Decomposition of a $3K_{8t}$ into H_2 Graphs

Dinesh G. Sarvate *
Department of Mathematics
College of Charleston
Charleston, SC 29424
U. S. A.
SarvateD@cofc.edu

Li Zhang †
Department of Mathematics
and Computer Science
The Citadel
Charleston, SC 29409
U. S. A.
li.zhang@citadel.edu

Abstract

An H_2 graph is a multigraph on three points with a double edge between a pair of distinct points and single edges between the other two pairs. In this paper we settle the H_2 graph decomposition problem, which was unfinished in a paper of Hurd and Sarvate, by decomposing a complete multigraph $3K_{8t}$ into H_2 graphs recursively.

1 Introduction

A graph can be decomposed into a collection of subgraphs such that every edge of the graph is contained in one of the subgraphs. Graph decomposition is an important problem in combinatorial design theory and it has many applications such as experimental designs, group testings, DNA library screening, scheduling and synchronous optical networks [6]. Decomposing a graph into simple graphs has been well studied in the literature. For a well written survey on the decomposition of a complete graph into simple graphs with small number of points and edges, see [1].

A *multigraph* is a graph where more than one edge between a pair of points is allowed. The decomposition of copies of a complete graph into proper multigraphs has not received much attention yet, see [3, 7, 9]. A complete multigraph λK_v ($\lambda \geq 1$) is a graph on v points with λ edges between every pair of distinct points. In this paper we study the decomposition of a $3K_{8t}$ ($t \geq 1$) into H_2 graphs (defined in section 1.1). A well studied combinatorial design (BIBD, which can also be used to find graph decompositions) is defined below. On the other hand,

*Thanks to the Department of Mathematics, College of Charleston for a Summer 2012 research grant.

†Thanks to The Citadel Foundation for its support during 2012 sabbatical.

BIBD itself can be considered as a decomposition of λK_v into complete graphs K_k.

Definition 1 Given a finite set V of v points and integers k and $\lambda \geq 1$, a *balanced incomplete block design* (BIBD), denoted as BIBD(v, k, λ), is a pair (V, B) where B is a collection of subsets (also called blocks) of V such that every block contains exactly $k < v$ points and every pair of distinct points is contained in exactly λ blocks.

The following definition and results from combinatorial designs are well-known, for example, see [4, 8, 10].

Definition 2 A *group divisible design* GDD$(g, u, k; \lambda_1, \lambda_2)$ is a collection of k-subsets(blocks) of a set V of v points such that each point appears in r blocks (called the*replication number*); the points of V are partitioned into u subsets (called groups) of size g each; any two points within the same group (called first associates) appear together in λ_1 blocks; any two points not in the same group (called second associates) appear together in λ_2 blocks.

If the blocks in a design can be partitioned into *resolution (or parallel) classes* such that the blocks of each class partition the set V, then the design is called *resolvable*. A resolvable GDD$(g, u, k; 0, \lambda)$ is denoted by RGDD$(g, u, k; 0, \lambda)$. Note that the number of resolution classes is equal to the replication number $r = \frac{\lambda(v-g)}{k-1}$.

Theorem 1 (Theorem 19.33 in [5]) A 3-RGDD (i.e., RGDD$(g, u, 3; 0, 1)$) of type g^u exists if and only if $u \geq 3$ and (1) $g \equiv 1, 5 \pmod 6$ and $u \equiv 3 \pmod 6$; (2) $g \equiv 3 \pmod 6$ and $u \equiv 1 \pmod 2$; (3) $g \equiv 2, 4 \pmod 6$ and $u \equiv 0 \pmod 3$, except for $g^u \in \{2^3, 2^6\}$; (4) $g \equiv 0 \pmod 6$, except for $g^u = 6^3$.

1.1 H_2 Graphs

Definition 3 An H_2 graph is a multigraph on three points with a double edge between a pair of distinct points and single edges between the other two pairs of distinct points.

If the set of points of an H_2 is $V = \{a, b, c\}$ and the double edge is between a and b, then we denote the H_2 graph by $\langle a, b, c\rangle_{H_2}$ (see figure 1). An $H_2(v, \lambda)$ is a decomposition of λK_v into H_2 graphs. In particular, an $H_2(8t, 3)$ is a decomposition of a $3K_{8t}$ graph into $3t(8t-1)$ H_2 graphs.

