal, and thus a Toeplitz matrix is determined by its first row and column.

On the other hand, let $H_n = [h_{i+j}]_{i,j=0}^{n-1}$ be a Hankel matrix. Every $n \times n$ Cauchy-Hankel matrix is of the form

$$H_n = \left[\frac{1}{g + (i+j)h} \right]_{i,j=1}^{n} \tag{1.4}$$

where $h \neq 0$, g and h are any numbers and g/h is not integer. Hankel matrices are symmetric.

Recently, there have been several papers on the norms of Cauchy-Toeplitz matrix and Cauchy-Hankel matrix [1,2,4,5]. Refs. [4,5] are related to the spectral norm of Cauchy-Toeplitz matrix. In [5], a lower bound for the spectral norm of Cauchy-Toeplitz matrix was obtained by Tyrtyshnikov taking $g = 1/2$ and $h = 1$ in the (1.3). Parter proved that singular values could be related to eigenvalues of certain Hermitian Toeplitz matrices corresponding to Laurent-Fourier series [4]. Gungor [9] obtained lower bounds for the spectral norm and Euclidean norm of Cauchy-Toeplitz and Cauchy-Hankel matrices in the forms (1.3) and (1.4) by taking $g = 1/2$ and $h = 1$ [8]. Solak, Turkmen and Bozkurt [7] obtained upper bounds for the l_p norm of the Hilbert matrix and its Hadamard square root.

In this paper, firstly, we have established lower and upper bounds for the Euclidean norm of a complex matrix A of order $n \times n$ using B matrix defined as in [8]. In Section 3, we have obtained upper and lower bounds for Euclidean norms of the Hilbert matrix and its Hadamard square root. In Section 4, we have established upper and lower bounds for Euclidean norms of Cauchy-Toeplitz and Cauchy-Hankel matrices in the forms (1.3) and (1.4) by taking $g = 1/2$ and $h = 1$. In Section 5, we have found upper bounds for spectral norms of the Hilbert matrix and its Hadamard square root. In Section 6, we have obtained upper bounds for spectral norms of Cauchy-Toeplitz and Cauchy-Hankel matrices in the forms (1.3) and (1.4) by taking $g = 1/2$ and $h = 1$. Consequently, we have given an example related to all of these bounds which are found.

Now, we give some preliminaries related to our study. Let A be an $n \times n$ complex matrix. Let $\sigma_i(A)$'s $(i = 1, ..., n)$ such that $\sigma_1(A) \geq$

$\sigma_2(A) \geq ... \geq \sigma_n(A)$ be the singular values of A. Its well known Euclidean norm of matrix A is

$$\|A\|_E^2 = \sigma_1^2(A) + \sigma_2^2(A) + ... + \sigma_n^2(A) \tag{1.5}$$

and also the spectral norm of the matrix A is

$$\|A\|_2 = \sqrt{\max_{1\leq i\leq n} \lambda_i(A^H A)}, \tag{1.6}$$

where λ_i is an eigenvalue of $A^H A$ and A^H is conjugate transpose of matrix A.

A function Ψ is called a psi (or digamma) function if

$$\Psi(x) = \frac{d}{d(x)} \{\log[\Gamma(x)]\}$$

where

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt.$$

The n th derivate of a psi function is called a polygamma function, i.e.

$$\Psi(n,x) = \frac{d}{dx^n}[\Psi(x)] = \frac{d}{dx^n}\left\{\frac{d}{dx}(\ln[\Gamma(x)])\right\} \text{ [5]}.$$

If $n = 0$ then $\Psi(n,x) = \Psi(x) = \frac{d}{dx}\{\ln[\Gamma(x)]\}$. On the other hand, if $a > 0$ and b are any numbers and n is a positive integer, then

$$\lim_{n\to\infty} \Psi(a, n+b) = 0 \text{ [3]}.$$

Euler-Mascheroni constant, γ, is defined by

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln(n)\right) = 0.577215664901533....$$

