Cubic 1-fault-tolerant hamiltonian graphs, Globally 3*-connected graphs, and Super 3-spanning connected graphs

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Abstract

A *k-container* $C(u,v)$ in a graph G is a set of k internal vertex-disjoint paths between vertices u and v. A k^**-container* $C(u,v)$ of G is a *k-container* such that $C(u,v)$ contains all vertices of G. A graph is *globally* k^**-connected* if there exists a k^*-container $C(u,v)$ between any two distinct vertices u and v. A k-regular graph G is *super k-spanning connected* if G is i^*-connected for $1 \leq i \leq k$. A graph G is 1*-fault-tolerant hamiltonian* if $G - F$ is hamiltonian for any $F \subseteq V \cup E$ and $|F| = 1$. In this paper, we prove that for cubic graphs, every super 3-spanning connected graph is glob-

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ally 3^*-connected and every globally 3^*-connected graph is 1-fault-tolerant hamiltonian. We present some examples of super 3-spanning connected graphs, some examples of globally 3^*-connected graphs that are not super 3-spanning connected graphs, some examples of 1-fault-tolerant hamiltonian graphs that are globally 1^*-connected but not globally 3^*-connected, and some examples of 1-fault-tolerant hamiltonian that are neither globally 1^*-connected nor globally 3^*-connected. Furthermore, we prove that there are infinitely many graphs in each such family.

Keywords: hamiltonian, connectivity, Menger Theorem.

1 Definitions and notations

For the graph definition and notation we follow [3]. The *connectivity* of G, $\kappa(G)$, is the minimum number of vertices whose removal leaves the remaining graph disconnected or trivial. Let $G = (V, E)$ be a graph with connectivity $\kappa(G)$. It follows from Menger's Theorem [11] that there are k *internal vertex-disjoint* paths joining any two vertices u and v for any $k \leq \kappa(G)$. A *k-container* $C(u, v)$ in a graph G is a set of k internal vertex-disjoint paths between u and v.

Albert, Aldred, and Holton [2] introduced a very interesting family of graphs, called globally 3^*-connected graphs. A *globally* 3^**-connected graph* G is defined to be a cubic graph such that there exists a 3-container between any two different vertices of G that spans all vertices of G. Lin et al. [10] extended this concept by introducing the k^*-container. $C(u, v)$ is called a k^**-container* of G if $C(u, v)$ is a *k-container* between u and v such that it spans all vertices of G. A graph G is *globally* k^**-connected* if G contains a k^*-container $C(u, v)$ between any two distinct vertices u and v of G. (Thus, a globally k^*-connected graph might not be a regular graph under the new definition. We shall refer the globally 3^*-connected graphs proposed by

Albert, Aldred, and Holton as cubic globally 3^*-connected graphs.) Obviously, a graph G is globally 1^*-connected if and only if it is hamiltonian connected. Moreover, a graph G is globally 2^*-connected if and only if it is hamiltonian. In addition, all globally 1^*-connected graphs except K_1 and K_2 are globally 2^*-connected. By considering the 3^*-container between two adjacent vertices, it is easy to see that every globally 3^*-connected graph is globally 2^*-connected.

A graph G is *super spanning connected* if G is k^*-connected for any k with $1 \leq k \leq \Delta(G)$. Since $\kappa(G) \leq \delta(G) \leq \Delta(G)$, any super spanning connected graph is a regular graph. A k-regular super spanning connected graph is called a *super k-spanning connected graph.* Recently, several families of graphs are proved to be super spanning connected [6, 10, 14]. Graph containers do exist in engineering design, information/telecommunication networks, and biological neural systems (see [1, 4] and their references). Thus the study of super spanning connected graphs plays a vital role in design and implementation of parallel routing and in efficient information transmission of large scale network systems. From the applicational point of view, k^*-containers can be used in multipath communication. The fact that graph connectivity and hamiltonicity are two really interesting problems in graph theory makes the spanning connectivity interesting as well. Recently, Lin et al. [10] prove that those graphs G with $\frac{n(G)}{2} + 1 \leq \delta(G) \leq n(G) - 2$ are k^*-connected for $1 \leq k \leq 2\delta(G) - n(G) + 2$. Moreover, $G - T$ is k^*-connected for $1 \leq k \leq 2\delta(G) - n(G) + 2 - |T|$ if T is a vertex subset with $|T| \leq 2\delta(G) - n(G) - 1$.

From our observation, super k-spanning connected graphs is closely related to $(k-2)$-fault tolerant hamiltonian graphs. Let $G = (V, E)$ be a graph and let $V' \subseteq V$ and $E' \subseteq E$. We use $G - V'$ to denote the subgraph of G induced by $V - V'$, and $G - E'$ the subgraph obtained by removing E' from G. Faults can be in the combination of vertices and edges. Let $F \subseteq V \cup E$. We use $G - F$ to denote the subgraph induced by $V - F$ and deleting the edges in F from the induced subgraph.

If $G - F$ is hamiltonian for any $F \subseteq V \cup E$ and $|F| = k$, then G is called a k-fault-tolerant hamiltonian graph. An n-vertex k-fault-tolerant hamiltonian graph is *optimal* if it contains the least number of edges among all n-vertex k-fault-tolerant hamiltonian graphs. Obviously, every k-fault-tolerant hamiltonian graph has at least $k+3$ vertices. Moreover, the degree of each vertex in a k-fault-tolerant hamiltonian graph is at least $k+2$. The similar concepts and their connection with fault-tolerance for bipartite graphs are recently published by Kao et al. [8].

There are numerous interesting problems we can ask about super k-spanning connected graphs. For example, we suspect whether every super k-spanning connected graph is $(k-2)$-fault-tolerant hamiltonian because all examples we have indicate that the statement is true [5, 7, 13]. The other question is the existence of any graph that is globally k^*-connected but not globally $(k-t)^*$-connected for some $1 \leq t \leq k-1$. To answer the latter question, we begin with $k = 3$ by finding examples.

In [2], it is proved that every globally 3^*-connected graph is 1-fault-tolerant hamiltonian graph. Thus, every super 3-spanning connected graph is 1-fault-tolerant hamiltonian. Moreover, every 1-fault-tolerant remains hamiltonian after the removal of any edge. Therefore, every 1-fault-tolerant hamiltonian is hamiltonian and hence is globally 2^*-connected. It is known that every hamiltonian connected graph except K_1 and K_2 is hamiltonian. Thus, every cubic globally 1^*-connected graph is globally 2^*-connected. In Figure 1, we use Venn diagram to illustrate the relation among cubic globally 1^*-connected graphs, cubic globally 2^*-connected graphs, cubic globally 3^*-connected graphs, and cubic 1-fault-tolerant hamiltonian graphs.

