The diameter and the chromatic number of almost self-complementary graphs

Ping An[a] Zhantao Huang[b] and Yinglie Jin[b*]
[a] State Key Laboratory of Reactor System Design Technology, Chengdu, P. R. China
[b] School of Mathematical Sciences and LPMC, Nankai University, Tianjin, P. R. China

Abstract

A graph G with an even number of vertices is an almost self-complementary, if it is isomorphic to one of its almost complements $G^c - M$, where M denotes a perfect matching in its complement G^c. In this paper, we show that the diameter of connected almost self-complementary graphs must be 2, 3 or 4. And we construct connected almost self-complementary graphs with $2n$ vertices having diameter 3 and 4 for each $n \geq 3$, and diameter 2 for each $n \geq 4$ respectively. In addition, we also obtain that for any almost self-complementary graph G_n with $2n$ vertices, $\lceil \sqrt{n} \rceil \leq \chi(G_n) \leq n$. By construction, we verify that the upper bound is attainable for each positive integer n as well as the lower bound when $\sqrt{n}$ is an integer.

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1 Introduction

A graph G is *self-complementary*, if it is isomorphic to its complement G^c. Similarly, a graph G with an even number of vertices is *almost self-complementary*, if it is isomorphic to a graph (called an *almost complement* of G) obtained from G^c by minus a perfect matching of G^c. Let M denote the perfect matching, then the almost complement of G depending on M can be written as $AC_M(G)$. Obviously, G is an almost complement of $AC_M(G)$ with the same matching M.

*Corresponding Author. E-mail: yljin@nankai.edu.cn.

Alspach [1], who first introduced almost self-complementary graphs, proposed the determination of all possible orders of almost self- complementary circulant graphs. Dobson and Šajna [3] solved this problem for a particularly "nice" subclass of almost self-complementary circulants. Later on, Potočnik and Šajna [5] investigated some properties of almost self-complementary graphs and gave necessary and sufficient conditions on the order of a regular graph, and constructed several types with special meanings.

In this paper, we mainly investigate the diameter and chromatic number of almost self-complementary graphs.

In section 2, we show that if G is a connected almost self-complementary graph with $2n$ vertices, the diameter of G must satisfy $2 \leq diam(G) \leq 4$. We also construct three types of connected almost self-complementary graphs with $2n$ vertices having diameter 2 for each integer $n \geq 4$, as well as diameter 3 and 4 for each integer $n \geq 3$.

In section 3, inspired by the idea in [2], we show that for an almost self-complementary graph G with $\chi(G) = k$, the number of vertices $|V(G)|$ must be less than or equal to $2k^2$. Furthermore, we obtain that for any almost self-complementary graph G with $2n$ vertices, the chromatic number of G must satisfy $\lceil \sqrt{n}\, \rceil \leq \chi(G) \leq n$. In addition, we verify that the upper bound is attainable for each positive integer n and the lower bound is attainable when $\sqrt{n}$ is an integer, by constructing almost self-complementary graphs with $|V(G)| = 2n$, $\chi(G) = n$ and $|V(G)| = 2n^2$, $\chi(G) = n$ respectively.

For any two vertices x, y of a graph G, $(x, y) \in E(G)$ denotes that x is adjacent to y in G.

2 The diameter of almost self-complementary graphs

Ringel [6] showed that the diameter of any self-complementary graph G must satisfy $2 \leq diam(G) \leq 3$. In this section, we investigate the diameter of almost self-complementary graphs. Let G be an almost self-complementary graph. If G is disconnected, the diameter of G is infinity. So we suppose G is connected in this section. Let $AC_M(G)$ be its almost complement depending on the perfect matching M in G^c such that $G \cong AC_M(G)$.