To minimize the numerical round-off errors in solving the system $Ax = b$, it is normally convenient that the rows of A be properly scaled before the solution procedure begins. One way is to premultiply by the diagonal matrix

$$D = \text{diag}\left\{\frac{\alpha_1}{r_1(A)}, \frac{\alpha_2}{r_2(A)}, ..., \frac{\alpha_n}{r_n(A)}\right\}, \tag{1.7}$$

where $r_i(A)$ is the Euclidean norm of the $i-$th row of A and $\alpha_1, \alpha_2, ..., \alpha_n$ are positive real numbers such that

$$\alpha_1^2 + \alpha_2^2 + ... + \alpha_n^2 = n. \tag{1.8}$$

Clearly, the Euclidean norm of the coefficient matrix $B = DA$ of the scaled system is equal to $\sqrt{n}$ and if $\alpha_1 = \alpha_2 = ... = \alpha_n = 1$ then each row of B is a unit vector in the Euclidean norm. Also, we can define $B = AD$,

$$D\text{=diag}\left\{\frac{\alpha_1}{c_1(A)}, \frac{\alpha_2}{c_2(A)}, ..., \frac{\alpha_n}{c_n(A)}\right\}, \qquad (1.9)$$

where $c_i(A)$ is the Euclidean norm of the i–th columm of A. Again, $\|B\|_E = \sqrt{n}$ and if $\alpha_1 = \alpha_2 = ... = \alpha_n = 1$ then each column of B is a unit vector in the Euclidean norm.

Since the matrices PA, AP and A have the same singular values for any permutation matrix P, we assume, without loss of generality, that the rows and columns of A are such that

$$r_1(A) \leq r_2(A) \leq ... \leq r_n(A), \qquad (1.10)$$

$$c_1(A) \leq c_2(A) \leq ... \leq c_n(A), \qquad (1.11)$$

and α_i's in (1.8) are ordered in such a way that

$$0 < \alpha_n \leq ... \leq \alpha_2 \leq \alpha_1. \qquad (1.12)$$

Let $A = (a_{ij})$ and $B = (b_{ij})$ be $n \times n$ matrices. The Hadamard product of A and B is defined by $A \circ B = (a_{ij}b_{ij})$. Let A, B and C be $m \times n$ matrices. If $A = B \circ C$ then

$$\|A\|_2 \leq r_n(B)\, c_n(C) \ . \qquad [11]$$

Theorem 1 [10] *Assume that A and B are two arbitrary $n \times n$ matrices with the singular values*

$$\sigma_n(A) \leq ... \leq \sigma_1(A), \ \sigma_n(B) \leq ... \leq \sigma_1(B).$$

Then the singular values

$$\sigma_n(AB) \leq ... \leq \sigma_1(AB)$$

of the matrix AB satisfy

$$\sigma_i(AB) = \theta_i \sigma_i(A) = \eta_i \sigma_i(B), \ \sigma_n(B) \leq \theta_i \leq \sigma_1(B), \ \sigma_n(A) \leq \eta_i \leq \sigma_1(A)$$

and

$$\sigma_i(AB) = w_i \sqrt{\sigma_i(A)\sigma_i(B)}, \ \sqrt{\sigma_n(A)\sigma_n(B)} \leq w_i \leq \sqrt{\sigma_1(A)\sigma_1(B)}$$

where $1 \leq i \leq n$.

2 Lower and upper bounds for the Euclidean norm of a complex matrix

Theorem 2 *Let* A *be an* $n \times n$ *complex matrix. Let* α_i*'s and* $r_i(A)$*'s (* $c_i(A)$*'s) be as in* (1.12) *and* (1.10), (1.11), *respectively. Then*

$$\frac{\sqrt{n.\max\{c_1^2(A),\ r_1^2(A)\}}}{\alpha_1} \leq \|A\|_E \leq \frac{\sqrt{n.\min\{c_n^2(A),\ r_n^2(A)\}}}{\alpha_n}. \quad (2.1)$$

Proof. Firstly, we can write Theorem $1(a)$ in the form

$$\sigma_i(AB) = \theta_i \sigma_i(A) = \eta_i \sigma_i(B)$$

where $\sigma_n(B) \leq \theta_i \leq \sigma_1(B)$, $\sigma_n(A) \leq \eta_i \leq \sigma_1(A)$, $1 \leq i \leq n$. Hence, we obtain

$$\sigma_n(B)\sigma_i(A) \leq \sigma_i(AB) \leq \sigma_1(B)\sigma_i(A) \quad (2.2)$$

and

$$\sigma_n(A)\sigma_i(B) \leq \sigma_i(AB) \leq \sigma_1(A)\sigma_i(B). \quad (2.3)$$