We consider the existence of graphs for all the possible regions in Figure 1. It is proved that there exist infinitely many graphs in regions 5 and 6 in [9]. In Sections 3, 4, 5 and 6, we will present some examples of graphs in regions 1, 2, 3 and 4, respectively. Furthermore, we prove that there exist infinite graphs in regions 2, 3 and 4.

2 Preliminaries

Theorem 1 *Every super 3-spanning connected graph is globally 3^*-connected. Every globally 3^*-connected graph is 1-fault-tolerant hamiltonian.*

Proof. By definition, every super 3-spanning connected graph is globally 3^*-connected.

Suppose that G is a globally 3^*-connected graph. Let F be a subset of $V \cup E$ with $|F| = 1$. Suppose that $e = (x, y)$ is an edge in F. Since G is globally 3^*-connected, there are three disjoint paths, P_1, P_2, and P_3 joining x to y. Note that G is cubic. One of P_1, P_2, and P_3, say P_3, is $\langle x, y\rangle$. Obviously, $P_1 \cup P_2$ forms a hamiltonian cycle of $G - e$. Suppose that v is a vertex in F. Let x and y be two distinct neighbors of v. Since G is

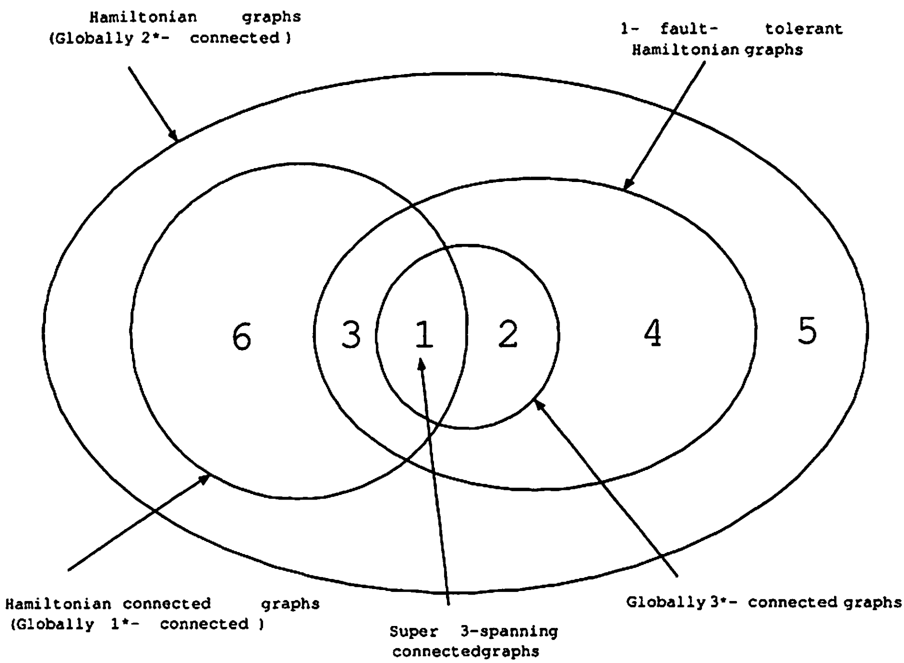

Figure 1: The set of cubic globally 1*-connected graphs corresponds to regions 1, 3 and 6; the set of cubic globally 2*-connected graphs corresponds to regions 1, 2, 3, 4, 5 and 6; the set of cubic globally 3*-connected graphs corresponds to regions 1 and 2; the set of cubic 1-fault-tolerant hamiltonian graphs corresponds to regions 1, 2, 3 and 4; the set of super 3-spanning connected graphs corresponds to region 1.

globally 3^*-connected, there are three disjoint paths, P_1, P_2, and P_3 joining x to y. Note that G is cubic. One of P_1, P_2, and P_3, say P_3, is $\langle x, v, y\rangle$. Obviously, $P_1 \cup P_2$ forms a hamiltonian cycle of $G-v$. Thus, every globally 3^*-connected graph is 1-fault-tolerant hamiltonian. □

Lemma 1 *Every 1-fault-tolerant hamiltonian graph is non-bipartite.*

Proof. Assume that G is a 1-fault-tolerant hamiltonian graph. Since $G-e$ is hamiltonian for any edge e of G, G contains a cycle of length $|V(G)|$. Since $G-v$ is hamiltonian for any vertex v in G, G contains a cycle of length $|V(G)|-1$. Obviously, G contains a cycle of odd length. Hence, G is non-bipartite. □

Lemma 2 *Assume that G is a cubic globally 1^*-connected graph. Let u and v be two distinct vertices in G with $d(u,v)=2$. Then either $G-u$ or $G-v$ is hamiltonian.*

Proof. Let w be the common neighbor of u and v. Since G is globally 1^*-connected, there exists a hamiltonian path P between u and v. Then P can be written as either $\langle u, w, P_1, v\rangle$ or $\langle u, P_2, w, v\rangle$. If $P=\langle u, w, P_1, v\rangle$, then $\langle w, P_1, v, w\rangle$ forms a hamiltonian cycle of $G-u$. If $P=\langle u, P_2, w, v\rangle$, then $\langle u, P_2, w, u\rangle$ forms a hamiltonian cycle of $G-v$. □

Let G_1 and G_2 be two graphs with $V(G_1)\cap V(G_2)=\emptyset$. Let $x\in V(G_1)$ with $\deg_{G_1}(x)=3$ and $y\in V(G_2)$ with $\deg_{G_2}(y)=3$. Let $N(x)=\{x_1, x_2, x_3\}$ be an ordered set of the neighbors of x and $N(y)=\{y_1, y_2, y_3\}$ an ordered set of the neighbors of y. *The 3-join* of G_1 and G_2 at x and y, denoted by $J(G_1, N(x); G_2, N(y))$, is the graph with $V(J(G_1, N(x); G_2, N(y)))=(V(G_1)-\{x\})\cup(V(G_2)-\{y\})$ and $E(J(G_1, N(x); G_2, N(y)))=(E(G_1)-$

$\{(x,x_i) \mid 1 \le i \le 3\}) \cup (E(G_2) - \{(y,y_i) \mid 1 \le i \le 3\}) \cup \{(x_i,y_i) \mid 1 \le i \le 3\}$. A graph G is called *a 3-join of* G_1 *and* G_2 if $G = J(G_1, N(x); G_2, N(y))$ for some vertices $x \in V(G_1)$ and $y \in V(G_2)$ with $\deg_{G_1}(x) = \deg_{G_2}(y) = 3$. We note that a different ordering of $N(x)$ and $N(y)$ generates a different 3-join of G_1 and G_2 at x and y. See Figure 2 for an illustration. In particular, let $G = (V, E)$ be a graph with a vertex x of degree 3. *The 3-vertex expansion of* G, is the graph $J(G, N(x); K_4, N(y))$ where $y \in V(K_4)$. Any vertex in $J(G, N(x); K_4, N(y)) - V(G)$ is called an *expanded vertex* of G at x. We have the following theorem.