Lemma 1. *For any two vertices $x, y \in V(G)$, $d_{AC_M(G)}(x, y) \leq 2$ if $d_G(x, y) \geq 4$.*

Proof. Since G is connected, let $xv_1v_2 \cdots v_my$ $(m \geq 3)$ denote the shortest path between x and y in G. If $(x, y) \in E(AC_M(G))$, then $d_{AC_M(G)}(x, y) =$

$1 \leq 2$. Otherwise, we have $(x, y) \in M$, $(x, v_2) \in E(AC_M(G))$ and $(y, v_2) \in E(AC_M(G))$, then $d_{AC_M(G)}(x, y) = 2$. □

Theorem 1. *G is a connected almost self-complementary graph, then $2 \leq diam(G) \leq 4$.*

Proof. Because G can't be complete graph, $diam(G) \geq 2$. On the other hand, suppose there exists a connected almost self-complementary graph with $diam(G) \geq 5$. For any two vertices $x, y \in V(G)$:
(1) If $d_G(x, y) \geq 3$, we claim $d_{AC_M(G)}(x, y) \leq 3$. Because, if $d_{AC_M(G)}(x, y) \geq 4$, we have $d_G(x, y) \leq 2$ according to the Lemma 1.
(2) If $d_G(x, y) = 2$, there exists a vertex z such that $(x, z) \notin E(G)$ and $(y, z) \notin E(G)$ since $diam(G) \geq 5$. Then $(x, y) \in E(AC_M(G))$, or $(x, y) \in M$, $(x, z) \in E(AC_M(G))$, $(y, z) \in E(AC_M(G))$. Hence $d_{AC_M(G)}(x, y) \leq 2$.
(3) If $d_G(x, y) = 1$. Let $I_1 = \{v : (x, v) \notin E(G)\}$, $I_2 = \{v : (y, v) \notin E(G)\}$, then we can easily obtain $|I_i| \geq 3$ $(i = 1, 2)$ and $|I_1 \cap I_2| \geq 2$ since $diam(G) \geq 5$.

(a) $|I_1 \cap I_2| \geq 3$, suppose $v_1, v_2, v_3 \in I_1 \cap I_2$. In $AC_M(G)$, both x and y are adjacent to at least two vertices in $\{v_1, v_2, v_3\}$ and $d_{AC_M(G)}(x, y) = 2$.

(b) $|I_1 \cap I_2| = 2$. In this situation, x, y, u_1, u_2 must be the vertices of a path P_5 : $z_0 z_1 z_2 z_3 z_4 z_5$, where $d_G(z_0, z_5) = 5$ and $\{x, y\} = \{z_1, z_2\}$ or $\{x, y\} = \{z_2, z_3\}$. In $AC_M(G)$, z_1 is adjacent to at least two vertices in $\{z_3, z_4, z_5\}$, z_2 is adjacent to at least two vertices in $\{z_0, z_4, z_5\}$, and z_3 is adjacent to at least two vertices in $\{z_0, z_1, z_5\}$. Hence $d_{AC_M(G)}(z_1, z_2) = 2$, $d_{AC_M(G)}(z_2, z_3) = 2$, namely, $d_{AC_M(G)}(x, y) = 2$.

Above all, we have $d_{AC_M(G)}(x, y) \leq 3$ for any two vertices x, y of G. So $diam(AC_M(G)) \leq 3$, contrary to $diam(G) \geq 5$ since $G \cong AC_M(G)$. Therefore, $2 \leq diam(G) \leq 4$. □

Potočnik and Šajna [5] presented all almost self-complementary graphs of order at most 6. The connected almost self-complementary graphs among them are listed in Figure 1 (dotted lines are the perfect matching M, and σ is an isomorphism such that $\sigma(G) = AC_M(G)$).

From Figure 1, we obtain that the diameter is 3 or 4 for the connected almost self-complementary graphs of order at most 6. In the following, we shall construct three types of connected almost self-complementary graphs G on $2n$ vertices such that $diam(G) = 3, 4$ for each integer $n \geq 3$ and $diam(G) = 2$ for $n \geq 4$ respectively. Let $V(G) = \{x_i, y_i : i = 1, 2, \cdots, n\}$ and the perfect matching in G^c be $M = \{(x_i, y_i) : i = 1, 2, \cdots, n\}$. We construct the graphs by defining the edge set $E(G)$ and the almost self-complementary isomorphism $\sigma : G \to AC_M(G)$ respectively.
Construction 1. For any positive integer $n \geq 3$, we construct a connected almost self-complementary graph G on $2n$ vertices with $diam(G) = 4$ as following.

