On the spectrum of unitary Euclidean graphs

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Abstract

We consider unitary graphs attached to $\mathbb{Z}_n^d$ using an analogue of the Euclidean distance. These graphs are shown to be integral when n is odd or the dimension d is even.

1 Introduction

Let Γ be an additive group. For $S \subseteq \Gamma$, $0 \notin S$ and $S^{-1} = \{-s : s \in S\} = S$, the Cayley graph $G = C(\Gamma, S)$ is the undirected graph having vertex set $V(G) = \Gamma$ and edge set $E(G) = \{(a, b) : a - b \in S\}$. The Cayley graph $G = C(\Gamma, S)$ is regular of degree $|S|$. For a positive integer $n > 1$, the unitary Cayley graph $X_n = C(\mathbb{Z}_n, \mathbb{Z}_n^*)$ is defined by the additive group of the ring $\mathbb{Z}_n$ of integers modulo n and the multiplicative group $\mathbb{Z}_n^*$ of its units. So X_n has vertex set $V(X_n) = \mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ and edge set

$$E(X_n) = \{(a, b) : a, b \in \mathbb{Z}_n, \gcd(a - b, n) = 1\}.$$

The graph X_n is regular of degree $|\mathbb{Z}_n^*| = \phi(n)$, where $\phi(n)$ denotes the Euler function. Unitary Cayley graphs are highly symmetric and have some remarkable properties connecting graph theory and number theory. More information about the unitary Cayley graphs can be found in Berrizbeitia and Giudici [3], Dejter and Giudici [5], Fuchs [6, 7], and Klotz and Sander [8].

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In this paper we study higher dimensional unitary Cayley graphs over $\mathbb{Z}_n^d$ using an analogue of the Euclidean distance. Precisely, we define for positive integers n and d the unitary Euclidean graph $T_n^{(d)}$ with vertex set $V(T_n^{(d)}) = \mathbb{Z}_n^d$ and edge set

$$E(T_n^{(d)}) = \left\{(a,b) : d(a,b) = \sum_{i=1}^{d}(a_i - b_i)^2 \in \mathbb{Z}_n^*\right\}. \tag{1.1}$$

Note that the Euclidean graph $E_R^{(d)}(r)$ over a finite ring R, $r \in R$, is the Cayley graph with vertex set $V(E_R^{(d)}(r)) = R^d$ and the edge set

$$E(E_R^{(d)}(r)) = \left\{(a,b) : d(a,b) = \sum_{i=1}^{d}(a_i - b_i)^2 = r\right\}.$$

In [12], Medrano, Myers, Stark and Terras studied the spectrum of the Euclidean graphs over finite fields and showed that these graphs are asymptotically Ramanujan graphs. In [13], these authors studied the same problem for the Euclidean graphs over rings $\mathbb{Z}_q$ for an odd prime power q. They showed that over rings, except for the smallest case, the graphs (with unit distance parameter) are not (asymptotically) Ramanujan.

In [2], Bannai, Shimabukuro and Tanaka showed that the Euclidean graphs over finite fields are always asymptotically Ramanujan for a more general setting (i.e. they replace the Euclidean distance by nondegenerated quadratic forms). Vinh recently applied these results to several interesting combinatorial problems, for example to tough Ramsey graphs (with P. Dung) [15], Szemeredi-Trotter type theorem and sum-product estimate [17] and the Erdös distance problem [16].

The main purpose of this paper is to study the spectrum of unitary Euclidean graphs. We will show that the spectrum of unitary Euclidean graphs consists entirely of integers when n is odd or the dimension d is even. A graph is called integral, if its spectrum consists entirely of integers. This property seems to be amazingly widespread among Cayley graphs on abelian groups. One of the first papers in this direction is due to L. Lovász [9], who proved that all Cayley graphs, (cube-like) graphs, on $\mathbb{Z}_2^d$ are integral where $\mathbb{Z}_n$ is the ring of integers modulo n. See also W. So [14] for related results.