Some examples of simple constructions of such decompositions are given below.

Theorem 2 *If there exists a BIBD$(v, 3, 1)$, then there exists an $H_2(v, 4)$.*

Proof: Replace each block $\{a, b, c\} \in B$ of a BIBD$(v, 3, 1)$ by three H_2 blocks, $\langle a, b, c\rangle_{H_2}$, $\langle c, a, b\rangle_{H_2}$ and $\langle b, c, a\rangle_{H_2}$. □

Corollary 1 *If $v \equiv 1, 3 (mod\ 6)$, then there exists an $H_2(v, 4)$.*

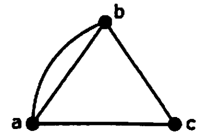

Figure 1: An H_2 Graph

Proof: If $v \equiv 1, 3 (\text{mod } 6)$, then a BIBD$(v, 3, 1)$ exists (see [4, 8, 10]). □

As a convention, we use the term *triangle* to denote a complete graph K_3. A minimum number of four triangles are needed to construct three H_2 graphs since three H_2 graphs have a total of 12 edges. Using triangles $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$ and $\{b, c, d\}$, we can construct three H_2 graphs $\langle a, b, d\rangle_{H_2}$, $\langle a, c, d\rangle_{H_2}$ and $\langle b, c, d\rangle_{H_2}$. In other words, one can say that $2K_4$ can be decomposed into three H_2 graphs. We use this idea to prove the following theorem (although a more general construction for an $H_2(v, 2)$ where $v \equiv 0, 1 (\text{mod } 4)$ has been given in [7], we use a different approach here).

Theorem 3 *If* $v \equiv 1, 4 (\textit{mod } 12)$, *then an* $H_2(v, 2)$ *exists.*

Proof: It is known that if $v \equiv 1, 4 (\text{mod } 12)$, then a BIBD$(v, 4, 1)$ exists. Each block $\{a, b, c, d\}$ in the BIBD$(v, 4, 1)$ can be used to construct three H_2 graphs (as above). Since each edge (pair) is contained in only one block in the BIBD$(v, 4, 1)$, an $H_2(v, 2)$ exists. □

1.2 Difference Sets for $H_2(8t, 3)$ Decompositions

One of the powerful techniques to construct combinatorial designs is based on *difference sets* and *difference families*; for example, see Stinson [10] for details and how to develop the difference sets to get the blocks of a design including the use of ("dummy") elements ∞'s (see Example 1 below).

Definition 4 Suppose $(G, +)$ is a finite group of order v in which the identity element is denoted "0". Let k and λ be positive integers such that $2 \leq k < v$. A (v, k, λ) difference set in $(G, +)$ is a subset $D \subseteq G$ that satisfies the following properties: 1. $|D| = k$, 2. the multiset $[x - y : x, y \in D, x \neq y]$ contains every element in $G \backslash \{0\}$ exactly λ times. A difference family is a collection $[D_1, \ldots, D_l]$ of k-subsets of G such that the multiset of the differences from all sets in the collection $[D_1, \ldots, D_l]$ together cover all nonzero differences λ times. .

In many cases G is taken as $(Z_v, +)$, the integers modulo v. For example, a $(7, 3, 1)$-difference set in $(Z_7, +)$ is $D = \{3, 0, 2\}$. Note $0 - 3 = 4, 2 - 3 = 6, 3 - 0 = 3, 2 - 0 = 2, 3 - 2 = 1$ and $0 - 2 = 5$, hence we get every element of $Z_7 \backslash \{0\}$ exactly once as a difference of two distinct elements in D.

To connect the difference set concept to an $H_2(8t, 3)$, we define the difference set $D = \langle a, b, c \rangle$ corresponding to the H_2 graph $\langle a, b, c \rangle_{H_2}$ as the difference set such that it gives $|a - b|$ twice (corresponding to a double edge between a and b), $|a - c|$ once (corresponding to a single edge between a and c) and $|b - c|$ once (corresponding to a single edge between b and c). For example, the difference set $\langle 3, 0, 2 \rangle$ gives the difference 3 twice, the difference 1 once and the difference 2 once. A graphical illustration is given in Figure 2.

