On the Spectral Radius of Cactuses with n Vertices and Edge Independence Number q *

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Abstract

A graph is a cactus if any two of its cycles have at most one common vertex. In this paper, we determine the graph with the largest spectral radius among all connected cactuses with n vertices and edge independence number q.

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1. Introduction

Let $G = (V, E)$ be a simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. Let A(G) be a (0, 1)-adjacency matrix of G. Since $A(G)$ is symmetric, its eigenvalues are real. Without loss of generality, we can write them as $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$ and call them the eigenvalues of G. The characteristic polynomial of G is just $\det(\lambda I - A(G))$, and is denoted by $P(G; \lambda)$. The largest eigenvalue $\lambda_1(G)$ is called the spectral radius of G, denoted by $\rho(G)$. If G is connected, then $A(G)$ is irreducible and by the Perron-Frobenius theory of non-negative matrices, $\rho(G)$ has multiplicity one and there exists a unique positive unit eigenvector corresponding to $\rho(G)$. we shall refer to such an eigenvector as the Perron vector of G.

Two distinct edges in a graph G are independent if they are not adjacent in G. A set of pairwise independent edges of G is called a matching in G, while a matching of maximum cardinality is a maximum matching in G denoted by $M(G)$ or M. The cardinality $|M|$ of a maximum matching M

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of G is commonly known as its edge independence number denoted by q. An edge $e = uv$ which belongs to M is called an M-saturated edge and both u and v are called M-saturated vertices.

The investigation on the spectral radius of graphs is an important topic in the theory of graph spectra. Recently, the problem concerning graphs with maximal or minimal spectral radius of a given class of graphs has been studied extensively. For related results, one may refer to [1, 3-11, 14, 15] and the references therein.

A graph is a cactus, or a treelike graph, if any two of its cycles have at most one common vertex. Cactuses have been studied by several authors, for example, one may see [3, 12]. Clearly, cactuses are a generalization of unicyclic graphs. In this paper we study the spectral radius of cactuses with n vertices and edge independence number q, and prove the following theorem.

Theorem 1. *Let $n \geq 8$, $q \geq 2$ and G be a connected cactus with n vertices and edge independence number q. Then*

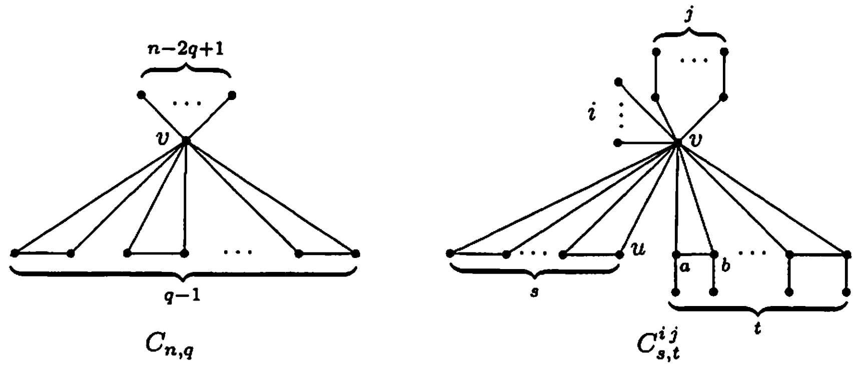
$$\rho(G) \leq \rho(C_{n,q}),$$

and the equality holds if and only if $G = C_{n,q}$, where $C_{n,q}$ is the cactus depicted in Figs. 1, $\rho(C_{n,q})$ is the largest root of the equation