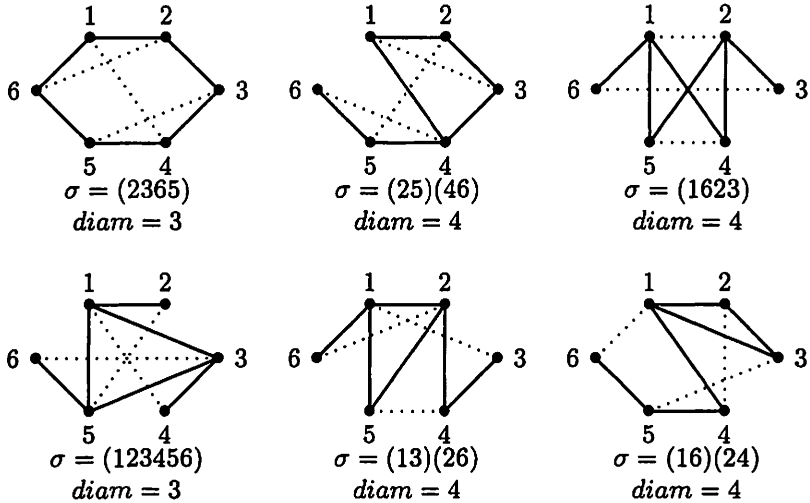

Figure 1: Connected almost self-complementary graphs with order at most 6

$$\begin{aligned} E(G) &= \{(x_1,x_2),(y_1,y_2),(y_2,x_i),(y_2,y_i) : 3 \le i \le n\} \\ &\cup \{(x_i,x_j),(x_i,y_j) : 2 \le i < j \le n\} \\ \sigma &= (x_1x_2y_1y_2)\prod_{i=3}^{n}(x_iy_i) \end{aligned}$$

We can verify G is connected. And for any two vertices $x, y \in V(G)$: $(x,y) \in E(G)$ if and only if $(\sigma(x),\sigma(y)) \in E(AC_M(G))$, namely $G \cong AC_M(G)$. Furthermore $d(x_1,y_1) = 4$, so $diam(G) = 4$.

Construction 2. For any positive integer $n \ge 3$, we construct a connected almost self-complementary graph G on $2n$ vertices with $diam(G) = 3$ as following.

$$\begin{aligned} E(G) &= \{(x_i,x_j),(x_i,y_j) : 1 \le i < j \le n\} \cup \{(y_1,x_n)\} \backslash \{x_1,y_n\} \\ \sigma &= \prod_{i=1}^{n}(x_iy_i) \end{aligned}$$

We can easily verify σ is an isomorphism: $G \to AC_M(G)$. The vertex x_1 is adjacent to all vertices in $V(G)$ except y_1, y_n and $d(x_1,y_n) = d(x_1,y_1) = 2, d(y_1,y_i) = 3$ for $i = 1, \cdots, n$. Hence $diam(G) = 3$.

Construction 3. For any positive integer $n \ge 4$, we construct a connected almost self-complementary graph G on $2n$ vertices with $diam(G) = 2$ as following.

$$
\begin{aligned}
E(G) &= \{(x_1, y_i), (y_1, y_i) : 2 \leq i \leq n\} \\
&\cup \{(x_i, x_j), (x_i, y_j) : 2 \leq i < j \leq n\} \cup \{(y_2, x_n)\} \backslash \{(x_2, y_n)\} \\
\sigma &= (x_1)(y_1) \prod_{i=2}^{n} (x_i y_i)
\end{aligned}
$$

We have $G \cong AC_M(G)$ and σ is the isomorphism. The subgraph on vertex set $V(G) \backslash \{x_1, y_1\}$ is a graph in Construction 2. Now $d(x_1, y_1) = d(y_i, y_j) = 2$ for $1 \leq i < j \leq n$. Hence $diam(G) = 2$.

3 The chromatic number of the almost self-complementary graphs

According to the conclusion in [4], we have $\sqrt{n} \leq \chi(G) \leq \frac{n+1}{2}$ for any self-complementary graph G with n vertices. In this section, we investigate the chromatic number of almost self-complementary graphs.