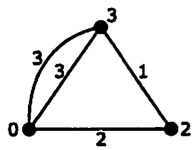

Figure 2: Difference set $\langle 3, 0, 2 \rangle$ corresponding to the H_2 graph $\langle 3, 0, 2 \rangle_{H_2}$

If we label the points of $3K_{8t}$ as the points in $V = \{\infty, 0, 1, 2, \ldots, 8t - 2\}$, then $V \backslash \{\infty\} = (Z_{8t-1}, +)$, the integers modulo $8t - 1$. The idea here is to construct a difference family $[D_1, \ldots, D_{3t}]$ where all differences in $\{1, 2, \ldots, \frac{8t-2}{2}\}$ appear exactly 3 times except one difference d which occurs twice in $3t - 1$ difference sets, and the missing difference d occurs in the difference set $\langle d, \infty, 0 \rangle$. Once we find such a difference family, we can expand the difference sets in the difference family modulo $8t - 1$ to obtain an $H_2(8t, 3)$ where each difference set or base block is expanded to obtain $8t - 2$ additional blocks. The total number of blocks after the expansion is $3t(8t - 1)$, each of which corresponds to an H_2 graph in the decomposition, and each edge between a pair of distinct points appears 3 times in these H_2 graphs as required.

Example 1 *For an $H_2(8, 3)$, we have $t = 1$, so we need 3 difference sets in a difference family where each difference in $\{1, 2, 3\}$ appears exactly 3 times. One such difference family is $[\langle 3, 0, 2 \rangle, \langle 2, 0, 3 \rangle, \langle 4, \infty, 5 \rangle]$. Next, we expand the difference sets modulo 7 to obtain an $H_2(8, 3)$.*

Hurd and Sarvate [7] proved that the necessary condition for existence of an $H_2(v, 3)$, $v(v - 1) \equiv 0 \pmod 8$, i.e., $v = 8t$ or $8t + 1$, for all $t \geq 1$ is sufficient for the existence of an $H_2(v, 3)$, except possibly for the cases $v = 8t$ and $24 \leq v \leq 1680$. In this paper, we resolve all these cases in the affirmative. In the next section, we obtain difference family solutions for $H_2(8t, 3)$, $t = 1, \ldots, 9$ and 19. The existence of these decompositions is needed for a recursive method developed in Section 4.

2 Initial Decompositions

According to [4], the cases for $t = 1, \ldots, 7$ are the *genuine exceptions*. We show the existence of an $H_2(8t, 3)$ for these exceptions as well as for $t = 8, 9$ and 19 as they are needed to complete the solution.

Example 2 $t = 1$: $[\langle 2,0,3\rangle, \langle 3,0,2\rangle, \langle 1,\infty,0\rangle]$.

Example 3 $t = 2$: $[\langle 3,0,4\rangle, \langle 4,0,6\rangle, \langle 5,0,3\rangle, \langle 6,0,7\rangle, \langle 7,0,5\rangle, \langle 1,\infty,0\rangle]$.

Example 4 $t = 3$: $[\langle 4,0,5\rangle, \langle 5,0,7\rangle, \langle 6,0,9\rangle, \langle 7,0,4\rangle, \langle 8,0,6\rangle, \langle 9,0,10\rangle, \langle 10,0,11\rangle, \langle 11,0,8\rangle, \langle 2,\infty,0\rangle]$.

Example 5 $t = 4$: $[\langle 5,0,6\rangle, \langle 6,0,8\rangle, \langle 7,0,10\rangle, \langle 8,0,12\rangle, \langle 9,0,5\rangle, \langle 10,0,7\rangle, \langle 11,0,9\rangle, \langle 12,0,15\rangle, \langle 13,0,14\rangle, \langle 14,0,13\rangle, \langle 15,0,11\rangle, \langle 2,\infty,0\rangle]$.

Example 6 $t = 5$: $[\langle 6,0,7\rangle, \langle 7,0,9\rangle, \langle 8,0,11\rangle, \langle 9,0,13\rangle, \langle 10,0,15\rangle, \langle 11,0,6\rangle, \langle 12,0,8\rangle, \langle 13,0,10\rangle, \langle 14,0,12\rangle, \langle 15,0,19\rangle, \langle 16,0,18\rangle, \langle 17,0,16\rangle, \langle 18,0,17\rangle, \langle 19,0,14\rangle, \langle 3,\infty,0\rangle]$.