$$\lambda^4 - n\lambda^2 - 2(q-1)\lambda + (n - 2q + 1) = 0.$$

2. Preliminaries

Denote by C_n and P_n the cycle and the path, respectively, each on n vertices. An internal path of G as a walk $v_0v_1 \ldots v_s$ ($s \geq 1$) such that the vertices v_0, $v_1, \ldots, v_s$ are distinct, $d(v_0) > 2$, $d(v_s) > 2$, and $d(v_i) = 2$, whenever $0 < i < s$. And s is called the length of the internal path. For $v \in V(G)$, $N(v)$ denotes the set of all neighbors of vertex v in G, and the degree of v, written by $d(v)$, is the cardinality $|N(v)|$ of $N(v)$. Let $G - x$ or $G - xy$ denote the graph that arises from G by deleting the vertex $x \in V(G)$ or edge $xy \in E(G)$. By $d(x, y)$ we will denote the distance between vertices x and y in G. A pendant vertex of G is a vertex of degree one. A pendant edge is an edge with which a pendant vertex is incident. Denote by $C(n, q)$ the set of all connected cactuses with n vertices and edge independence number q, and by $C(n, k, q)$ the set of all connected cactuses with n vertices, k cycles and edge independence number q. $C_{s,t}^{ij} \in C(n, k, q)$ is depicted in Figs. 1, where $s + t = k$. The terminology not defined here can be found in [2].



Figs. 1. $C_{n,q}$ and $C^{ij}_{s,t}$

In order to complete the proof of our main result, we need following lemmas.

Lemma 1 [14, 11]. *Let G be a connected graph and $\rho(G)$ be the spectral radius of $A(G)$. Let u, v be two vertices of G and d_v be the degree of vertex v. Suppose $v_1, v_2, \ldots, v_s \in N(v)\backslash N(u)\,(1 \le s \le d_v)$ and $x = (x_1, x_2, \ldots, x_n)$ is the Perron vector of $A(G)$, where x_i corresponds to the vertex $v_i\,(1 \le i \le n)$. Let G^* be the graph obtained from G by deleting the edges vv_i and adding the edges $uv_i\,(1 \le i \le s)$. If $x_u \ge x_v$, then $\rho(G) < \rho(G^*)$.*

By Lemma 1, we obtain easily following Lemmas 2 and 3 which may be regard as immediate consequences of Lemma 1.

Lemma 2 [9]. *Let G be a connected graph and let $e = uv$ be a non-pendant edge of G with $N(u) \cap N(v) = \emptyset$. Let G^* be the graph obtained from G by deleting the edge uv, identifying u with v, and adding a pendant edge to $u(= v)$. Then $\rho(G) < \rho(G^*)$.*

Lemma 3 [9]. *Let G, G', G'' be three connected graphs pairwise disjoint. Suppose that u, v are two vertices of G, u' is a vertex of G' and u'' is a vertex of G''. Let G_1 be the graph obtained from G, G', G'' by identifying, respectively, u with u' and v with u''. Let G_2 be the graph obtained from G, G', G'' by identifying vertices u, u', u''. Let G_3 be the graph obtained from G, G', G'' by identifying vertices v, u', u''. Then either $\rho(G_1) < \rho(G_2)$ or $\rho(G_1) < \rho(G_3)$.*

The following two lemmas are often used to calculate the characteristic polynomials of graphs.

Lemma 4 [13]. *Let u be a vertex of G, and let $C(u)$ be the set of all cycles* *containing u. The characteristic polynomial of G satisfies*

$$P(G;\lambda) = \lambda P(G-u;\lambda) - \sum_{v\in N(u)} P(G-u-v;\lambda) - 2\sum_{Z\in C(u)} P(G\backslash V(Z);\lambda).$$

Lemma 5 [13]. *Let $e = uv$ be an edge of G, and let C(u) be the set of all cycles containing e. The characteristic polynomial of G satisfies*

$$P(G;\lambda) = P(G-e;\lambda) - P(G-u-v;\lambda) - 2\sum_{Z\in C(u)} P(G\backslash V(Z);\lambda).$$

Lemma 6. *Let $n \geq 8$, $s \geq 0$, $t \geq 1$ and $C_{s,t}^{i,j} \in C(n,k,q)$ be the graph shown in Figs. 1. Then*

$$\rho(C_{s+1,t-1}^{i,j+1}) > \rho(C_{s,t}^{i,j}).$$

Proof. Applying Lemma 5 to the edges uv and ab of $C_{s+1,t-1}^{i,j+1}$ and $C_{s,t}^{i,j}$, respectively, we have