By applying $B = AD$ and $B = DA$ matrices to (2.2) and (2.3), respectively, we have

$$\sigma_n(D)\sigma_i(A) \leq \sigma_i(B) \leq \sigma_1(D)\sigma_i(A).$$

Hence, (2.1) is obvious. ■

Corollary 3 *Let* A *be an* $n \times n$ *complex matrix. Let* α_i*'s and* $r_i(A)$*'s (* $c_i(A)$*'s) be as in* (1.12) *and* (1.10) ((1.11)), *respectively. Then*

$$\frac{\sqrt{\max\{c_1^2(A),\ r_1^2(A)\}}}{\alpha_1} \leq \|A\|_2. \quad (2.4)$$

Proof. Since

$$\frac{1}{\sqrt{n}}\|A\|_E \leq \|A\|_2, \quad (2.5)$$

then we obtain (2.4) from (2.1) . ■

3 Lower and upper bounds for norms of the Hilbert matrix and its Hadamard square root

Theorem 4 *Let the matrix* H *and* α_i*'s be as in* (1.1) *and* (1.12), *respectively. Then*

$$\frac{\sqrt{n(-\Psi(1,2n) + \Psi(1,n))}}{\alpha_1} \leq \|H\|_E \leq \frac{\sqrt{n\left(-\Psi(1,n+1) + \frac{\pi^2}{6}\right)}}{\alpha_n}. \quad (3.1)$$

Proof. Let the matrix H be as in (1.1). We have

$$c_1(H) = r_1(H) = \sqrt{-\Psi(1,2n) + \Psi(1,n)}$$

and

$$c_n(H) = r_n(H) = \sqrt{-\Psi(1,n+1) + \frac{\pi^2}{6}}.$$

By applying these equalities to (2.1), we obtain (3.1).

Corollary 5 *Let the matrix H and α_i's be as in* (1.1) *and* (1.12), *respectively. Then*

$$\frac{\sqrt{-\Psi(1,2n) + \Psi(1,n)}}{\alpha_1} \leq \|H\|_2 \,. \tag{3.2}$$

■

Proof. The proof is obvious from (2.5) and (3.1). ■

Theorem 6 *Let the matrix $H^{\circ 1/2}$ and α_i's be as in* (1.2) *and* (1.12), *respectively. Then*

$$\frac{\sqrt{n(\Psi(2n) - \Psi(n))}}{\alpha_1} \leq \left\|H^{\circ 1/2}\right\|_E \leq \frac{\sqrt{n(\Psi(n+1) + \gamma)}}{\alpha_n}. \tag{3.3}$$

Proof. For the matrix $H^{\circ 1/2}$ in (1.2) we have

$$c_1\left(H^{\circ 1/2}\right) = r_1\left(H^{\circ 1/2}\right) = \sqrt{\Psi(2n) - \Psi(n)}$$

and

$$c_n\left(H^{\circ 1/2}\right) = r_n\left(H^{\circ 1/2}\right) = \sqrt{\Psi(n+1) + \gamma}.$$

These equalities substitute in (2.1), then we obtain (3.3). ■

Corollary 7 *Let the matrix $H^{\circ 1/2}$ and α_i's be as in* (1.2) *and* (1.12), *respectively. Then*

$$\frac{\sqrt{\Psi(2n) - \Psi(n)}}{\alpha_1} \leq \left\|H^{\circ 1/2}\right\|_2 . \tag{3.4}$$

Proof. The proof is obvious from (2.5) and (3.3). ■

4 Lower and upper bounds for the Euclidean norms of Cauchy-Toeplitz and Cauchy-Hankel matrices

We substitute $g = 1/2$ and $h = 1$ in T_n and H_n matrices in the forms (1.3) and (1.4).