Theorem 2. *Let G be an almost self-complementary graph with chromatic number $\chi(G) = k$, then $|V(G)| \leq 2k^2$.*

Proof. Let M be the perfect matching in G^c such that $G \cong AC_M(G)$. Suppose $|V(G)| \geq 2k^2 + 1$. Since there are k colors, there are at least $2k+1$ vertices, which are independent in G, equipped with the same color. The subgraph in $AC_M(G)$ on those $2k+1$ vertices contains a complete graph of order $2k+1$ minus a maximal matching, and we have $\chi(AC_M(G)) \geq k+1$. Hence $\chi(G) \geq k+1$ since $G \cong AC_M(G)$, a contradiction. Therefore $|V(G)| \leq 2k^2$. □

Theorem 3. *For any almost self-complementary graph G with $2n$ vertices, we have $\lceil \sqrt{n} \rceil \leq \chi(G) \leq n$.*

Proof. Let $\chi(G) = k$. Suppose $k \leq \lceil \sqrt{n} \rceil - 1 < \sqrt{n}$. According to Theorem 2, $|V(G)| \leq 2k^2 < 2n$ contrary to $|V(G)| = 2n$.

On the other hand, let M be the perfect matching in G^c, such that $G \cong AC_M(G)$. We equip the two adjacent vertices in M with the same color and obtain a proper coloring of G. Hence $\chi(G) \leq n$. □

Next, by construction, we show that the upper bound is attainable for any integer $n \geq 2$ as well as the lower bound when $\sqrt{n}$ is an integer.

Construction 4. For any integer $n \geq 2$, we construct an almost self-complementary graph G of order $2n$ such that $\chi(G) = n$.

$$V(G) = \{x_i, y_i : i = 1, \cdots, n\} \qquad M = \{(x_i, y_i) : i = 1, \cdots, n\}$$
$$E(G) = \{(x_i, x_j), (x_i, y_j) : 1 \leq i < j \leq n\} \quad \sigma = \prod_{i=1}^{n} (x_i y_i)$$

We can easily verify σ is an isomorphism: $G \to AC_M(G)$ and G is an almost self-complementary graph. G contains a complete subgraph K_n, so $\chi(G) \geq n$. On the other hand, we can set x_i and y_i with the same color i $(1 \leq i \leq n)$, then we obtain $\chi(G) \leq n$. Hence $\chi(G) = n$.

Construction 5. For any integer $n \geq 1$, we construct a graph G such that $\chi(G) = n$ and $|V(G)| = 2n^2$.

$$V(G) = \{x_{i,j}, y_{i,j} : i, j = 1, \cdots, n\} \quad M = \{(x_{i,j}, y_{i,j}) : i, j = 1, \cdots, n\}$$

1. If n is even, we define a mapping $\sigma : V(G) \to V(G)$ as follows:

$$\begin{aligned} \sigma \;=\; & \prod_{\substack{t \text{ odd} \\ 1 \leq t \leq n-1}} (x_{t,t}\; x_{t+1,t}\; x_{t+1,t+1}\; x_{t,t+1})(y_{t,t}\; y_{t+1,t}\; y_{t+1,t+1}\; y_{t,t+1}) \\ \times & \prod_{\substack{i \text{ odd} \\ 1 \leq i \leq n-3 \\ i+2 \leq j \leq n}} (x_{i,j}\; x_{j,i}\; x_{i+1,j}\; x_{j,i+1})(y_{i,j}\; y_{j,i}\; y_{i+1,j}\; y_{j,i+1}) \end{aligned}$$

We can easily verify that σ contains $2 \times \left(\frac{n}{2} + \sum_{i=1}^{n-3} (n - i - 1) \right) = \frac{1}{2} n^2$ cycles, namely $4 \times \frac{1}{2} n^2 = 2n^2$ elements of $V(G)$. On the other hand, for any vertex a_{ij} in $V(G)$, a_{ij} belongs to a unique cycle of σ and a_{ij} has a unique image and preimage under the action of σ. Hence σ is a permutation.