$$\begin{aligned} P(C_{s+1,t-1}^{i,j+1};\lambda) &= P(C_{s,t-1}^{i,j+2};\lambda) - \lambda^{i+1}(\lambda^2-1)^{s+j+1}P((t-1)P_4;\lambda) \\ &\quad -2\lambda^i(\lambda^2-1)^{s+j+1}P((t-1)P_4;\lambda), \\ P(C_{s,t}^{i,j};\lambda) &= P(C_{s,t-1}^{i,j+2};\lambda) - \lambda^2 P(C_{s,t-1}^{i,j};\lambda) \\ &\quad -2\lambda^{i+2}(\lambda^2-1)^{s+j}P((t-1)P_4;\lambda). \end{aligned}$$

When $t = 1$, applying Lemma 4 to the vertex v of $C_{s,t-1}^{i,j}$, we have

$$\begin{aligned} & P(C_{s,t}^{i,j};\lambda) - P(C_{s+1,t-1}^{i,j+1};\lambda) \\ = \ & \lambda^{i+1}(\lambda^2-1)^{s+j+1} - \lambda^2 P(C_{s,0}^{i,j};\lambda) - 2\lambda^i(\lambda^2-1)^{s+j} \\ = \ & \lambda^i(\lambda^2-1)^{s+j-1}[(i+j+2s-1)\lambda^3 + (2s-2)\lambda^2 - (i-1)\lambda + 2] \\ \geq \ & \lambda^i(\lambda^2-1)^{s+j-1}[2(i+j+2s-1)\lambda^2 + (2s-2)\lambda^2 - (i-1)\lambda + 2] \\ = \ & \lambda^i(\lambda^2-1)^{s+j-1}[2(i+j+3s-2)\lambda^2 - (i-1)\lambda + 2] > 0 \end{aligned}$$

for all $\lambda \geq \rho(C_{s,t}^{i,j}) \geq 2$. So

$$\rho(C_{s+1,t-1}^{i,j+1}) > \rho(C_{s,t}^{i,j}).$$

When $t \geq 2$, applying Lemma 4 to the vertex v of $C_{s,t-1}^{i,j}$, we have

$$\begin{aligned} & P(C_{s,t}^{i,j};\lambda) - P(C_{s+1,t-1}^{i,j+1};\lambda) \\ = \ & \lambda^{i+1}(\lambda^2-1)^{s+j+1}P((t-1)P_4;\lambda) - \lambda^2 P(C_{s,t-1}^{i,j};\lambda) \\ & -2\lambda^i(\lambda^2-1)^{s+j}P((t-1)P_4;\lambda) \\ = \ & \lambda^i(\lambda^2-1)^{s+j-1}(\lambda^4-3\lambda^2+1)^{t-2}f(\lambda). \end{aligned}$$

where

$$\begin{aligned} f(\lambda) &= (2s+2t+i+j-3)\lambda^7 + (2s+2t-4)\lambda^6 \\ &\quad -(6s+4t+4i+3j-8)\lambda^5 - (6s+2t-10)\lambda^4 \\ &\quad +(2s+2t+4i+j-6)\lambda^3 + (2s-8)\lambda^2 - (i-1)\lambda + 2 \\ &\geq 7(2s+2t+i+j-3)\lambda^5 + 7(2s+2t-4)\lambda^4 \\ &\quad -(6s+4t+4i+3j-8)\lambda^5 - (6s+2t-10)\lambda^4 \\ &\quad +(2s+2t+4i+j-6)\lambda^3 + (2s-8)\lambda^2 - (i-1)\lambda + 2 \\ &= (8s+10t+3i+4j-13)\lambda^5 + (8s+12t-18)\lambda^4 \\ &\quad +(2s+2t+4i+j-6)\lambda^3 + (2s-8)\lambda^2 - (i-1)\lambda + 2 \\ &\geq (58s+72t+25i+29j-97)\lambda^3 + (58s+84t-134)\lambda^2 \\ &\quad -(i-1)\lambda + 2 > 0 \end{aligned}$$

for all $\lambda \geq \rho(C_{s,t}^{i,j}) \geq \sqrt{7}$. So

$$\rho(C_{s+1,t-1}^{i,j+1}) > \rho(C_{s,t}^{i,j}).$$

This completes the proof.