Theorem 8 *Let the matrix* T_n *and* α_i*'s be as in* (1.3) *and* (1.12), *respectively. Then*

$$a \leq \|T_n\|_E \leq \begin{cases} b, & \text{if } n \text{ odd} \\ c, & \text{if } n \text{ even} \end{cases} \tag{4.1}$$

where

$$a = \frac{\sqrt{n\left(\Psi\left(1, \frac{1}{2}+n\right)+\frac{\pi^2}{2}\right)}}{\alpha_1},$$

$$b = \frac{\sqrt{n\left(-\Psi\left(1, \frac{n}{2}\right)+\Psi\left(1, \frac{-n}{2}\right)\right)}}{\alpha_n}$$

and

$$c = \frac{\sqrt{n\left(-\Psi\left(1, \frac{n+1}{2}\right)+\Psi\left(1, \frac{1-n}{2}\right)\right)}}{\alpha_n}.$$

Proof. For the matrix T_n in (1.3) we have

$$c_{\min}(T_n) = r_{\min}(T_n) = \sqrt{\left(\Psi\left(1, \frac{1}{2}+n\right)+\frac{\pi^2}{2}\right)}.$$

If n is odd, the equality become

$$c_{\max}(T_n) = r_{\max}(T_n) = \sqrt{\left(-\Psi\left(1, \frac{n}{2}\right)+\Psi\left(1, \frac{-n}{2}\right)\right)}.$$

If n is even, we have

$$c_{\max}(T_n) = r_{\max}(T_n) = \sqrt{\left(-\Psi\left(1, \frac{n+1}{2}\right)+\Psi\left(1, \frac{1-n}{2}\right)\right)}.$$

These equalities substitute in (2.1), then we obtain (4.1). ■

Corollary 9 *Let the matrix* T_n *and* α_i*'s be as in* (1.3) *and* (1.12), *respectively. Then*

$$\frac{\sqrt{\Psi\left(1, \frac{1}{2}+n\right)+\frac{\pi^2}{2}}}{\alpha_1} \leq \|T_n\|_2 . \tag{4.2}$$

Proof. The proof is obvious from (2.5) and (4.1). ■

Theorem 10 *Let the matrix* H_n *and* α_i*'s be as in* (1.4) *and* (1.8), *respectively. Then*

$$\frac{\sqrt{n\left(-\Psi\left(1,2n+\frac{3}{2}\right)+\Psi\left(1,n+\frac{3}{2}\right)\right)}}{\alpha_1} \leq \|H_n\|_E \leq \frac{\sqrt{n\left(-\Psi\left(1,n+\frac{5}{2}\right)-\frac{40}{9}+\frac{\pi^2}{2}\right)}}{\alpha_n}. \tag{4.3}$$

Proof. For the matrix H_n in (1.4) we have

$$c_1(H_n) = r_1(H_n) = \sqrt{-\Psi\left(1,2n+\frac{3}{2}\right)+\Psi\left(1,n+\frac{3}{2}\right)}$$

and

$$c_n(H_n) = r_n(H_n) = \sqrt{-\Psi\left(1,n+\frac{5}{2}\right)-\frac{40}{9}+\frac{\pi^2}{2}}.$$

These equalities substitute in (2.1), then we obtain (4.3). ∎

Corollary 11 *Let the matrix* H_n *and* α_i*'s be as in* (1.4) *and* (1.12), *respectively. Then*

$$\frac{\sqrt{-\Psi\left(1,2n+\frac{3}{2}\right)+\Psi\left(1,n+\frac{3}{2}\right)}}{\alpha_1} \leq \|H_n\|_2 . \tag{4.4}$$

5 Upper bounds for spectral norms of the Hilbert matrix and its Hadamard square root

Theorem 12 *Let the matrix* H *be as in* (1.1) . *Then*

$$\|H\|_2 \leq \sqrt{(\Psi(n+2)-1+\gamma)(\Psi(n+1)+\gamma)}. \tag{5.1}$$

Proof. Let $H = A \circ B$ such that

$$A = \begin{bmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{n} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \cdots & \frac{1}{\sqrt{n+1}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n+1}} & \cdots & \frac{1}{\sqrt{2n-1}} \end{bmatrix}, B = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \cdots & \frac{1}{\sqrt{n+1}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n+1}} & \cdots & \frac{1}{\sqrt{2n-1}} \end{bmatrix}. \tag{5.2}$$