The rest of this paper is organized as follows. In Section 2 we summarize preliminary facts that will be used throughout the paper. In Section 3 we prove the main theorem of this paper stating that unitary Euclidean graphs are integral if n is odd or the dimension d is even. We conjecture that the result is also true for the odd dimensional case.

2 Preliminaries

We define the following conventions which will be used throughout the paper.

- The notation $\delta(\mathcal{P})$ where $\mathcal{P}$ is some property, means that it's 1 if $\mathcal{P}$ is satisfied, 0 otherwise.

- For any $k \in \mathbb{Z}_n^*$ we define $I_n(k)$ to be the unique element $I_n(k) \in \mathbb{Z}_n^*$ satisfying
$$kI_n(k) \equiv 1 \mod n.$$

- For any integer k and $x = (x_1, \ldots, x_d) \in \mathbb{Z}_n^d$, $k \mid x$ if and only if $k \mid x_i$ for all $1 \leq i \leq d$.

- The exponential:
$$e_n(x) = \exp\{2\pi i x/n\}.$$

- We write $n = p_1^{r_1} \ldots p_t^{r_t}$ for the prime decomposition of the positive integer n. For any nonempty subset $I \subseteq \{1, \ldots, t\}$, set $p_I = \Pi_{i \in I} p_i$ and $n_I = n/p_I$. We define the integral square root of a positive integer n to be the largest integer d such that $d \mid x$ for any x satisfying $n \mid x^2$. It is easy to show that if $n = p_1^{r_1} \ldots p_t^{r_t}$ then $d = p_1^{\lfloor \frac{r_1+1}{2} \rfloor} \ldots p_t^{\lfloor \frac{r_t+1}{2} \rfloor}$. We define $n_I^{'}$ to be the integral square root of n_I and $p_I^{'} = n/n_I^{'}$. It is clear that $p_I \mid p_I^{'}$ and $n | n_I p_I^{'}$ for any nonempty subset I of $\{1, \ldots, t\}$.

In the following discussion, we will need to consider Gauss, Ramanujan and Gauss character sums over rings. For $c \in \mathbb{Z}_n^*$ define the Gauss sum

$$G_n(c) = \sum_{k \in \mathbb{Z}_n} e_n(ck^2). \tag{2.1}$$

If n odd, then

$$G_n(1) = \varepsilon_n \sqrt{n}, \tag{2.2}$$

where

$$\varepsilon_n = \begin{cases} 1 & \text{if } n \equiv 1(\mod 4) \\ i & \text{if } n \equiv 3(\mod 4). \end{cases}$$

If $(c, n) = 1$ and n odd, then

$$G_n(c) = \left(\frac{c}{n}\right) G_n(1), \tag{2.3}$$

here the Jacobi symbol $\left(\frac{c}{n}\right)$ is defined as follows:

- If $n = p$ is prime then

$$\left(\frac{c}{p}\right) = \begin{cases} 1 & p \nmid c, \quad c \text{ is square mod } p \\ -1 & p \nmid c, \quad c \text{ is nonsquare mod } p \\ 0 & p \mid c. \end{cases}$$

- If $n = p_1^{r_1} \dots p_t^{r_t}$ is the prime decomposition of n then

$$\left(\frac{c}{n}\right) = \left(\frac{c}{p_1}\right)^{r_1} \cdots \left(\frac{c}{p_t}\right)^{r_t}.$$

If $(c, n) = 1$ and $n \equiv 2 \mod 4$, then

$$G_n(c) = 0. \tag{2.4}$$

If $(c, n) = 1$ and $n \equiv 0 \mod 4$, then

$$G_n(c) = \left(\frac{n}{c}\right)(1 + i^c)\sqrt{n}. \tag{2.5}$$

For more information about the Gauss sums, we refer the reader to Section 1.5 in [4].

For $c \in \mathbb{Z}_n$ define the Ramanujan sum

$$r(c, n) = \sum_{(k,n)=1} e_n(kc). \tag{2.6}$$

For positive integers n, r, Ramanujan sums have only integral values. Precisely, we have (cf. Corollary 2.4 of [11])

$$r(c, n) = \mu(t_c)\frac{\phi(n)}{\phi(t_c)}, \tag{2.7}$$

where $t_c = \frac{n}{\gcd(c,n)}$. Here μ denotes the Möbius function.