3. The Proof of Theorem 1

The proof of Theorem 1 follows immediately from the following theorem and the fact that the spectral radius strictly increases if we add an edge to a connected graph.

Theorem 2. *Let $n \geq 8$, $q \geq k \geq 1$ and G be a connected cactus with n vertices, k cycles and edge independence number q. If $n = 2k+1$, then*

$$\rho(G) \leq \frac{1+\sqrt{8k+1}}{2},$$

and the equality holds if and only if $G = C_{k,0}^{0,0}$. If $n \neq 2k+1$, then

$$\rho(G) \leq \rho(C_{k,0}^{n-2q+1,\,q-k-1}),$$

and the equality holds if and only if $G = C_{k,0}^{n-2q+1,\,q-k-1}$, where $\rho(C_{k,0}^{n-2q+1,\,q-k-1})$ is the largest root of the equation

$$\lambda^4 - (n-q+k+1)\lambda^2 - 2k\lambda + (n-2q+1) = 0.$$

Proof. Choose $G \in C(n,k,q)$ such that the spectral radius of G is as large as possible. Denote the vertex set of G by $\{v_1, v_2, \ldots, v_n\}$ and the

Perron vector of G by $x = (x_1, x_2, \ldots, x_n)$, where x_i corresponds to the vertex v_i $(1 \le i \le n)$. Assume that M is a maximum matching in G. Then $|M| = q$ and there are three cases for a non-pendant edge $e = uv$ in G: (1) $e = uv$ is an M-saturated edge; (2) $e = uv$ has exactly one M-saturated vertex; (3) $e = uv$ is not M-saturated but both u and v are M-saturated vertices. In the following, we will first prove some facts.

Fact 1. Each edge $e = uv$ of case (1) or case (2) in G is a pendant edge unless e belongs to a cycle of length three. There exists no internal path of length greater than one in G unless the path belongs to a cycle of length three.

Proof of Fact 1. If there exists a non-pendant edge $e = uv$ of case (1) or case (2) in G such that $e = uv$ does not belong to any cycle in G or belongs to a cycle of length greater than three, then $N(u) \cap N(v) = \emptyset$. Carrying out the transformation described in Lemma 2, we can transform G into a graph $G^* \in C(n, k, q)$ such that edge $e = uv$ is a pendant edge. by Lemma 2, we have $\rho(G) < \rho(G^*)$. This contradicts the definition of G.

If there exists an internal path of length greater than one in G such that the path does not belong to any cycle of length three, then there exists a non-pendant edge of case (1) or case (2) in the path, a contradiction.

Fact 2. There exists a maximum matching of G such that any M-saturated edge $e = uv$ belonging to a cycle $C_3 = uvwu$ of length three has $d(u) = d(v) = 2$.

Proof of Fact 2. For any M-saturated edge $e = uv$ belonging to a cycle $C_3 = uvwu$ of length three, by Lemma 3, at most one of u and v has degree greater than two. Assume that $d(u) \ge 3$. If w is a M-saturated vertex, then $d(w) \ge 3$. Applying Lemma 3 to vertices w and u, we can obtain a graph $G^* \in C(n, k, q)$ such that $\rho(G^*) > \rho(G)$, a contradiction. If w is not M-saturated, then $d(w) = 2$. Otherwise, applying Lemma 3 to vertices w and u, we derive a contradiction similarly. Let

$$M' = M - \{uv\} + \{wv\}.$$

then M' is also a maximum matching of G and $d(v) = d(w) = 2$.

Fact 3. All cycles of the graph G have exactly one common vertex.