Since $\|H\|_2 \leq r_2(A)\, c_1(B)$ where

$$r_2(A) = r_{\max}(A) = \sqrt{\Psi(n+2)-1+\gamma}$$

and

$$c_1(B) = c_{\max}(B) = \sqrt{\Psi(n+1)+\gamma}$$

we have (5.1). ∎

Theorem 13 *Let the matrix $H^{\circ 1/2}$ be as in* (1.2). *Then*

$$\left\| H^{\circ 1/2} \right\|_2 \leq 2\sqrt{n} - 1. \tag{5.3}$$

Proof. Let $H^{\circ 1/2} = A \circ B$ such that

$$A = \begin{bmatrix} 1 & \frac{1}{\sqrt{2}} & \cdots & \frac{1}{\sqrt{n}} \\ \frac{1}{\sqrt{\sqrt{2}}} & \frac{1}{\sqrt{\sqrt{3}}} & \cdots & \frac{1}{\sqrt{\sqrt{n+1}}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{\sqrt{n}}} & \frac{1}{\sqrt{\sqrt{n+1}}} & \cdots & \frac{1}{\sqrt{\sqrt{2n-1}}} \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \frac{1}{\sqrt{\sqrt{2}}} & \frac{1}{\sqrt{\sqrt{3}}} & \cdots & \frac{1}{\sqrt{\sqrt{n+1}}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{\sqrt{n}}} & \frac{1}{\sqrt{\sqrt{n+1}}} & \cdots & \frac{1}{\sqrt{\sqrt{2n-1}}} \end{bmatrix}.$$

Since $\sum_{k=1}^{n} \frac{1}{\sqrt{k}} \leq 2\sqrt{n} - 1$ (see [12], p. 191) and $\left\| H^{\circ 1/2} \right\|_2 \leq r_2(A)\, c_1(B)$ where $r_2(A) = r_{\max}(A) = \sqrt{\sum_{k=2}^{n+1} \frac{1}{\sqrt{k}}}$ and $c_1(B) = c_{\max}(B) = \sqrt{\sum_{k=1}^{n} \frac{1}{\sqrt{k}}}$, we have

$$\left\| H^{\circ 1/2} \right\|_2 \leq \sqrt{\sum_{k=2}^{n+1} \frac{1}{\sqrt{k}}} \sqrt{\sum_{k=1}^{n} \frac{1}{\sqrt{k}}}$$

and

$$\left\| H^{\circ 1/2} \right\|_2 \leq \sqrt{\sum_{k=2}^{n+1} \frac{1}{\sqrt{k}}} \sqrt{\sum_{k=1}^{n} \frac{1}{\sqrt{k}}} \leq \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \leq 2\sqrt{n} - 1.$$

Thus the proof is completed. ■

6 Upper bounds for spectral norms of the Cauchy-Toeplitz and Cauchy-Hankel matrices

Theorem 14 *Let the matrix T_n be as in* (1.3). *Then, for $n \leq 75$ we have*

$$\|T_n\|_2 \leq \begin{cases} d.e, & \text{if } n \text{ odd} \\ d.f, & \text{if } n \text{ even} \end{cases} \tag{5.4}$$

where

$$d=\frac{1}{2}\sqrt{-4\Psi\left(1,-\frac{1}{2}+n\right)+16+2\pi^2},$$

$$e=\sqrt{-1+2\gamma+4\ln(2)+\Psi\left(-\frac{n}{2}\right)+\Psi\left(\frac{n}{2}\right)}$$

and

$$f=\sqrt{-1+2\gamma+4\ln(2)+\Psi\left(\frac{1}{2}-\frac{n}{2}\right)+\Psi\left(\frac{1}{2}+\frac{n}{2}\right)}.$$

For $n>75$ *we have*

$$\|T_n\|_2\leq\begin{cases} j.k, & \text{if } n \text{ odd}\\ j.l, & \text{if } n \text{ even}\end{cases} \tag{5.5}$$

where

$$j=\frac{1}{3}\sqrt{6+9\Psi\left(-\frac{1}{2}+n\right)+9\gamma+18\ln(2)},$$

$$k=\sqrt{-1+2\gamma+4\ln(2)+\Psi\left(-\frac{n}{2}\right)+\Psi\left(\frac{n}{2}\right)}$$

and

$$l=\sqrt{-1+2\gamma+4\ln(2)+\Psi\left(\frac{1}{2}-\frac{n}{2}\right)+\Psi\left(\frac{1}{2}+\frac{n}{2}\right)}.$$