Theorem 2 [15] *Assume that* G_1 *and* G_2 *are two cubic graphs. Let* G *be a 3-join of* G_1 *and* G_2. *Then* G *is a 1-fault-tolerant hamiltonian graph if both* G_1 *and* G_2 *are 1-fault-tolerant hamiltonian graphs.*

Let G be a cubic graph and x be a vertex of G with $N(x) = \{x_1, x_2, x_3\}$. We say that x is *nice* in G if it satisfies the following properties:
(1) $G - (x, x_i)$ *is hamiltonian for all* $i \in \{1,2,3\}$; (2) *for any* $i \in \{1,2,3\}$ *and for any vertex* u *of* G *with* $u \notin \{x, x_i\}$, *there exists a hamiltonian path of* $G - (x, x_i)$ *joining* u *to* x_i.

Theorem 3 [2] *Assume that both* G_1 *and* G_2 *are cubic graphs,* x *is a vertex in* G_1, *and* y *is a vertex in* G_2. *Then* $J(G_1, N(x); G_2, N(y))$ *is globally* 3^**-connected if and only if both* G_1 *and* G_2 *are globally* 3^**-connected.*

Theorem 4 [9] *Assume that* G_1 *is a globally* 1^**-connected graph with a nice vertex* x. *Let* K_4 *be the complete graph defined on* $\{y, y_1, y_2, y_3\}$ *and*

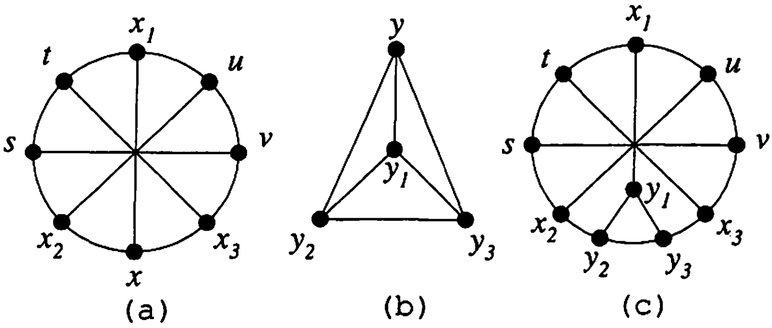

Figure 2: The graphs (a) G, (b) K_4 and (c) $J(G, N(x); K_4, N(y))$.

$G = J(G_1, N(x); K_4, N(y))$ for some $N(x)$ and $N(y)$. Then G is globally 1^*-connected. Moreover, y_1, y_2 and y_3 are nice vertices of G.

Theorem 5 [9] *Assume that G_1 is a graph with a degree 3 vertex x and two distinct vertices a and b such that $a \neq x$ and $b \neq x$. Let G_2 be a graph with a degree 3 vertex y such that $G_2 - (y, y_i)$ is hamiltonian for all $y_i \in N(y)$. Let $G = J(G_1, N(x); G_2, N(y))$ for some $N(x)$ and $N(y)$. Then there exists a hamiltonian path of G joining a to b if and only if there exists a hamiltonian path of G_1 joining a to b.*

3 Examples of super 3-spanning connected graphs

In this section, we present three families of super 3-spanning connected graphs. Throughout this section, we use $\oplus$ and $\ominus$ to denote addition and subtraction in integer modular n, Z_n.

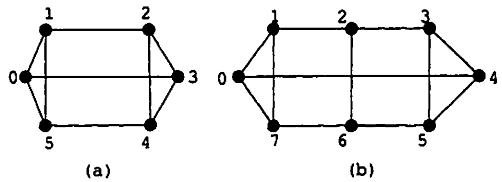

Figure 3: The ladder graphs (a) $L(3)$ and (b) $L(4)$.

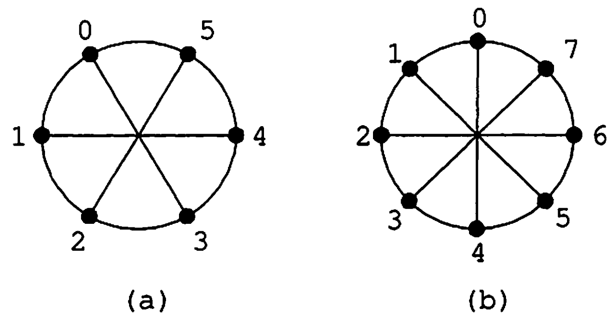

Figure 4: The projective cycle graphs (a) $PJ(3)$ and (b) $PJ(4)$.

3.1 The ladder graphs

Assume that n and k are two positive integers with $n = 2k$ and $k \geq 2$. The *ladder graph* $L(k)$ is the graph with the vertex set $\{0, 1, \ldots, 2k-1\}$ and the edge set $\{(i, 2k-i) \mid 1 \leq i < k\} \cup \{(i, i \oplus 1) \mid 0 \leq i \leq 2k-1\} \cup \{(0, k)\}$. The ladder graphs $L(3)$ and $L(4)$ are shown in Figure 3. It is proved that any $L(k)$ is globally 1^*-connected [12]. Moreover, it is proved that any $L(k)$ is globally 3^*-connected [2]. We have the following theorem.

Theorem 6 *Any $L(k)$ is super 3-spanning connected.*

3.2 The projective cycle graphs

Assume that n and k are two positive integers with $n = 2k$ and $k \geq 2$. The *projective cycle graph* $PJ(k)$ is the graph with the vertex set $\{0, 1, \ldots, 2k-1\}$ and the edge set $\{(i, k+i) \mid 0 \leq i < k\} \cup \{(i, i \oplus 1) \mid 0 \leq i < 2k\}$. The projective cycle graphs $PJ(3)$ and $PJ(4)$ are shown in Figure 4. It is easy to see that $PJ(k)$ is a bipartite graph if and only if k is odd. By Lemma 1, any $PJ(k)$ is not super 3-spanning connected if k is odd. It is proved in [12] that any $PJ(k)$ with an even integer k is globally 1^*-connected. The following theorem can be obtained by showing that $PJ(k)$ is globally 3^*-connected if k is even. Readers can easily construct the 3^*-container between any given pair of different vertices of $PJ(k)$ by brute force.