For $c \in \mathbb{Z}_n$ define the Gauss character sum for the quadratic character by

$$G_n(\chi, c) = \sum_{k \in \mathbb{Z}_n} \left(\frac{k}{n}\right) e_n(ck). \tag{2.8}$$

Then it can be shown that

$$G_n(\chi, c) = \left(\frac{c}{n}\right) G_n(\chi, 1). \tag{2.9}$$

If n is square-free and $(c, n) = 1$ then

$$G_n(c) = G_n(\chi, c). \tag{2.10}$$

In general we have the following relation for the Gauss character sum and the Gauss sum.

Lemma 2.1 *For any odd positive integer n and $c \in \mathbb{Z}_n$ we have $n \mid G_n(1)G_n(\chi, c)$.*

Proof From (2.9), it is sufficient to assume $c = -b^2$ for some $b \in \mathbb{Z}_n^*$. We have

$$\begin{aligned}\sum_{k\in\mathbb{Z}_n, x\in\mathbb{Z}_n} e_n(bx+kx^2) &= \sum_{x\in\mathbb{Z}_n} e_n(bx) \sum_{k\in\mathbb{Z}_n} e_n(x^2k) \\ &= \sum_{x\in\mathbb{Z}_n} e_n(bx)\delta(n|x^2)n \\ &= nA,\end{aligned}$$

for some integer A. Thus, by the Inclusion-Exclusion principle, we have

$$\begin{aligned} nA &= \sum_{k\in\mathbb{Z}_n, x\in\mathbb{Z}_n} e_n(bx+kx^2) \\ &= \sum_{\substack{(k,n)=1\\ x\in\mathbb{Z}_n}} e_n(bx+kx^2) - \sum_{I\subseteq\{1,\dots,t\}} (-1)^{|I|} \sum_{\substack{s\in\mathbb{Z}_{n_I}\\ x\in\mathbb{Z}_n}} e_n(bx+p_I s x^2) \\ &= \sum_{\substack{(k,n)=1\\ x\in\mathbb{Z}_n}} e_n(k(x+I_n(2k)b)^2 - I_n(4k)b^2) \\ &\quad - \sum_{I\subseteq\{1,\dots,t\}} (-1)^{|I|} n_I \sum_{x\in\mathbb{Z}_n} \delta(n_I|x^2) e_n(bx) \\ &= \sum_{\substack{(k,n)=1\\ x\in\mathbb{Z}_n}} e_n(kx^2 - I_n(4k)b^2) - \sum_{I\subseteq\{1,\dots,t\}} (-1)^{|I|} n_I \sum_{s\in\mathbb{Z}_{p_I'}} e_{p_I'}(bs) \\ &= \sum_{(k,n)=1} G_n(k) e_n(-I_n(4k)b^2) - \sum_{I\subseteq\{1,\dots,t\}} (-1)^{|I|} n_I \delta(p_I'|b) p_I' \\ &= \sum_{(k,n)=1} G_n(1) \left(\frac{k}{n}\right) e_n(-I_n(4k)b^2) - nB \\ &= G_n(1) \sum_{(k,n)=1} \left(\frac{I_n(4k)}{n}\right) e_n(-I_n(4k)b^2) - nB \\ &= G_n(1) G_n(\chi, c) - nB, \end{aligned}$$

for some integer B because $n | n_I p_I'$. The lemma follows. □

3 Unitary Euclidean graphs

Let

$$S_d(n) := \{x = (x_1, \dots, x_d) \in \mathbb{Z}_n^d | d(x,0) = \sum_1^d x_i^2 \in \mathbb{Z}_n^*\}. \quad (3.1)$$

Then the unitary Euclidean graph $T_n^{(d)}$ is the Cayley graph $C(\mathbb{Z}_n^d, S_d(n))$. Recall that the eigenvalues of Cayley graphs of abelian groups can be computed easily in terms of the characters of the group. This result, described in, e.g., [10], implies that the eigenvalues of the unitary Euclidean graph $T_n^{(d)}$ are all the numbers

$$\lambda_b = \sum_{x \in S_d(n)} e_n({}^t b.x), \tag{3.2}$$

where $b \in \mathbb{Z}_n^d$. We will show that λ_b is an integer for any $b \in \mathbb{Z}_n^d$ if n is odd or d is even.