Proof. We can partition the matrix T_n as in (5.2). Hence, $r_{\max}(A)=r_1(A)$ for $n\leq 75$ and $r_{\max}(A)=r_2(A)$ for $n>75$ where

$$r_1(A)=\frac{1}{2}\sqrt{-4\Psi\left(1,-\frac{1}{2}+n\right)+16+2\pi^2}$$

and

$$r_2(A)=\frac{1}{3}\sqrt{6+9\Psi\left(-\frac{1}{2}+n\right)+9\gamma+18\ln(2)}.$$

On the other hand, we obtain that if n is odd

$$c_{\max}(B)=c_{\frac{n+1}{2}+1}(B)=\sqrt{-1+2\gamma+4\ln(2)+\Psi\left(-\frac{n}{2}\right)+\Psi\left(\frac{n}{2}\right)},$$

and if n is even

$$c_{\max}(B)=c_{\frac{n}{2}+1}(B)=\sqrt{-1+2\gamma+4\ln(2)+\Psi\left(\frac{1}{2}-\frac{n}{2}\right)+\Psi\left(\frac{1}{2}+\frac{n}{2}\right)}.$$

Consequently, we complete the proof using $\|T_n\|_2\leq r_{\max}(A)\,c_{\max}(B)$ inequality. ■

Theorem 15 *Let the matrix* H_n *be as in* (1.4)*. Then*

$$\|H_n\|_2 \leq s.t \tag{5.6}$$

where

$$s = \frac{1}{15}\sqrt{225\Psi\left(n+\frac{7}{2}\right) - 690 + 225\gamma + 450\ln(2)}$$

and

$$t = \frac{1}{15}\sqrt{-465 + 225\Psi\left(n+\frac{5}{2}\right) + 225\gamma + 450\ln(2)}$$

Proof. We can separate the matrix H_n as in (5.2). Hence, we find that $r_{\max}(A)=r_1(A)$ and $c_{\max}(B)=c_1(B)$ where

$$r_1(A) = \frac{1}{15}\sqrt{225\Psi\left(n+\frac{7}{2}\right) - 690 + 225\gamma + 450\ln(2)}$$

and

$$c_1(B) = \frac{1}{15}\sqrt{-465 + 225\Psi\left(n+\frac{5}{2}\right) + 225\gamma + 450\ln(2)}.$$

Since $\|H_n\|_2 \leq r_1(A)\,c_1(B)$, then we have (5.6). ∎

7 Numerical Results

We will take $\alpha_1 = \alpha_2 = ... = \alpha_n = 1$ in the following examples.

Example 16 *For Euclidean norm and spectral norm of the matrix* H *in* (1.1)*, we have the following values:*

n	$\sqrt{n\left(-\Psi(1,2n)+\Psi(1,n)\right)}$	$\|H\|_E$	$\sqrt{n\left(-\Psi(1,n+1)+\frac{\pi^2}{6}\right)}$
1	1	1	1
5	0.7620912678	1.580906263	2.705190484
10	0.7341356330	1.785527123	3.936708944
20	0.7204959276	1.969813453	5.650067686
50	0.712431279	2.190011373	9.014246318
70	0.7109056921	2.265522786	10.68421703
100	0.7097637513	2.342915545	12.78664890
150	0.7088769155	2.427902440	15.67620604

n	$\sqrt{-\Psi(1,2n)+\Psi(1,n)}$	$\Vert H\Vert_2$
1	1	1
5	0.340817575	1.567050691
10	0.231540711	1.751919670
20	0.161107787	1.907134720
50	0.100752997	2.076296683
70	0.084969482	2.129987511
100	0.709763751	2.182696098
150	0.057879557	2.237881172

Example 17 *For Euclidean norm and spectral norm of the matrix* $H^{\circ 1/2}$ *in* (1.2), *we obtain the following values:*

n	$\sqrt{n}\,(\Psi(2n)-\Psi(n))$	$\Vert H^{\circ 1/2}\Vert_E$	$\sqrt{n}\,(\Psi(n+1)+\gamma)$
1	1	1	1
5	1.930848155	2.540934711	3.378855822
10	2.680991240	3.657243232	5.411994322
20	3.757135564	5.218441843	8.482617114
50	5.908350783	8.295614384	14.99867551
70	6.983637691	9.825598761	18.39289463
100	8.340584092	11.75290118	22.77581506
150	10.20894186	14.40295066	28.95992210

n	$\sqrt{\Psi(2n)-\Psi(n)}$	$\Vert H^{\circ 1/2}\Vert_2$
1	1	1
5	0.863501546	2.533602599
10	0.847803870	3.638302962
20	0.840121052	5.180584414
50	0.835566980	8.219123246
70	0.834704357	9.730118667
100	0.834058409	11.63380279
150	0.833556611	14.25184035