Theorem 7 *$PJ(k)$ is super 3-spanning connected if and only if k is even.*

3.3 The generalized Petersen graph $P(n, 1)$

The *generalized Petersen graph* $P(n, k)$ is the graph with the vertex set $\{i \mid 0 \leq i < n\} \cup \{i' \mid 0 \leq i < n\}$ and the edge set $\{(i, i \oplus 1) \mid 0 \leq i < n\} \cup \{(i, i') \mid 0 \leq i < n\} \cup \{(i', (i \oplus k)') \mid 0 \leq i < n\}$. It is easy to see that $P(n, 1)$ is bipartite if and only if n is even. By Lemma 1, any $P(n, 1)$ is not super 3-spanning connected if n is even. It is proved in [2] that $P(n, 1)$ is globally 3^*-connected if n is odd. Moreover, it is proved in [12] that $P(n, 1)$ is globally 1^*-connected if n is odd. We have the following theorem.

Theorem 8 *$P(n, 1)$ is super 3-spanning connected if and only if n is odd.*

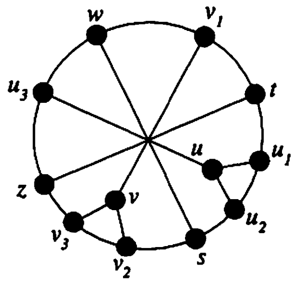

Figure 5: The graph T.

4 Examples of cubic 1-fault-tolerant hamiltonian graphs that are globally 3^*-connected but not globally 1^*-connected

Let T be the graph in Figure 5. Obviously, T is obtained from $PJ(4)$ by a sequence of 3-vertex expansion of $PJ(4)$. More precisely, T is obtained by a 3-vertex expansion on vertex v of the graph in Figure 2(c). Since $PJ(4)$ is super 3^*-connected, it is globally 3^*-connected. Since K_4 is isomorphism to $PJ(2)$, K_4 is globally 3^*-connected. With Theorem 3, T is globally 3^*-connected.

Lemma 3 *There exists no hamiltonian path between u and v in T.*

Proof. Suppose that T has a hamiltonian path, P, between u and v.

Case 1 P contains (u, u_1). Then $\{(u, u_2), (u, u_3)\} \not\subset P$, and P contains $\langle u, u_1, u_2, s\rangle$ and $\langle w, u_3, z, t, v_1\rangle$. Hence, $(z, v_3) \notin P$, and $\langle v_2, v_3, v\rangle \in P$. Therefore, $(v_1, w) \in P$. Thus, P contains a cycle $\langle w, u_3, z, t, v_1, w\rangle$, which is impossible.

Case 2 P contains (u, u_2). Then $\{(u, u_1), (u, u_3)\} \not\subset P$, and P contains $\langle u, u_2, u_1, t\rangle$ and $\langle z, u_3, w, s, v_2\rangle$. Since $(w, v_1) \notin P$, $\langle t, v_1, v\rangle \subset P$. Then $(v, v_3) \notin P$, and $\langle v_2, v_3, z\rangle \in P$. Thus, P contains a cycle $\langle z, u_3, w, s, v_2, v_3, z\rangle$, which is impossible.

Case 3 P contains (u, u_3). Then $\{(u, u_1), (u, u_2)\} \not\subset P$ and P contains $\langle s, u_2, u_1, t\rangle$.

(3.1) P contains (u_3, w). Then $\langle t, z, v_3\rangle \in P$. Since $(t, v_1) \notin P$, $\langle w, v_1, v\rangle \in P$. Therefore, $\{(v, v_2), (v, v_3)\} \not\subset P$ and $\langle s, v_2, v_3\rangle \in P$. Thus, P has a cycle $\langle s, u_2, u_1, t, z, v_3, v_2, s\rangle$. This is a contradiction.

(3.2) P contains (u_3, z). Then $(u_3, w) \notin P$ and P contains $\langle v_1, w, s\rangle$. Since $(s, v_2) \notin P$, P contains $\langle v, v_2, v_3, z\rangle$ and $(v_1, t) \in P$. Thus, P has a cycle $\langle s, u_2, u_1, t, v_1, w, s\rangle$. This is a contradiction.

The lemma is proved. □.

With Lemma 3, T is not globally 1^*-connected. Hence, T is a graph that is globally 3^*-connected, but not super 3-spanning connected.

Let w be the vertex of T shown in Figure 5. With Lemma 3 and Theorem 5, there exists no hamiltonian path between vertices u and v in $J(T, N(w); K_4, N(y))$. Thus, $J(T, N(w); K_4, N(y))$ is not globally 1^*-connected. Now, we recursively define a sequence of graphs as follows: Let $G_1 = T$, $x_1 = w$ and $G_2 = J(G_1, N(x_1); K_4, N(y))$. Suppose that we have defined $G_1, G_2, ..., G_i$ with $i \geq 2$. Let x_i be any expanded vertex of G_{i-1} at x_{i-1}. We define G_{i+1} as $J(G_i, N(x_i); K_4, N(y))$. Recursively

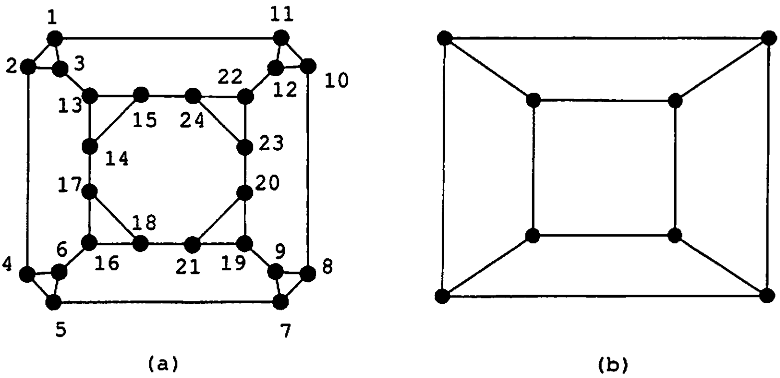

Figure 6: The graphs (a) Q and (b) Q_3.

applying Theorems 5 and 3, G_i is globally 3^*-connected but not hamiltonian connected for $i \geq 2$. With Theorem 1, G_i is 1-fault-tolerant hamiltonian for $i \geq 2$. Hence, we have the following theorem.