For positive integers n_1, n_2, n and $b \in \mathbb{Z}_{n_1 n_2}^d$, we define

$$\begin{aligned} f_{n_1,n_2}(b) &= \sum_{k \in \mathbb{Z}_{n_2}, x \in \mathbb{Z}_{n_1 n_2}^d} e_{n_1 n_2}({}^t b.x + n_1 k {}^t x.x) \\ f_n(b) &= \sum_{k \in \mathbb{Z}_n, x \in \mathbb{Z}_n^d} e_n({}^t b.x + k {}^t x.x) \\ g_n(b) &= \sum_{(k,n)=1, x \in \mathbb{Z}_n^d} e_n({}^t b.x + k {}^t x.x). \end{aligned}$$

We first need some lemmas.

Lemma 3.1 *For any n_1, n_2 positive integers and $b \in \mathbb{Z}_{n_1 n_2}^d$ we have*

$$f_{n_1,n_2}(b) = \delta(n_1 | b) n_1^d f_{n_2}(c), \tag{3.3}$$

where $c \in \mathbb{Z}_{n_2}^d$ satisfies $b = n_1 c$ given that $\delta(n_1|b)$.

Proof We can write $x = n_2 x_1 + x_2$ uniquely where $x_1 \in \mathbb{Z}_{n_1}^d$ and $x_2 \in \mathbb{Z}_{n_2}^d$. Then

$$\begin{aligned} f_{n_1,n_2}(b) &= \sum_{k \in \mathbb{Z}_{n_2}, x \in \mathbb{Z}_{n_1 n_2}^d} e_{n_1 n_2}({}^t b.x + n_1 k {}^t x.x) \\ &= \sum_{k \in \mathbb{Z}_{n_2}, x_1 \in \mathbb{Z}_{n_1}^d, x_2 \in \mathbb{Z}_{n_2}^d} e_{n_1}({}^t b.x_1) e_{n_1 n_2}({}^t b.x_2 + n_1 k {}^t x_2.x_2) \\ &= \delta(n_1|b) n_1^d \sum_{k \in \mathbb{Z}_{n_2}, x_2 \in \mathbb{Z}_{n_2}^d} e_{n_1 n_2}({}^t b.x_2 + n_1 k {}^t x_2.x_2) \\ &= \delta(n_1|b) n_1^d f_{n_2}(c). \end{aligned}$$

This completes the proof of the lemma. □

Lemma 3.2 *Suppose that $n = p_1^{r_1} \dots p_t^{r_t}$ is the prime decomposition of a positive integer n. Then*

$$f_n(b) = g_n(b) - \sum_{I \subseteq \{1,\dots,t\}} (-1)^{|I|} f_{p_I, n_I}(b). \tag{3.4}$$

Proof By the Inclusion-Exclusion principle, we have

$$\begin{aligned} f_n(b) &= \sum_{k \in \mathbb{Z}_n, x \in \mathbb{Z}_n^d} e_n({}^t b.x + k {}^t x.x) \\ &= \sum_{(k,n)=1, x \in \mathbb{Z}_n^d} e_n({}^t b.x + k {}^t x.x) \\ &\quad - \sum_{I \subseteq \{1,\dots,t\}} (-1)^{|I|} \sum_{s \in \mathbb{Z}_{n_I}, x \in \mathbb{Z}_n^d} e_n({}^t b.x + p_I s {}^t x.x) \\ &= g_n(b) - \sum_{I \subseteq \{1,\dots,t\}} (-1)^{|I|} f_{p_I, n_I}(b), \end{aligned}$$

completing the lemma. □

Lemma 3.3 *Suppose that $n = p_1^{r_1} \dots p_t^{r_t}$ is the prime decomposition of a positive integer n. For any $b \in \mathbb{Z}_n^d$ then*

$$\lambda_b = \sum_{x \in \mathbb{Z}_n^d : \gcd({}^t x.x, n)=1} e_n({}^t b.x) = \delta(n|b) n^d + \sum_{I \subseteq \{1,\dots,t\}} \frac{(-1)^{|I|}}{p_I} \delta(n_I|b) n_I^d f_{p_I}(b). \tag{3.5}$$