Example 18 *For Euclidean norm and spectral norm of the matrix* T_n *in* (1.3), *we have the following values:*

n	a	$\Vert T_n\Vert_E$	c
2	3.293982380	3.527668414	4
10	7.095575494	9.389605035	9.731864996
20	9.984780219	13.61887703	13.90666781
50	15.73976102	21.90272099	22.12421193
70	18.61279498	26.00862001	26.20825404
100	22.23691102	31.17407718	31.3522020
150	27.22536182	38.26868659	38.42448073

n	a	$\|T_n\|_E$	b
3	3.974352763	4.636090306	5.033222957
5	5.066628363	6.340772176	6.726399068
15	8.661504678	11.69193376	12.00134391
25	11.15212634	15.30932629	15.57998231
55	16.50497178	22.99753429	23.21265519
75	19.26421943	26.93804649	27.13337163
105	22.78495608	31.95424116	32.12955369

n	$\sqrt{\Psi\left(1, \frac{1}{2}+n\right)+\frac{\pi^2}{2}}$	$\|T_n\|_2$
2	2.329197278	3.070367517
5	2.265865087	3.141589238
10	2.243817987	3.141592654
20	2.232664730	3.141592654
50	2.225938350	3.141592655
70	2.224654507	3.141592655
100	2.223691102	3.141592654
150	2.222941485	3.141592655

where a, b and c are as in Theorem8.

Example 19 *Let*

$$x = \sqrt{n\left(-\Psi\left(1, 2n+\frac{3}{2}\right)+\Psi\left(1, n+\frac{3}{2}\right)\right)},$$

$$y = \sqrt{n\left(-\Psi\left(1, n+\frac{5}{2}\right)-\frac{40}{9}+\frac{\pi^2}{2}\right)}$$

and

$$z = \sqrt{-\Psi\left(1, 2n+\frac{3}{2}\right)+\Psi\left(1, n+\frac{3}{2}\right)}.$$

For Euclidean norm and spectral norm of the matrix H_n in (1.4), *we have the following values:*

n	x	$\|H_n\|_E$	y
1	0.4	0.4	0.4
5	0.6141574492	0.930512018	1.318601265
10	0.6575454725	1.174318993	2.017603855
20	0.6814839896	1.404290252	2.982988539
50	0.6966457419	1.680614853	4.853491421
70	0.6996051574	1.774381680	5.775191463
100	0.7018400888	1.869665496	6.932199628
150	0.7035875505	1.973249681	8.518616380

n	z	$\|H_n\|_2$
1	0.4	0.4
5	0.2746595610	0.928688586
10	0.2079341358	1.167692602
20	0.1523844526	1.387415415
50	0.0985205856	1.639373809
70	0.0836188099	1.720877333
100	0.0701840088	1.801238715
150	0.0574476829	1.885540666

Example 20 *For spectral norm of the matrix* T_n *in* (1.3), *we have the following values:*

n	$\|T_n\|_2$	$d.f$
10	3.141592654	7.366097824
20	3.141592654	8.179833685
50	3.141592655	9.136873637
70	3.141592655	9.462565390

n	$\|T_n\|_2$	$d.e$
1	2	2
3	5.564437344	5.564437344
5	3.141589238	6.412027625
15	3.141592654	7.850792583
25	3.141592654	8.422840878
55	3.141592655	9.230201877
75	3.141592655	9.527818181

n	$\|T_n\|_2$	$j.l$
76	3.141592655	8.419138280
80	3.141592655	8.589459599
90	3.141592655	8.662279969
100	3.141592654	8.813621669
120	3.141592654	9.075231689
150	3.141592655	9.394989966

n	$\|T_n\|_2$	$j.k$
77	3.141592655	8.437880157
85	3.141592655	8.580065696
95	3.141592654	8.739913278
105	3.141592655	8.883626136
125	3.141592654	9.133735532
155	3.141592655	9.441924362