Proof of Fact 3. We first prove that any two cycles of the graph G have one common vertex. Assume, on the contrary, that there are two disjoint cycles C_p and C_q. Let $v_1 v_2 \ldots v_l$ be a shortest path joining the cycles C_p and C_q of length $l - 1$, where $l \ge 2$, the vertex v_1 belongs to the cycle C_p and the vertex v_l belongs to the cycle C_q. Note, any path joining the cycles C_p and C_q starts from v_1 and ends to v_l (in the opposite case G is not a cactus). Without loss of generality we may assume that $x_1 \geq x_l$. Denote by v_{l+1} and v_{l+2} the neighbors of v_l which belong to C_q. By Fact 1 and Fact 2, neither v_lv_{l+1} nor v_lv_{l+2} is M-saturated. Let

$$G^* = G - \{ v_lv_{l+1}, v_lv_{l+2}\} + \{ v_1v_{l+1}, v_1v_{l+2} \}.$$

Then $G^* \in C(n,k,q)$ and by Lemma 1 we have $\rho(G^*) > \rho(G)$, a contradiction. Hence, any two cycles have one common vertex.

Secondly, we prove that any three cycles have exactly one common vertex. In the opposite case the graph G is not a cactus, because there exist cycles which have more than one common vertex.

From the above arguments, we have that all cycles of the graph G have exactly one common vertex.

Fact 4. In the graph G, any tree T attached to a vertex v of one of the cycles consists of only some pendant edges and pendant paths of length two.

Proof of Fact 4. In the opposite case, there exists a tree T with root $v_i \in C_p$ such that T does not consist of only some pendant edges and pendant paths of length two. Then there exists a vertex v_j of T such that $d(v_i, v_j) = 1$ and $d(v_j) \geq 3$. Otherwise, there exists an internal path of length greater than one in T, a contradiction. Furthermore, there exists an M-saturated pendant edge at v_i and v_j, respectively. Otherwise, v_iv_j is a non-pendant edge of case (1) or case (2). Denote by v_iv_{i+1} and v_jv_{j+1} the two M-saturated pendant edges. Let

$$N(v_i) \setminus \{ v_iv_j, v_iv_{i+1}\} = \{ v_ix_1, \dots, v_ix_a\}$$

and

$$N(v_j) \setminus \{ v_jv_i, v_jv_{j+1}\} = \{ v_jy_1, \dots, v_jy_b\}.$$

If $x_i \geq x_j$, let

$$G^* = G - \{ v_jy_1, \dots, v_jy_b \} + \{ v_iy_1, \dots, v_iy_b \}.$$

If $x_i < x_j$, let

$$G^* = G - \{ v_ix_1, \dots, v_ix_a \} + \{ v_jx_1, \dots, v_jx_a \}.$$

Then $G^* \in C(n,k,q)$. By Lemma 1, we have $\rho(G) < \rho(G^*)$, a contradiction.

Fact 5. Let v_1 be the common vertex of all cycles in G. Then any pendant edge which is not M-saturated and any pendant path of length two is attached to v_1.

Proof of Fact 5. In the opposite case, by Lemma 3 we can obtain a graph $G^* \in C(n,k,q)$ such that $\rho(G) < \rho(G^*)$, a contradiction.

Fact 6. All cycles of G have length three.

Proof of Fact 6. Suppose, on the contrary, that there exists a cycle C_p of length $p \geq 4$. Let $Cp = v_1 v_2 \dots v_p v_1$ and let $w_1, w_2, \dots, w_l \in N(v_1) \backslash V(C_p)$. Then there exists an M-saturated pendant edge at v_1 and v_2 respectively. Otherwise $v_1 v_2$ is a non-pendant edge of Case 1 or Case 2, a contradiction. Let $v_1 w_1$ and $v_2 v_t$ be the M-saturated pendant edges. Applying Lemma 1 to vertices v_1 and v_2, similarly to the proof of Fact 4, we can obtain a graph $G^* \in C(n,k,q)$ such that $\rho(G) < \rho(G^*)$, a contradiction.