Theorem 9 *There are infinite cubic 1-fault-tolerant hamiltonian graphs that are globally* 3^**-connected but not globally* 1^**-connected.*

5 Examples of cubic 1-fault-tolerant hamiltonian graphs that are globally 1^*-connected but not globally 3^*-connected

Let Q be the graph as in Figure 6(a). By brute force, we can check that Q is 1-fault-tolerant hamiltonian and globally 1^*-connected. (See Fact 1 in Appendix.) It is obvious that Q can be obtained from the 3-dimensional hypercube Q_3 by a sequence of 3-vertex expansions of Q_3. Note that Q_3 is bipartite. By Lemma 1, Q_3 is not 1-fault-tolerant hamiltonian. By Theorem 1, Q_3 is not globally 3^*-connected. With Theorem 3, Q is not

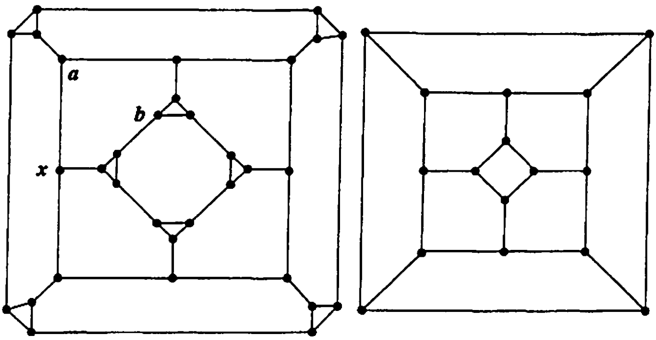

Figure 7: The graphs (a) M and (b) $P(8,2)$.

globally 3*-connected.

Let x be the vertex 1 of Q shown in Figure 6(a). By brute force, we can check that x is nice in Q. (See Fact 2 in Appendix.) Moreover, by Theorem 4, any expanded vertex of Q at 1 is nice. Now, we recursively define a sequence of graphs as follows: Let $G_1 = Q$, $x_1 = x$ and $G_2 = J(G_1, N(x_1); K_4, N(y))$. Suppose that we have defined $G_1, G_2, ..., G_i$ with $i \geq 2$. Let x_i be any expanded vertex of G_{i-1} at x_{i-1}. We define G_{i+1} as $J(G_i, N(x_i); K_4, N(y))$. Recursively applying Theorems 2, 4, and 3, G_i is 1-fault-tolerant hamiltonian and globally 1*-connected but not globally 3*-connected for $i \geq 2$. Hence, we have the following theorem.

Theorem 10 *There are infinite cubic 1-fault-tolerant hamiltonian graphs that are globally 1*-connected but not globally 3*-connected.*

6 Examples of cubic 1-fault-tolerant hamiltonian graphs that are neither globally 1^*-connected nor globally 3^*-connected

Let M be the graph as in Figure 7(a). Obviously, M can be obtained from $P(8,2)$, shown in Figure 7(b), by a sequence of 3-vertex expansions of $P(8,2)$. It is proved in [2] that $P(8,2)$ is not globally 3^*-connected. Using Theorem 3, M is not globally 3^*-connected. Yet, it is proved in [9] that M is 1-fault-tolerant hamiltonian but not globally 1^*-connected.

Let x, a, and b be three distinct vertices of M shown in Figure 7(a). It is proved in [9] that there exists no hamiltonian path between a and b in $J(M, N(x); K_4, N(y))$. Thus, $J(M, N(x); K_4, N(y))$ is not globally 1^*-connected. Let $G_1 = M$, $x_1 = x$ and $G_2 = J(G_1, N(x_1); K_4, N(y))$. Suppose that we have defined $G_1, G_2, \ldots, G_i$ with $i \geq 2$. Let x_i be any expanded vertex of G_{i-1} at x_{i-1}. We define G_{i+1} as $J(G_i, N(x_i); K_4, N(y))$. Recursively applying Theorems 2, 5 and 3, G_i is a 1-fault-tolerant hamiltonian graph that is neither globally 1^*-connected nor globally 3^*-connected for every $i \geq 2$. Hence, we have the following theorem.

Theorem 11 *There are infinite cubic 1-fault-tolerant hamiltonian graphs that are neither globally* 1^**-connected nor globally* 3^**-connected.*

7 Conclusion

In this paper, we prove that for cubic graphs, every super 3-spanning connected graph is globally 3^*-connected and every globally 3^*-connected graph is 1-fault-tolerant hamiltonian. We present some examples of super 3-spanning connected graphs, some examples of globally 3^*-connected graphs that are not super 3-spanning connected graphs, some examples of 1-fault-tolerant hamiltonian graphs that are globally 1^*-connected but not globally 3^*-connected, and some examples of 1-fault-tolerant hamiltonian that are neither globally 1^*-connected nor globally 3^*-connected. Furthermore, we show that there are infinitely many graphs in each such family. We fail to find a planar globally 3^*-connected graph that is not globally 1^*-connected. We conjecture that every 1-fault-tolerant hamiltonian graph that is globally 3^*-connected but not globally 1^*-connected is obtained from T by a sequence of 3-vertex expansions.

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8 Appendix

Fact 1 *Q is 1-fault-tolerant hamiltonian and globally* 1^**-connected.*

Proof. We first check Q is 1-fault-tolerant hamiltonian.

With the symmetric property of Q, we can verify that Q is 1-fault-tolerant hamiltonian by showing that $Q - f$ has a hamiltonian cycle for $f \in \{1, (1,2), (1,11)\}$. The corresponding hamiltonian cycles are listed below.

$Q - f$	Hamiltonian cycle in $Q - f$
$Q - 1$	$\langle 2,4,5,6,16,17,18,21,20,19,9,7,8,10,11,12,22,23,24,15,14,13,3,2 \rangle$
$Q - (1,2)$	$\langle 1,11,10,12,22,23,24,15,13,14,17,16,18,21,20,19,9,8,7,5,6,4,2,3,1 \rangle$
$Q - (1,11)$	$\langle 1,3,13,14,15,24,23,22,12,11,10,8,7,9,19,20,21,18,17,16,6,5,4,2,1 \rangle$

In the following, we check that Q is globally 1^*-connected. With the symmetric property of Q, we only need to find a hamiltonian path between 1 and x for any $x \neq 1$. The corresponding hamiltonian paths are listed below.