Example 7 $t = 6$: $[\langle 7,0,8\rangle, \langle 8,0,10\rangle, \langle 9,0,12\rangle, \langle 10,0,14\rangle, \langle 11,0,16\rangle, \langle 12,0,18\rangle, \langle 13,0,7\rangle, \langle 14,0,9\rangle, \langle 15,0,11\rangle, \langle 16,0,13\rangle, \langle 17,0,15\rangle, \langle 18,0,22\rangle, \langle 19,0,20\rangle, \langle 20,0,23\rangle, \langle 21,0,19\rangle, \langle 22,0,21\rangle, \langle 23,0,17\rangle, \langle 5,\infty,0\rangle]$.

Example 8 $t = 7$: $[\langle 8,0,9\rangle, \langle 9,0,11\rangle, \langle 10,0,13\rangle, \langle 11,0,15\rangle, \langle 12,0,17\rangle, \langle 13,0,19\rangle, \langle 14,0,21\rangle, \langle 15,0,8\rangle, \langle 16,0,10\rangle, \langle 17,0,12\rangle, \langle 18,0,14\rangle, \langle 19,0,16\rangle, \langle 20,0,18\rangle, \langle 21,0,26\rangle, \langle 22,0,25\rangle, \langle 23,0,24\rangle, \langle 24,0,23\rangle, \langle 25,0,27\rangle, \langle 26,0,22\rangle, \langle 27,0,20\rangle, \langle 6,\infty,0\rangle]$.

Example 9 $t = 8$: $[\langle 9,0,10\rangle, \langle 10,0,12\rangle, \langle 11,0,14\rangle, \langle 12,0,16\rangle, \langle 13,0,18\rangle, \langle 14,0,20\rangle, \langle 15,0,22\rangle, \langle 16,0,24\rangle, \langle 17,0,9\rangle, \langle 18,0,11\rangle, \langle 19,0,13\rangle, \langle 20,0,15\rangle, \langle 21,0,17\rangle, \langle 22,0,19\rangle, \langle 23,0,21\rangle, \langle 24,0,31\rangle, \langle 25,0,30\rangle, \langle 26,0,29\rangle, \langle 27,0,25\rangle, \langle 28,0,27\rangle, \langle 29,0,28\rangle, \langle 30,0,26\rangle, \langle 31,0,23\rangle, \langle 6,\infty,0\rangle]$.

Example 10 $t = 9$: $[\langle 10,0,11\rangle, \langle 11,0,13\rangle, \langle 12,0,15\rangle, \langle 13,0,17\rangle, \langle 14,0,19\rangle, \langle 15,0,21\rangle, \langle 16,0,23\rangle, \langle 17,0,25\rangle, \langle 18,0,27\rangle, \langle 19,0,10\rangle, \langle 20,0,12\rangle, \langle 21,0,14\rangle, \langle 22,0,16\rangle, \langle 23,0,18\rangle, \langle 24,0,20\rangle, \langle 25,0,22\rangle, \langle 26,0,24\rangle, \langle 27,0,35\rangle, \langle 28,0,34\rangle, \langle 29,0,33\rangle, \langle 30,0,28\rangle, \langle 31,0,32\rangle, \langle 32,0,31\rangle, \langle 33,0,30\rangle, \langle 34,0,29\rangle, \langle 35,0,26\rangle, \langle 7,\infty,0\rangle]$.