where d, e, f, j, k *and* l *are as in Theorem14.*

Example 21 *For spectral norm of the matrix* H_n *in* (1.4), *we have the following values:*

n	$\|H_n\|_2$	$s.t$
1	0.4	0.534522484
5	0.928688586	1.342041979
10	1.167692602	1.866181652
20	1.387415415	2.464305738
50	1.639373809	3.321609166
70	1.720877333	3.647231374
100	1.801238715	3.996142247
150	1.885540666	4.396030327

where s *and* t *are as in Theorem 15.*

Example 22 *For spectral norm of the matrix* H *i n* (1.1), *we have the following values:*

n	$\|H\|_2$	$\sqrt{(\Psi(n+2)-1+\gamma)(\Psi(n+1)+\gamma)}$
3	1.408318927	1.409294543
5	1.567050691	1.819569546
10	1.751919670	2.432315074
20	1.907134720	3.085014086
50	2.076296683	3.978927375
70	2.129987511	4.311791100
100	2.182696098	4.666140608
150	2.237881172	5.070221639

Example 23 *For spectral norm of the matrix* $H^{\circ 1/2}$ *in* (1.2), *we have the following values:*

n	$\|H^{\circ 1/2}\|_2$	$2\sqrt{n}-1$
3	1.920845897	2.464101616
5	2.533602599	3.472135956
10	3.638302962	5.324555320
20	5.180584414	7.944271912
50	8.219123246	13.14213562
70	9.730118667	15.73320053
100	11.63380279	19
150	14.25184035	23.49489743

We have seen that the bounds for $\alpha_1 = \alpha_2 = ... = \alpha_n = 1$ are better than those for α_i's $(i = 1, ..., n)$ such that $\alpha_1^2 + \alpha_2^2 + ... + \alpha_n^2 = n$.

For example, let $\alpha_1 = 0.5$, $\alpha_2 = 0.9$, $\alpha_3 = 1.1$, $\alpha_4 = 1.3$ and $\alpha_5 = \sqrt{1.04}$. For these values, we find $0.5862240520 \leq \|H\|_E \leq 5.410380968$ and $1.485267813 \leq \|H^{\circ 1/2}\|_E \leq 6.757711644$, respectively.

Acknowledgement 24 *This work is supported by coordinating office of Selcuk University Scientic Research Projects.*

References

[1] D. Bozkurt, On the l_p norms of Cauchy-Toeplitz matrices, Linear and Multilinear 44 (1998) 341-346 .

[2] D. Bozkurt, On the bounds for the l_p norm of almost Cauchy-Toeplitz matrix, Turkish Journal of Mathematics 20 (4) (1996) 545-552 .

[3] R. Moenck, On computing closed forms for summations, in: Proceedings of the MACSYMA User's Conference , 1977, 225-236 .

[4] S. V. Parter, On the distribution of the singular values of Toeplitz matrices, Linear Algebra and its Appl. 80 (1986) 115-130 .

[5] E. E. Tyrtyshnikov, Cauchy-Toeplitz matrices and some applications, Linear Algebra and its Applications 149 (1991) 1-18 .

[6] R. Reams, Hadamard inverses, square roots and product of almost semidefinite matrices, Linear Algebra and Its Applications 288 (2000) 35-43 .

[7] S. Solak , R. Turkmen and D. Bozkurt, On GCD, LCM and Hilbert matrices and their applications, Applied Mathematics and Computation (2003).

[8] Oscar Rojo, Further bounds for the smallest singular value and the spectral condition number, Computers and Mathematics with Applications, Vol. 38, N° 7-8, (1999) 215 - 228.

[9] A. D. Gungor, Lower bounds for the norms of Cauchy-Toeplitz and Cauchy-Hankel matrices, Applied Mathematics and Computation (to appear).

[10] S. Yongzhong, Estimations for the eigenvalues of the product of matrices (Chinese), J. Nanjing Norm. Univ. Nat. Sci. Ed. 16(2) (1994) 10-13.

[11] R. Mathias, The Spectral Norm of a Nonnegative Matrix, Linear Algebra and its Applications 131 (1990) 269-284 .

[12] D. S. Mitrinovic, Analytic Inequalities, New York 1970.