$\{1,x\}$	Hamiltonian path between 1 and x
$\{1,2\}$	$\langle 1,3,13,14,15,24,23,22,12,11,10,8,7,9,19,20,21,18,17,16,6,5,4,2 \rangle$
$\{1,3\}$	$\langle 1,2,4,5,6,16,17,18,21,20,19,9,7,8,10,11,12,22,23,24,15,14,13,3 \rangle$
$\{1,4\}$	$\langle 1,2,3,13,14,15,24,23,22,12,11,10,8,7,9,19,20,21,18,17,16,6,5,4 \rangle$
$\{1,5\}$	$\langle 1,2,3,13,14,15,24,23,22,12,11,10,8,7,9,19,20,21,18,17,16,6,4,5 \rangle$
$\{1,6\}$	$\langle 1,2,3,13,15,14,17,16,18,21,19,20,23,24,22,12,11,10,8,9,7,5,4,6 \rangle$
$\{1,7\}$	$\langle 1,2,3,13,14,15,24,23,22,12,11,10,8,9,19,20,21,18,17,16,6,4,5,7 \rangle$
$\{1,8\}$	$\langle 1,2,3,13,15,14,17,18,16,6,4,5,7,9,19,21,20,23,24,22,12,11,10,8 \rangle$
$\{1,9\}$	$\langle 1,2,3,13,14,15,24,23,22,12,11,10,8,7,5,4,6,16,17,18,21,20,19,9 \rangle$
$\{1,10\}$	$\langle 1,2,3,13,15,14,17,18,16,6,4,5,7,8,9,19,21,20,23,24,22,12,11,10 \rangle$
$\{1,11\}$	$\langle 1,2,3,13,14,15,24,22,23,20,19,21,18,17,16,6,4,5,7,9,8,10,12,11 \rangle$
$\{1,12\}$	$\langle 1,2,3,13,14,15,24,22,23,20,19,21,18,17,16,6,4,5,7,9,8,10,11,12 \rangle$
$\{1,13\}$	$\langle 1,3,2,4,5,6,16,17,18,21,20,19,9,7,8,10,11,12,22,23,24,15,14,13 \rangle$
$\{1,14\}$	$\langle 1,3,2,4,5,6,16,17,18,21,20,19,9,7,8,10,11,12,22,23,24,15,13,14 \rangle$
$\{1,15\}$	$\langle 1,3,2,4,6,5,7,9,8,10,11,12,22,24,23,20,19,21,18,16,17,14,13,15 \rangle$
$\{1,16\}$	$\langle 1,2,3,13,15,14,17,18,21,19,20,23,24,22,12,11,10,8,9,7,5,4,6,16 \rangle$
$\{1,17\}$	$\langle 1,3,2,4,5,6,16,18,21,20,19,9,7,8,10,11,12,22,23,24,15,13,14,17 \rangle$
$\{1,18\}$	$\langle 1,2,3,13,15,14,17,16,6,4,5,7,9,8,10,11,12,22,24,23,20,19,21,18 \rangle$
$\{1,19\}$	$\langle 1,2,3,13,14,15,24,23,22,12,11,10,8,9,7,5,4,6,16,17,18,21,20,19 \rangle$
$\{1,20\}$	$\langle 1,2,3,13,14,15,24,23,22,12,11,10,8,9,7,5,4,6,16,17,18,21,19,20 \rangle$
$\{1,21\}$	$\langle 1,2,3,13,15,14,17,18,16,6,4,5,7,9,8,10,11,12,22,24,23,20,19,21 \rangle$
$\{1,22\}$	$\langle 1,2,3,13,14,15,24,23,20,19,21,18,17,16,6,4,5,7,9,8,10,11,12,22 \rangle$
$\{1,23\}$	$\langle 1,2,3,13,14,15,24,22,12,11,10,8,9,7,5,4,6,16,17,18,21,19,20,23 \rangle$
$\{1,24\}$	$\langle 1,3,2,4,6,5,7,9,8,10,11,12,22,23,20,19,21,18,16,17,14,13,15,24 \rangle$

□

Fact 2 *The vertex* 1 *is nice in* Q.

Proof. First, we prove that $Q - (1, x_i)$ is hamiltonian for any $x_i \in N(1)$. The corresponding hamiltonian cycles are listed below.

$Q - (1, x_i)$	The hamiltonian cycles of $Q - (1, x_i)$.
$Q - (1, 11)$	$\langle 1, 3, 13, 14, 15, 24, 23, 22, 12, 11, 10, 8, 7, 9, 19, 20, 21, 18, 17, 16, 6, 5, 4, 2, 1 \rangle$
$Q - (1, 2)$	$\langle 1, 11, 12, 10, 8, 7, 9, 19, 21, 20, 23, 22, 24, 15, 13, 14, 17, 18, 16, 6, 5, 4, 2, 3, 1 \rangle$
$Q - (1, 3)$	$\langle 1, 11, 12, 10, 8, 9, 7, 5, 4, 6, 16, 17, 18, 21, 19, 20, 23, 22, 24, 15, 14, 13, 3, 2, 1 \rangle$

In the following, we list all hamiltonian paths between u and x_i in $Q - (1, x_i)$ for any $u \in V(Q) - \{1, x_i\}$ and $x_i \in N(1)$. The corresponding hamiltonian paths are listed below.