Proof By the Inclusion-Exclusion principle, we have

$$\begin{aligned} \lambda_b &= \sum_{x \in \mathbb{Z}_n^d} e_n({}^t b.x) + \sum_{I \subseteq \{1,\dots,t\}} (-1)^{|I|} \sum_{{}^t x.x \in p_I \mathbb{Z}_{n_I}} e_n({}^t b.x) \\ &= \delta(n|b) n^d \\ &\quad + \sum_{I \subseteq \{1,\dots,t\}} \left\{ \frac{(-1)^{|I|}}{n} \sum_{k \in \mathbb{Z}_{n_I}, h \in \mathbb{Z}_n, x \in \mathbb{Z}_n^d} e_n({}^t b.x + h({}^t x.x - p_I k)) \right\} \\ &= \delta(n|b) n^d \\ &\quad + \sum_{I \subseteq \{1,\dots,t\}} \left\{ \frac{(-1)^{|I|}}{n} \sum_{x \in \mathbb{Z}_n^d, h \in \mathbb{Z}_n} \left(e_n({}^t b.x + h {}^t x.x) \sum_{k \in \mathbb{Z}_{n_I}} e_{n_I}(-hk) \right) \right\} \\ &= \delta(n|b) n^d + \sum_{I \subseteq \{1,\dots,t\}} \left\{ \frac{(-1)^{|I|}}{p_I} \sum_{x \in \mathbb{Z}_n^d, h \in n_I \mathbb{Z}_{p_I}} e_n({}^t b.x + h {}^t x.x) \right\}. \end{aligned}$$

From Lemma 3.1, it follows that

$$\begin{aligned}\lambda &= \delta(n|b)n^d + \sum_{I\subseteq\{1,\ldots,t\}} \frac{(-1)^{|I|}}{p_I} f_{n_I,p_I}(b) \\ &= \delta(n|b)n^d + \sum_{I\subseteq\{1,\ldots,t\}} \frac{(-1)^{|I|}}{p_I} \delta(n_I|b)n_I^d f_{p_I}(b).\end{aligned}$$

This concludes the proof of the lemma. □

If n is odd, we can write $g_n(b)$ in terms of the Gauss and Ramanujan sums.

Lemma 3.4 *a) For any odd positive integer n we have*

$$g_n(b) = \begin{cases} (G_n(1))^d G_n(\chi, -{}^tb.b) & d \text{ odd} \\ (G_n(1))^d r(-{}^tb.b, n) & d \text{ even.} \end{cases} \tag{3.6}$$

b) Furthermore, $n \mid g_n(b)$ for all odd integers n.

Proof a) We have

$$\begin{aligned}\sum_{\substack{(k,n)=1\\ x\in\mathbb{Z}_n^d}} e_n({}^tb.x &+ k^tx.x) \\ &= \sum_{(k,n)=1, x\in\mathbb{Z}_n^d} e_n(k^t(x + I_n(2k)b).(x + I_n(2k)b) - I_n(4k)^tb.b) \\ &= \sum_{(k,n)=1, x\in\mathbb{Z}_n^d} e_n(k^tx.x - I_n(4k)^tb.b) \\ &= \sum_{(k,n)=1} G_n^d(k) e_n(-I_n(4k)^tb.b) \\ &= (G_n(1))^d \sum_{(k,n)=1} \left(\frac{k}{n}\right)^d e_n(-I_n(4k)^tb.b) \\ &= (G_n(1))^d \sum_{(k,n)=1} \left(\frac{I_n(4k)}{n}\right)^d e_n(-I_n(4k)^tb.b) \\ &= (G_n(1))^d \sum_{(k,n)=1} \left(\frac{k}{n}\right)^d e_n(-{}^tb.bk) \\ &= \begin{cases} (G_n(1))^d G_n(\chi, -{}^tb.b) & d \text{ odd} \\ (G_n(1))^d r(-{}^tb.b, n) & d \text{ even} \end{cases}\end{aligned}$$