Example 11 $t = 19$: $[\langle 20,0,21\rangle, \langle 21,0,23\rangle, \langle 22,0,25\rangle, \langle 23,0,27\rangle, \langle 24,0,29\rangle, \langle 25,0,31\rangle, \langle 26,0,33\rangle, \langle 27,0,35\rangle, \langle 28,0,37\rangle, \langle 29,0,39\rangle, \langle 30,0,41\rangle, \langle 31,0,43\rangle, \langle 32,0,45\rangle, \langle 33,0,47\rangle, \langle 34,0,49\rangle, \langle 35,0,51\rangle, \langle 36,0,53\rangle, \langle 37,0,55\rangle, \langle 38,0,57\rangle, \langle 39,0,20\rangle, \langle 40,0,22\rangle, \langle 41,0,24\rangle, \langle 42,0,26\rangle, \langle 43,0,28\rangle, \langle 44,0,30\rangle, \langle 45,0,32\rangle, \langle 46,0,34\rangle, \langle 47,0,36\rangle, \langle 48,0,38\rangle, \langle 49,0,40\rangle, \langle 50,0,42\rangle, \langle 51,0,44\rangle, \langle 52,0,46\rangle, \langle 53,0,48\rangle, \langle 54,0,50\rangle, \langle 55,0,52\rangle, \langle 56,0,54\rangle, \langle 57,0,74\rangle, \langle 58,0,73\rangle, \langle 59,0,72\rangle, \langle 60,0,71\rangle, \langle 61,0,70\rangle, \langle 62,0,69\rangle, \langle 63,0,68\rangle, \langle 64,0,67\rangle, \langle 65,0,64\rangle, \langle 66,0,65\rangle, \langle 67,0,63\rangle, \langle 68,0,66\rangle, \langle 69,0,75\rangle, \langle 70,0,62\rangle, \langle 71,0,61\rangle, \langle 72,0,60\rangle, \langle 73,0,59\rangle, \langle 74,0,58\rangle, \langle 75,0,56\rangle, \langle 18,\infty,0\rangle]$.

3 New Tools

In this section, we develop new procedures for constructing $H_2(8t, \lambda)$s.

Procedure SPLIT($\{b_1, b_2, b_3\}, a$): Take any triangle (or block) $\{b_1, b_2, b_3\}$ and a new point a, we construct three H_2 graphs $\langle a, b_1, b_2\rangle_{H_2}$, $\langle a, b_2, b_3\rangle_{H_2}$ and $\langle a, b_3, b_1\rangle_{H_2}$. This implies that SPLIT($\{b_1, b_2, b_3\}, a$) results in three H_2 graphs such that each of the three pairs ($\{a, b_1\}$, $\{a, b_2\}$, $\{a, b_3\}$) involving the new point a appears three times and the three pairs ($\{b_1, b_2\}$, $\{b_2, b_3\}$, $\{b_1, b_3\}$) of the original triangle appear once. A graphical illustration of SPLIT($\{b_1, b_2, b_3\}, a$) is shown in figure 3.

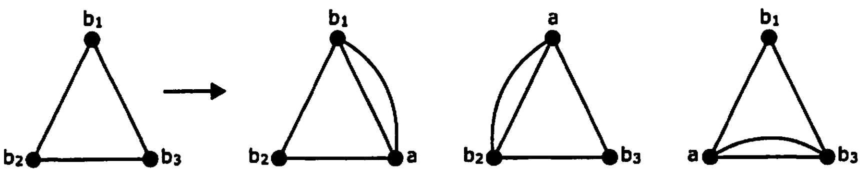

Figure 3: SPLIT($\{b_1, b_2, b_3\}, a$) results in three H_2 graphs $\langle a, b_1, b_2\rangle_{H_2}$, $\langle a, b_2, b_3\rangle_{H_2}$ and $\langle a, b_3, b_1\rangle_{H_2}$.

Procedure THREE-FOR-CLASS($P, A, n = 3r$): Given a resolvable design with block size $k = 3$, $\lambda = 1$ and replication number r, let $P = \{P_1, \ldots, P_r\}$ be the collection of the r resolution classes, $A = \{a_1, \ldots, a_{3r}\}$ be a set of new points (so $A \cap V = \emptyset$). Let $V = \{b_1, b_2, \ldots, b_v\}$ and w be the number of blocks in each class. For each $i = 1, 2, \ldots, r$, let $P_i = \{B_{i1}, B_{i2}, \ldots, B_{iw}\}$ where each B_{ij} ($j = 1, \ldots, w$) represents a block of three points from V in P_i.

For $i = 1, \ldots, r$ and $j = 1, \ldots, w$, we perform procedures SPLIT(B_{ij}, a_i), SPLIT(B_{ij}, a_{r+i}) and SPLIT(B_{ij}, a_{2r+i}), respectively. Recall that SPLIT(B_{ij}, a_i) produces three H_2 graphs where each of the three pairs involving a_i and a point in B_{ij} appears three times and each of the three pairs involving two points in B_{ij} appear once. After performing these $3wr$ procedures, we get a family of $9wr$ H_2 graphs where each pair of distinct points in V appears three times since each pair is contained in exactly one block in some resolution class (note we assumed that $\lambda = 1$) and it is produced three times from the three SPLIT procedures (where each procedure produces the pair once). Furthermore, every pair $\{a_i, b_j\}$ ($i = 1, \ldots, 3r$ and $j = 1, \ldots, v$) occurs three times since each a_i is used in one resolution class P_i that contains every point in V exactly once.