$\{u, 11\}$	The hamiltonian paths between u and 11 in $Q - (1, 11)$ for any $u \notin \{1, 11\}$.
$\{2, 11\}$	$\langle 2, 1, 3, 13, 14, 15, 24, 22, 23, 20, 19, 21, 18, 17, 16, 6, 4, 5, 7, 9, 8, 10, 12, 11 \rangle$
$\{3, 11\}$	$\langle 3, 1, 2, 4, 6, 5, 7, 8, 9, 19, 20, 21, 18, 16, 17, 14, 13, 15, 24, 23, 22, 12, 10, 11 \rangle$
$\{4, 11\}$	$\langle 4, 2, 1, 3, 13, 14, 15, 24, 22, 23, 20, 19, 21, 18, 17, 16, 6, 5, 7, 9, 8, 10, 12, 11 \rangle$
$\{5, 11\}$	$\langle 5, 7, 8, 9, 19, 20, 21, 18, 17, 16, 6, 4, 2, 1, 3, 13, 14, 15, 24, 23, 22, 12, 10, 11 \rangle$
$\{6, 11\}$	$\langle 6, 16, 18, 17, 14, 15, 13, 3, 1, 2, 4, 5, 7, 8, 9, 19, 21, 20, 23, 24, 22, 12, 10, 11 \rangle$
$\{7, 11\}$	$\langle 7, 8, 9, 19, 20, 21, 18, 17, 16, 6, 5, 4, 2, 1, 3, 13, 14, 15, 24, 23, 22, 12, 10, 11 \rangle$
$\{8, 11\}$	$\langle 8, 9, 7, 5, 6, 4, 2, 1, 3, 13, 15, 14, 17, 16, 18, 21, 19, 20, 23, 24, 22, 12, 10, 11 \rangle$
$\{9, 11\}$	$\langle 9, 8, 7, 5, 6, 4, 2, 1, 3, 13, 15, 14, 17, 16, 18, 21, 19, 20, 23, 24, 22, 12, 10, 11 \rangle$
$\{10, 11\}$	$\langle 10, 8, 9, 7, 5, 6, 4, 2, 1, 3, 13, 15, 14, 17, 16, 18, 21, 19, 20, 23, 24, 22, 12, 11 \rangle$
$\{12, 11\}$	$\langle 11, 10, 8, 9, 7, 5, 6, 4, 2, 1, 3, 13, 15, 14, 17, 16, 18, 21, 19, 20, 23, 24, 22, 12 \rangle$
$\{13, 11\}$	$\langle 11, 12, 10, 8, 7, 9, 19, 21, 20, 23, 22, 24, 15, 14, 17, 18, 16, 6, 5, 4, 2, 1, 3, 13 \rangle$
$\{14, 11\}$	$\langle 11, 12, 10, 8, 9, 7, 5, 6, 4, 2, 1, 3, 13, 15, 24, 22, 23, 20, 19, 21, 18, 16, 17, 14 \rangle$
$\{15, 11\}$	$\langle 11, 12, 10, 8, 9, 7, 5, 6, 4, 2, 1, 3, 13, 14, 17, 16, 18, 21, 19, 20, 23, 22, 24, 15 \rangle$
$\{16, 11\}$	$\langle 11, 12, 10, 8, 9, 7, 5, 6, 4, 2, 1, 3, 13, 14, 15, 24, 22, 23, 20, 19, 21, 18, 17, 16 \rangle$
$\{17, 11\}$	$\langle 11, 12, 10, 8, 9, 7, 5, 6, 4, 2, 1, 3, 13, 14, 15, 24, 22, 23, 20, 19, 21, 18, 16, 17 \rangle$
$\{18, 11\}$	$\langle 11, 12, 10, 8, 7, 9, 19, 21, 20, 23, 22, 24, 15, 14, 13, 3, 1, 2, 4, 5, 6, 16, 17, 18 \rangle$
$\{19, 11\}$	$\langle 11, 10, 12, 22, 24, 23, 20, 21, 18, 16, 17, 14, 15, 13, 3, 1, 2, 4, 6, 5, 7, 8, 9, 19 \rangle$
$\{20, 11\}$	$\langle 11, 12, 10, 8, 7, 9, 19, 21, 18, 17, 16, 6, 5, 4, 2, 1, 3, 13, 14, 15, 24, 22, 23, 20 \rangle$
$\{21, 11\}$	$\langle 11, 12, 10, 8, 7, 9, 19, 20, 23, 22, 24, 15, 14, 13, 3, 1, 2, 4, 5, 6, 16, 17, 18, 21 \rangle$
$\{22, 11\}$	$\langle 11, 12, 10, 8, 9, 7, 5, 6, 4, 2, 1, 3, 13, 15, 14, 17, 16, 18, 21, 19, 20, 23, 24, 22 \rangle$
$\{23, 11\}$	$\langle 11, 12, 10, 8, 7, 9, 19, 20, 21, 18, 17, 16, 6, 5, 4, 2, 1, 3, 13, 14, 15, 24, 22, 23 \rangle$
$\{24, 11\}$	$\langle 11, 12, 10, 8, 9, 7, 5, 6, 4, 2, 1, 3, 13, 15, 14, 17, 16, 18, 21, 19, 20, 23, 22, 24 \rangle$

$\{u, 2\}$	The hamiltonian paths between u and 2 in $Q - (1, 2)$ for any $u \notin \{1, 2\}$.
$\{3, 2\}$	$\langle 2, 4, 6, 5, 7, 8, 9, 19, 20, 21, 18, 16, 17, 14, 13, 15, 24, 23, 22, 12, 10, 11, 1, 3 \rangle$
$\{4, 2\}$	$\langle 2, 3, 1, 11, 12, 10, 8, 7, 9, 19, 21, 20, 23, 22, 24, 15, 13, 14, 17, 18, 16, 6, 5, 4 \rangle$
$\{5, 2\}$	$\langle 2, 3, 1, 11, 12, 10, 8, 7, 9, 19, 21, 20, 23, 22, 24, 15, 13, 14, 17, 18, 16, 6, 4, 5 \rangle$
$\{6, 2\}$	$\langle 2, 3, 1, 11, 10, 12, 22, 23, 24, 15, 13, 14, 17, 16, 18, 21, 20, 19, 9, 8, 7, 5, 4, 6 \rangle$
$\{7, 2\}$	$\langle 2, 3, 1, 11, 12, 10, 8, 9, 19, 21, 20, 23, 22, 24, 15, 13, 14, 17, 18, 16, 6, 4, 5, 7 \rangle$
$\{8, 2\}$	$\langle 2, 4, 5, 6, 16, 18, 17, 14, 15, 13, 3, 1, 11, 10, 12, 22, 24, 23, 20, 21, 19, 9, 7, 8 \rangle$
$\{9, 2\}$	$\langle 2, 3, 1, 11, 12, 10, 8, 7, 5, 4, 6, 16, 18, 17, 14, 13, 15, 24, 22, 23, 20, 21, 19, 9 \rangle$
$\{10, 2\}$	$\langle 2, 4, 5, 6, 16, 18, 17, 14, 15, 13, 3, 1, 11, 12, 22, 24, 23, 20, 21, 19, 9, 7, 8, 10 \rangle$
$\{11, 2\}$	$\langle 2, 4, 6, 5, 7, 9, 8, 10, 12, 22, 24, 23, 20, 19, 21, 18, 16, 17, 14, 15, 13, 3, 1, 11 \rangle$
$\{12, 2\}$	$\langle 2, 4, 6, 5, 7, 9, 8, 10, 11, 1, 3, 13, 15, 14, 17, 16, 18, 21, 19, 20, 23, 24, 22, 12 \rangle$
$\{13, 2\}$	$\langle 2, 3, 1, 11, 12, 10, 8, 9, 7, 5, 4, 6, 16, 17, 18, 21, 19, 20, 23, 22, 24, 15, 14, 13 \rangle$
$\{14, 2\}$	$\langle 2, 3, 1, 11, 12, 10, 8, 9, 7, 5, 4, 6, 16, 17, 18, 21, 19, 20, 23, 22, 24, 15, 13, 14 \rangle$
$\{15, 2\}$	$\langle 2, 3, 1, 11, 10, 12, 22, 24, 23, 20, 21, 19, 9, 8, 7, 5, 4, 6, 16, 18, 17, 14, 13, 15 \rangle$
$\{16, 2\}$	$\langle 2, 3, 1, 11, 10, 12, 22, 23, 24, 15, 13, 14, 17, 18, 21, 20, 19, 9, 8, 7, 5, 4, 6, 16 \rangle$
$\{17, 2\}$	$\langle 2, 3, 1, 11, 12, 10, 8, 9, 7, 5, 4, 6, 16, 18, 21, 19, 20, 23, 22, 24, 15, 13, 14, 17 \rangle$
$\{18, 2\}$	$\langle 2, 3, 1, 11, 12, 10, 8, 9, 7, 5, 4, 6, 16, 17, 14, 13, 15, 24, 22, 23, 20, 19, 21, 18 \rangle$
$\{19, 2\}$	$\langle 2, 3, 1, 11, 12, 10, 8, 9, 7, 5, 4, 6, 16, 18, 17, 14, 13, 15, 24, 22, 23, 20, 21, 19 \rangle$
$\{20, 2\}$	$\langle 2, 3, 1, 11, 10, 12, 22, 23, 24, 15, 13, 14, 17, 18, 16, 6, 4, 5, 7, 8, 9, 19, 21, 20 \rangle$
$\{21, 2\}$	$\langle 2, 3, 1, 11, 12, 10, 8, 9, 7, 5, 4, 6, 16, 18, 17, 14, 13, 15, 24, 22, 23, 20, 19, 21 \rangle$
$\{22, 2\}$	$\langle 2, 4, 6, 5, 7, 9, 8, 10, 12, 11, 1, 3, 13, 15, 14, 17, 16, 18, 21, 19, 20, 23, 24, 22 \rangle$
$\{23, 2\}$	$\langle 2, 3, 1, 11, 10, 12, 22, 24, 15, 13, 14, 17, 18, 16, 6, 4, 5, 7, 8, 9, 19, 21, 20, 23 \rangle$
$\{24, 2\}$	$\langle 2, 3, 1, 11, 10, 12, 22, 23, 20, 21, 19, 9, 8, 7, 5, 4, 6, 16, 18, 17, 14, 13, 15, 24 \rangle$