This completes the first part of the lemma.

b) We have

$$\begin{aligned} g_n(b) &= \begin{cases} (G_n(1))^d G_n(\chi, -{}^t b.b) & d \text{ odd} \\ (G_n(1))^d r(-{}^t b.b, n) & d \text{ even} \end{cases} \\ &= \begin{cases} \varepsilon_n^{d-1} n^{(d-1)/2} G_n(1) G_n(\chi, -{}^t b.b) & d \text{ odd} \\ \varepsilon_n^d n^{d/2} r(-{}^t b.b, n) & d \text{ even} \end{cases} \end{aligned}$$

Part b) follows immediately from Lemma 2.1 and the fact that the Ramanujan sums have only integral values. □

We also can write $g_n(b)$ in form of the Gauss sums when n is even.

Lemma 3.5 *a) Suppose that n is a squarefree even integer and $b = \{b_i\}_1^n \in \mathbb{Z}_n^d$, then*

$$g_n(b) = \begin{cases} 0 & \text{there exists } i \text{ such that } 2|b_i \\ \sum_{(k,n)=1} e_{4n}(-I_{4n}(k)^t b.b)(G_{4n}(k)/2)^d & 2 \nmid b_i \text{ for all } i. \end{cases}$$

b) Suppose that $2 \mid d$. For any squarefree even integer n and any vector $b \in Z_n^d$ we have $n \mid g_n(b)$.

Proof a) We have

$$\begin{aligned} g_n(b) &= \sum_{(k,n)=1} \sum_{x \in Z_n^d} e_n({}^t b.x + k^t x.x) \\ &= \sum_{(k,n)=1} \prod_{i=1}^{d} \left(\sum_{x \in Z_n} e_n(b_i x + kx^2) \right). \end{aligned}$$

Suppose that there exists i such that $2|b_i$. If we write $b_i = 2c$ then

$$\begin{aligned} \sum_{x \in Z_n} e_n(b_i x + kx^2) &= \sum_{x \in Z_n} e_n(k(x + cI_n(k))^2 - I_n(k)c^2) \\ &= e_n(-I_n(k)c^2) G_n(k) \\ &= 0, \end{aligned}$$

where the last line follows from (2.4). Thus, $g_n(b) = 0$ if there exists i such that $2|b_i$. Now, suppose that $2 \nmid b_i$ for all $1 \leqslant i \leqslant d$. We have

$$\begin{aligned} g_n(b) &= \sum_{(k,n)=1, x \in Z_n^d} e_{4n}(4^t b.x + 4k^t x.x) \\ &= \sum_{(k,n)=1, x \in Z_n^d} e_{4n}(k^t(2x + I_{4n}(k)b).(2x + I_{4n}(k)b) - I_{4n}(k)^t b.b) \\ &= \sum_{(k,n)=1} e_{4n}(-I_{4n}(k)^t b.b) \left\{ \prod_{i=1}^{d} \left(\sum_{x \in Z_n} e_{4n}(k(2x + I_{4n}(k)b_i)^2 \right) \right\}. \end{aligned}$$

Let $a_i = I_{4n}(k)b_i$, then a_i is odd. Set

$$S_i = \sum_{x \in Z_n} e_n(k(2x + a_i)^2). \tag{3.7}$$

Substitute $x = x + n$ into S_i, we have

$$\begin{aligned} S_i &= \frac{1}{2} \sum_{x \in Z_{2n}} e_{4n}(k(2x + a_i)^2) \\ &= \frac{1}{2} \left(\sum_{x \in Z_{4n}} e_{4n}(kx^2) - \sum_{x \in 2Z_{2n}} e_{4n}(kx^2) \right) \\ &= \frac{1}{2} \left(G_{4n}(k) - \sum_{x \in Z_{2n}} e_n(kx^2) \right) \\ &= \frac{1}{2}(G_{4n}(k) - 2G_n(k)) = \frac{1}{2} G_{4n}(k), \end{aligned}$$

because $G_n(k) = 0$ for $n \equiv 2 \bmod 4$ and $(k, n) = 1$. Therefore, we have

$$g_n(b) = \sum_{(k,n)=1} e_{4n}(-I_{4n}(k)^t b.b)(G_{4n}(k)/2)^d.$$

This concludes the proof of part a).