By Facts 1-6, we have $G = C_{s,t}^{i\,j}$. If $n = 2k+1$, then $q = k$ and $i = j = t = 0$, namely $G = C_{k,0}^{0,0}$. Applying Lemma 4 to the vertex v of $C_{k,0}^{0,0}$, we have

$$P(C_{k,0}^{0,0}; \lambda) = (\lambda+1)(\lambda^2 - \lambda - 2k)(\lambda^2 - 1)^{k-1},$$

and so

$$\rho(G) = \frac{1 + \sqrt{8k+1}}{2}.$$

If $n \neq 2k+1$, by Lemma 6 we further have $G = C_{k,0}^{n-2q+1,\,q-k-1}$. Applying Lemma 4 to the vertex v of $C_{k,0}^{n-2q+1,\,q-k-1}$, we have

$$P(C_{k,0}^{n-2q+1,\,q-k-1}; \lambda) = \lambda^{n-2q}(\lambda^2 - 1)^{q-2} f(\lambda),$$

where

$$f(\lambda) = \lambda^4 - (n - q + k + 1)\lambda^2 - 2k\lambda + n - 2q + 1.$$

So $\rho(C_{k,0}^{n-2q+1,\,q-k-1})$ is the largest root of the equation

$$\lambda^4 - (n - q + k + 1)\lambda^2 - 2k\lambda + n - 2q + 1 = 0.$$

This completes the proof.

Let $k = 1$. By Theorem 2 we have the following corollary which is one of the main results of [9] and [15].

Corollary 1. *Let $n \geq 8$, $q \geq 2$ and G be a unicyclic graph with n vertices and edge independence number q. Then*

$$\rho(G) \leq \rho(C_{1,0}^{n-2q+1,\,q-2}),$$

and the equality holds if and only if $G = C_{1,0}^{n-2q+1,q-2}$, *where* $\rho(C_{1,0}^{n-2q+1,q-2})$ *is the largest root of the equation*

$$\lambda^4 - (n-q+2)\lambda^2 - 2\lambda + n - 2q + 1 = 0.$$

References

[1] A. Berman, X. D. Zhang, On the spectral radius of graphs with cut vertices, *J. Combin. Theory Ser. B,* **83** (2001) 233-240.

[2] J. A. Bondy, U. S. R. Murty, Graph Theory with Applications, The Macmillan Press Ltd, 1976.

[3] B. Borovićanin, M. Petrović, On the index of cactuses with n vertices, *Publ. Inst. Math. (Beograd),* **79(93)** (2006) 13-18.

[4] R. A. Brualdi, E. S. Solheid, On the spectral radius of connected graphs, *Publ. Inst. Math. (Beograd),* **39 (53)** (1986) 45-54.

[5] A. Chang, F. Tian, On the spectral radius of unicyclic graphs with perfect matchings, *Linear Algebra Appl.,* **370** (2003) 237-250.

[6] J. M. Guo, J. Y. Shao, On the spectral radius of trees with fixed diameter, *Linear Algebra Appl.,* **413** (2006) 131-147.

[7] S. G. Guo, The spectral radius of unicyclic and bicyclic graphs with n vertices and k pendant vertices, *Linear Algebra Appl.,* **408** (2005) 78-85.

[8] S. G. Guo, On the spectral radius of bicyclic graphs with n vertices and diameter d, *Linear Algebra Appl.,* **422** (2007) 119-132.

[9] S. G. Guo, On the spectral radius of unicyclic graphs with n vertices and edge independence number q, *Ars Comb.,* **83** (2007) 279-287.

[10] Y. P. Hou, J. S. Li, Bounds on the largest eigenvalues of trees with a given size of matching, *Linear Algebra Appl.,* **342** (2002) 203-217.

[11] H. Liu, M. Lu, F. Tian, On the spectral radius of graphs with cut edges, *Linear Algebra Appl.,* **389** (2004) 139-145.

[12] Z. Radosavljević, M. Rašajski, A class of reflexive cactuses with four cycles, *Univ. Beograd Publ. Elektrotehn. Fak., Ser. Mat.,* **14** (2003) 64-85.

[13] A. Schwenk, Computing the characteristic polynomial of a graph, In: R. Bary, F. Harary ed., *Graphs and Combinatorics-Lecture Notes in Mathematics,* No. **406**, Berlin: Springer-Verlag, 1974, 153-172.

[14] B. F. Wu, E. L. Xiao, Y. Hong, The spectral radius of trees on k pendant vertices, *Linear Algebra Appl.,* **395** (2005) 343-349.

[15] A. M. Yu, F. Tian. On the spectral radius of unicyclic graphs, *MATCH Commun. Math. Comput. Chem.,* **53** (2005) 441-448.