$\{u, 3\}$	The hamiltonian paths between u and 3 in $Q - (1, 3)$ for any $u \notin \{1, 3\}$.
$\{2, 3\}$	$\langle 2, 1, 11, 12, 10, 8, 9, 7, 5, 4, 6, 16, 17, 18, 21, 19, 20, 23, 22, 24, 15, 14, 13, 3 \rangle$
$\{4, 3\}$	$\langle 3, 2, 1, 11, 12, 10, 8, 7, 9, 19, 21, 20, 23, 22, 24, 15, 13, 14, 17, 18, 16, 6, 5, 4 \rangle$
$\{5, 3\}$	$\langle 3, 2, 1, 11, 12, 10, 8, 7, 9, 19, 21, 20, 23, 22, 24, 15, 13, 14, 17, 18, 16, 6, 4, 5 \rangle$
$\{6, 3\}$	$\langle 3, 2, 1, 11, 10, 12, 22, 23, 24, 15, 13, 14, 17, 16, 18, 21, 20, 19, 9, 8, 7, 5, 4, 6 \rangle$
$\{7, 3\}$	$\langle 3, 2, 1, 11, 12, 10, 8, 9, 19, 21, 20, 23, 22, 24, 15, 13, 14, 17, 18, 16, 6, 4, 5, 7 \rangle$
$\{8, 3\}$	$\langle 3, 13, 14, 15, 24, 23, 22, 12, 10, 11, 1, 2, 4, 5, 6, 16, 17, 18, 21, 20, 19, 9, 7, 8 \rangle$
$\{9, 3\}$	$\langle 3, 2, 1, 11, 12, 10, 8, 7, 5, 4, 6, 16, 18, 17, 14, 13, 15, 24, 22, 23, 20, 21, 19, 9 \rangle$
$\{10, 3\}$	$\langle 3, 13, 14, 15, 24, 23, 22, 12, 11, 1, 2, 4, 5, 6, 16, 17, 18, 21, 20, 19, 9, 7, 8, 10 \rangle$
$\{11, 3\}$	$\langle 3, 13, 14, 15, 24, 23, 22, 12, 10, 8, 7, 9, 19, 20, 21, 18, 17, 16, 6, 5, 4, 2, 1, 11 \rangle$
$\{12, 3\}$	$\langle 3, 13, 15, 14, 17, 18, 16, 6, 5, 4, 2, 1, 11, 10, 8, 7, 9, 19, 21, 20, 23, 24, 22, 12 \rangle$
$\{13, 3\}$	$\langle 3, 2, 1, 11, 12, 10, 8, 9, 7, 5, 4, 6, 16, 17, 18, 21, 19, 20, 23, 22, 24, 15, 14, 13 \rangle$
$\{14, 3\}$	$\langle 3, 2, 1, 11, 12, 10, 8, 9, 7, 5, 4, 6, 16, 17, 18, 21, 19, 20, 23, 22, 24, 15, 13, 14 \rangle$
$\{15, 3\}$	$\langle 3, 2, 1, 11, 10, 12, 22, 24, 23, 20, 21, 19, 9, 8, 7, 5, 4, 6, 16, 18, 17, 14, 13, 15 \rangle$
$\{16, 3\}$	$\langle 3, 2, 1, 11, 10, 12, 22, 23, 24, 15, 13, 14, 17, 18, 21, 20, 19, 9, 8, 7, 5, 4, 6, 16 \rangle$
$\{17, 3\}$	$\langle 3, 2, 1, 11, 12, 10, 8, 9, 7, 5, 4, 6, 16, 18, 21, 19, 20, 23, 22, 24, 15, 13, 14, 17 \rangle$
$\{18, 3\}$	$\langle 3, 2, 1, 11, 12, 10, 8, 9, 7, 5, 4, 6, 16, 17, 14, 13, 15, 24, 22, 23, 20, 19, 21, 18 \rangle$
$\{19, 3\}$	$\langle 3, 2, 1, 11, 12, 10, 8, 9, 7, 5, 4, 6, 16, 18, 17, 14, 13, 15, 24, 22, 23, 20, 21, 19 \rangle$
$\{20, 3\}$	$\langle 3, 2, 1, 11, 10, 12, 22, 23, 24, 15, 13, 14, 17, 18, 16, 6, 4, 5, 7, 8, 9, 19, 21, 20 \rangle$
$\{21, 3\}$	$\langle 3, 2, 1, 11, 12, 10, 8, 9, 7, 5, 4, 6, 16, 18, 17, 14, 13, 15, 24, 22, 23, 20, 19, 21 \rangle$
$\{22, 3\}$	$\langle 3, 13, 15, 14, 17, 18, 16, 6, 5, 4, 2, 1, 11, 12, 10, 8, 7, 9, 19, 21, 20, 23, 24, 22 \rangle$
$\{23, 3\}$	$\langle 3, 2, 1, 11, 10, 12, 22, 24, 15, 13, 14, 17, 18, 16, 6, 4, 5, 7, 8, 9, 19, 21, 20, 23 \rangle$
$\{24, 3\}$	$\langle 3, 2, 1, 11, 10, 12, 22, 23, 20, 21, 19, 9, 8, 7, 5, 4, 6, 16, 18, 17, 14, 13, 15, 24 \rangle$

□