b) If there exists i such that $2|b_i$ then from part a), $g_n(b) = 0$. Suppose that $2 \nmid b_i$ for all i. Substitute (2.5) into part a). Note that k odd so $(1 + i^k)^2 = 2i^k$ and $(1 + i^k)^4 = -4$. We consider two cases.

Case 1: Suppose that $d = 4d_1$. Since $2 \nmid b_i$ for all i, we have $4 |^t b.b$.

$$\begin{aligned} g_n(b) &= \sum_{(k,n)=1} e_{4n}(-I_{4n}(k)^t b.b)(1 + i^k)^{4d_1} n^{2d_1} \\ &= (-4)^{d_1} n^{2d_1} \sum_{(k,n)=1} e_n \left(\frac{-^t b.b}{4} I_n(k) \right) \\ &= (-4)^{d_1} n^{2d_1} r(-^t b.b/4, n). \end{aligned} \tag{3.8}$$

Case 2: Suppose that $d = 4d_1 + 2$. Since $2 \nmid b_i$ for all i, we have $-^t b.b = 2c$

for some odd c. Let $n = 2m$ for some m odd. We have

$$\begin{aligned} g_n(b) &= \sum_{(k,n)=1} e_{2n}(I_{2n}(k)c)(1+i^k)^{4d_1+2}n^{2d_1+1} \\ &= (-1)^{d_1}(2n)^{2d_1+1} \sum_{(k,n)=1} e_{2n}(I_{2n}(k)c)i^k \\ &= (-1)^{d_1}(2n)^{2d_1+1} \sum_{(k,2m)=1} e_{4m}((I_{4m}(k)c)i^{I_{4m}(k)} \\ &= (-1)^{d_1}(2n)^{2d_1+1} \sum_{(k,2m)=1} e_{4m}(I_{4m}(k)(c+m)) \\ &= (-1)^{d_1}(2n)^{2d_1+1} \sum_{(k,2m)=1} e_{2m}(I_{2m}(k)(c+m)/2) \\ &= (-1)^{d_1}(2n)^{2d_1+1} r((c+m)/2, n) \end{aligned} \tag{3.9}$$

since $I_{4m}(k) \equiv k \bmod 4$ and $I_{4m}(k) \equiv I_{2m}(k) \bmod 2m$.

Part b) follows immediately from (3.8), (3.9) and the fact that Ramanujan sums have only integral values. □

We are now ready to prove the main result of the paper.

Theorem 3.6 *Suppose that n is an odd integer or d is an even integer then all eigenvalues of the unitary Euclidean graph $T_n^{(d)}$ are integers.*

Proof We consider two cases.

Case 1: Suppose that n is odd. From Lemmas 3.1, 3.2 and 3.4, we can show by induction that $n \mid f_n(b)$ for any positive odd integer n and for any vector $b \in \mathbb{Z}_n^d$. Together with Lemma 3.3, we have λ_b's are integral for all $b \in \mathbb{Z}_n^d$.

Case 2: Suppose that d is even. From Lemmas 3.2, 3.4 and 3.5, we can show by induction that $n \mid f_n(b)$ for any squarefree positive integer n and for any vector $b \in \mathbb{Z}_n^d$. Since p_I is squarefree for all $I \subseteq \{1, \ldots, n\}$, from Lemma 3.3, we have λ_b's are integral for all $b \in \mathbb{Z}_n^d$. This completes the proof of the theorem. □

We conjecture that the same result also holds for odd dimensional cases.

Conjecture 3.7 *For any positive integers n and d all eigenvalues of the unitary Euclidean graph $T_n^{(d)}$ are integers.*

From Theorem 3.6, the remaining open case is: n even and d odd.

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