We can see that in the procedure THREE-FOR-CLASS(P, A, n), we introduce three new points in A to the blocks in each resolution class to create H_2 graphs, thus $n = 3r$ if there are r resolution classes in P. The resulting H_2 graphs contain three edges between each pair of distinct points in V and three edges between every pair of points where one point is from V and the other point is from A.

Procedure COPY($\{b_1, b_2, b_3\}, a$): Given a triangle (or block) $\{b_1, b_2, b_3\}$ and a new point a. A graphical illustration of COPY($\{b_1, b_2, b_3\}, a$) is shown in figure 4. This procedure constructs three H_2 graphs $\langle b_1, b_2, b_3\rangle_{H_2}$, $\langle b_1, b_3, b_2\rangle_{H_2}$ and

$\langle a, b_2, b_3\rangle_{H_2}$. As a result each edge (or pair) of the original triangle appears three times, and the edge between a and b_2 appears twice, and the edge between a and b_3 appears once and no edge is created between a and b_1.

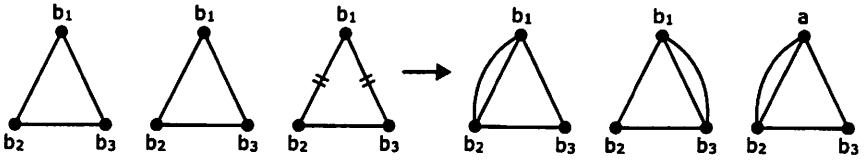

Figure 4: COPY($\{b_1, b_2, b_3\}, a$) results in three H_2 graphs $\langle b_1, b_2, b_3\rangle_{H_2}$, $\langle b_1, b_3, b_2\rangle_{H_2}$ and $\langle a, b_2, b_3\rangle_{H_2}$.

Lemma 1 [2] (Agrawal's Lemma) In every binary equi-replicate design of constant block size k (hence $bk = vr$ and $b = mv$), the treatments in each block can be rearranged such that in the k by b array, formed with blocks as columns, every treatment occurs in each row exactly m times.

Procedure GROUP-TRIPLE($P, A, s = \frac{r}{3}$): Let $V = \{b_1, \ldots, b_v\}$ and $A = \{a_1, \ldots, a_s\}$ where $A \cap V = \emptyset$. Let $P = \{P_1, \ldots, P_r\}$ be the collection of $r = 3s$ resolution classes with blocks of size $k = 3$, and w be the number of blocks in each class. For each $i = 1, 2, \ldots, r$, we let $P_i = \{B_{i1}, B_{i2}, \ldots, B_{iw}\}$ where each B_{ij} ($j = 1, \ldots, w$) represents a block of three points from V in P_i. We apply Agrawal's lemma to define the fourth procedure called GROUP-TRIPLE($P, A, s = \frac{r}{3}$) as follows:

As $r = 3s$, we can group three resolution classes together and get s groups : $\{P_1, P_2, P_3\}, \{P_4, P_5, P_6\}, \ldots, \{P_{3s-2}, P_{3s-1}, P_{3s}\}$. Let $G_i = \{P_{3i-2}, P_{3i-1}, P_{3i}\}$ for $1 \leq i \leq s$.

We apply Agrawal's lemma to rearrange the points in the blocks of each group so that every point in V comes exactly once in each of the three rows ($k = 3$ and $m = 1$). For $i = 1, \ldots, s$ and $j = 1, \ldots, w$, we perform procedures COPY($B_{(3i-2)j}, a_i$), COPY($B_{(3i-1)j}, a_i$) and COPY($B_{(3i)j}, a_i$), to obtain H_2 graphs. In these resulting H_2 graphs, it is clear that every pair between two distinct points in V occurs three times. Also, for $i = 1, \ldots, s$ and $j = 1, \ldots, v$, the edge between a_i and b_j appears three times since each point b_j appears exactly once in each of the three rows in group i (from Agrawal's lemma), and no edge between a_i and b_j was created when b_j appears in the first row, two edges between a_i and b_j were created when b_j appears in the 2nd row and one edge between a_i and b_j was created when b_j appears in the third row.

The resulting H_2 graphs contain three edges between each pair of distinct points from V and three edges between every pair of points where one point is from V and the other point is from A.

Note that since each procedure of THREE-FOR-CLASS(P, A, n) and GROUP-TRIPLE(P, A, n) produces H_2 graphs that contain three edges between every pair of distinct points from V and three edges between every pair of points where one point is from V and the other point is from A, we only need to focus on the number n of new points in A in these two procedures. In THREE-FOR-CLASS(P, A, n), $n = 3r$ if there are r resolution classes in P. In GROUP-TRIPLE(P, A, n), $n = s$ if $r = 3s$.

4 A Recursive Construction

In this section we prove that an $H_2(8t, 3)$ exists for all $t \geq 1$.

Theorem 4 *For $s > 0$ and $0 \leq i \leq 8s$, if an $H_2(8s + 8i, 3)$ and an $H_2(24s, 3)$ exist, then an $H_2(80s + 8i, 3)$ also exists.*

Proof: Let $v = 72s$ points in V for some $s > 0$. For each $i \in \{0, \ldots, 8s\}$, suppose an $H_2(8s + 8i, 3)$ and an $H_2(24s, 3)$ exist. Since $24s \equiv 0 (\text{mod } 6)$, by case (4) in Theorem 1, GDD$(24s, 3, 3; 0, 1)$ is resolvable, and there are $\frac{72s-24s}{3-1} = 24s$ resolution classes on V. Obtain an $H_2(24s, 3)$ for the $24s$ points in each of the three groups. This takes care of three edges between any two distinct points from V having first associates from the resulting H_2 graphs.

Next we divide the $24s$ resolution classes into two parts P^1 and P^2 where P^1 consists of $24s - 3i$ resolution classes and P^2 consists of $3i$ resolution classes. Perform the procedure GROUP-TRIPLE(P^1, A^1, n^1) on the $24s - 3i$ resolution classes in P^1, then $n^1 = \frac{24s-3i}{3} = 8s - i$. Next we perform THREE-FOR-CLASS(P^2, A^2, n^2) on the $3i$ resolution classes in P^2, then $n^2 = 3 \times 3i = 9i$. The total number of new points is $n' = n^1 + n^2 = 8s - i + 9i = 8s + 8i$. This takes care of three edges between any two points from V having second associates and three edges between any two points where one point is in V and the other point is a new point in $A^1 \cup A^2$ from the resulting H_2 graphs.

Finally we obtain an $H_2(8s + 8i, 3)$ on the $n' = 8s + 8i$ new points in $A^1 \cup A^2$. This takes care of three edges between any two distinct new points from the resulting H_2 graphs. Combine all the H_2 graphs obtained, we have an $H_2(v + n', 3) = H_2(72s + 8s + 8i, 3) = H_2(80s + 8i, 3)$ since these H_2 graphs contain three edges between any pair of distinct points in $V \cup A^1 \cup A^2$. □

Corollary 2 *An $H_2(8t, 3)$ exists for all $t \geq 1$.*

Proof: We know that an $H_2(8t, 3)$ exists for $1 \leq t \leq 9$ and $t = 19$ since a difference family solution was provided for each case at the beginning of this section. Let $s = 1$, an $H_2(24s, 3) = H_2(24, 3)$ exists. For $0 \leq i \leq 8$, $8 \leq 8s + 8i \leq 72$, an $H_2(8s + 8i, 3)$ exists. By Theorem 4, an $H_2(80s + 8i, 3)$ exists where $80 \leq 80s + 8i \leq 144$, i.e. an $H_2(8t, 3)$ exists for $10 \leq t \leq 18$.

Next we let $s = 2$, an $H_2(24s, 3) = H_2(48, 3)$ exists. For $0 \leq i \leq 16$, $16 \leq 8s + 8i \leq 144$, thus an $H_2(8s + 8i, 3)$ exists. By Theorem 4, an $H_2(80s + 8i, 3)$ exists where $160 \leq 80s + 8i \leq 288$, i.e. an $H_2(8t, 3)$ exists for $20 \leq t \leq 36$ (although $t = 19$ is not covered here, we know that an $H_2(152, 3)$ exists from the difference family solution provided earlier). Now we have an $H_2(8t, 3)$ exists for all $1 \leq t \leq 36$.

Similarly, let $s = 3$, an $H_2(24s, 3) = H_2(72, 3)$ exists. For $0 \leq i \leq 16$, $24 \leq 8s + 8i \leq 216$, thus an $H_2(8s + 8i, 3)$ exists. By Theorem 4, an $H_2(80s + 8i, 3)$ exists where $240 \leq 80s + 8i \leq 432$, i.e. an $H_2(8t, 3)$ exists for $30 \leq t \leq 54$.

We can see that in step w where $w \geq 2$, $s = w$ and $0 \leq i \leq 8w$. Since $24s = 24w < 72w$ and $8s + 8i = 8w + 8i \leq 8w + 8 \times 8w = 72w$ and $72w \leq 80(w - 1) + 8 \times 8(w - 1) = 144w - 144$ (note $w \geq 2$), this implies that the assumptions on the existence of an $H_2(8s + 8i, 3)$ and an $H_2(24s, 3)$ in step w are already shown by the results from step $w - 1$ and earlier steps (notice that $80(w - 1) + 8 \times 8(w - 1)$ is the maximum value for which the existence of decomposition into H_2 graphs can be shown in step $w - 1$). That is, by Theorem 4 and the difference family solutions for $1 \leq t \leq 9$ and $t = 19$, we can obtain any $H_2(8t, 3)$ from the results obtained in the previous steps. In other words, an $H_2(8t, 3)$ exists for all $t \geq 1$. □

Theorem 5 *The necessary condition for existence of an $H_2(v, 3)$ ($v(v - 1) \equiv 0$ (mod 8)) is sufficient for the existence of an $H_2(v, 3)$.*

Proof: The necessary condition implies that $v = 8t$ or $8t + 1$ for all $t \geq 1$. Hurd and Sarvate [7] have shown that the necessary condition is sufficient for the existence of an $H_2(8t + 1, 3)$ (also for an $H_2(8t, 3)$ except possibly for the cases $24 \leq 8t \leq 1680$). By Corollary 2, we conclude that the necessary condition is also sufficient for the existence of an $H_2(8t, 3)$. □

5 Summary

We provided difference family solutions to an $H_2(8t, 3)$ for $1 \leq t \leq 9$ and $t = 19$ where $1 \leq t \leq 7$ cases are the *genuine exceptions* in [4]. Next we developed two procedures THREE-FOR-CLASS(P, A, n) and GROUP-TRIPLE(P, A, n) for finding a construction of a $H_2(8t, 3)$. We concluded that an $H_2(8t, 3)$ exists for all $t \geq 1$ by a recursive construction, which implies that the necessary condition $v \equiv 0$ (mod 8) is sufficient for the existence of an $H_2(v, 3)$.

References

[1] P. Adams, D. Bryant, and M. Buchanan. A survey on the existence of G-Designs. *Journal of Combinatorial Designs*, 5:373–410, 2008.

[2] H. Agrawal. Some generalizations of distinct representives with applications to statistical designs. *Ann. Math. Statis.*, 37:25–528, 1966.

[3] H. Chan. Stanton graph decompositions. *Bulletin of the ICA*, 64:21–29, 2012.

[4] C. J. Colbourn and D. H. Dinitz (eds). *Handbook of Combinatorial Designs (2nd Edition)*. Chapman and Hall, CRC Press, Boca Raton, 2007.

[5] C. J. Colbourn and A. Rosa. *Triple System*. Oxford Science Publications, Clarendon Press, Oxford, 1999.

[6] H. Fu. Graph decomposition and its applications. National Chiao Tung University. URL:www.math.nsysu.edu.tw/conference/20050805/940805-1.ppt, 2005.

[7] S. P. Hurd and D. G. Sarvate. Graph decompositions of $K(v, \lambda)$ into modified triangles using Langford and Skolem sequences. Submitted and Under Review, 2011.

[8] C. C. Lindner and C. A. Rodger. *Design Theory (2nd Edition)*. CRC Press, Boca Raton, 2009.

[9] N. Punim. Loop designs. *J. Combin. Math. Combin. Comput.*, 78:261–272, 2011.

[10] D. R. Stinson. *Combinatorial Designs: Constructions and Analysis*. Springer, New York, 2004.