Next we define the edge set $E(G)$ based on each cycle in σ. Denote $V(X_i) = \{x_{i,j}, y_{i,j} : j = 1, \cdots, n\}$ for $i = 1, \cdots, n$; $V(Y_j) = \{x_{i,j}, y_{i,j} : i = 1, \cdots, n\}$ for $j = 1, \cdots, n$.

$$V(G) = V(X_1) \cup V(X_2) \cup \cdots \cup V(X_n) = V(Y_1) \cup V(Y_2) \cup \cdots \cup V(Y_n)$$

be two partitions of the vertex set of G into disjoint subsets.

For any pair of vertices $a_{i,j}$, $b_{s,t}$ in $V(G)$ such that $(a_{i,j}, b_{s,t}) \notin M$, let $(\sigma^0(a_{i,j})\; \sigma(a_{i,j})\; \sigma^2(a_{i,j})\; \sigma^3(a_{i,j}))$, $(\sigma^0(b_{s,t})\; \sigma(b_{s,t})\; \sigma^2(b_{s,t})\; \sigma^3(b_{s,t}))$ for the two cycles containing $a_{i,j}, b_{s,t}$ respectively, where $\sigma^0(a_{i,j}) = a_{i,j}$, $\sigma^0(b_{s,t}) = b_{s,t}$.

If there exist $p(0 \le p \le 3)$ and $j_0(1 \le j_0 \le n)$ such that $\sigma^p(a_{i,j}) \in V(Y_{j_0}), \sigma^p(b_{s,t}) \in V(Y_{j_0})$, let

$$(\sigma^{p+1}(a_{i,j}), \sigma^{p+1}(b_{s,t})) \in E(G) \text{ and } (\sigma^{p+3}(a_{i,j}), \sigma^{p+3}(b_{s,t})) \in E(G)$$

Otherwise, order

$$\{a_{i,j},\ \sigma(a_{i,j}),\ \sigma^2(a_{i,j}),\ \sigma^3(a_{i,j}),\ b_{s,t},\ \sigma(b_{s,t}),\ \sigma^2(b_{s,t}),\ \sigma^3(b_{s,t})\}$$

by the subscripts of the elements lexicographically. Let c be the smallest element under this ordering, namely there is $q(0 \le q \le 3)$ such that $\sigma^q(a_{i,j}) = c$ or $\sigma^q(b_{s,t}) = c$. We let

$$(\sigma^q(a_{i,j}), \sigma^q(b_{s,t})) \in E(G) \text{ and } (\sigma^{q+2}(a_{i,j}), \sigma^{q+2}(b_{s,t})) \in E(G)$$

We claim $E(G)$ is well-defined, since for any two vertices $a_{i,j}, b_{s,t}$, if $j = t$ and $i \neq s$, $\sigma(a_{i,j})$ and $\sigma(b_{s,t})$ cannot belong to the same set $V(Y_j)(1 \le j \le n)$ under the action of σ and the same holds to $\sigma^3(a_{i,j})$ and $\sigma^3(b_{s,t})$.

At the same time, for any edge $\alpha \in E(G)$, $\sigma\alpha \in E(AC_M(G))$. Conversely, for each $\beta \in E(AC_M(G))$, $\sigma^{-1}\beta \in E(G)$. It means $\sigma(G) = AC_M(G), \sigma(M) = M$, and thus G is an almost self-complementary graph.

We have $\chi(G) \le n$ since each subgraph $Y_j(1 \le j \le n)$ is an independent set. On the other hand, each subgraph $X_i(1 \le i \le n)$ is a complete graph on $2n$ vertices minus a perfect matching. So $\chi(G) \ge n$. Consequently, $\chi(G) = n$.

2. If n is odd, we define a mapping $\varphi : V(G) \to V(G)$ as follows:

$$\varphi = \{ \prod_{\substack{t \text{ odd} \\ 1 \le t \le n-1}} (x_{t,t}\ x_{t+1,t}\ x_{t+1,t+1}\ x_{t,t+1})(y_{t,t}\ y_{t+1,t}\ y_{t+1,t+1}\ y_{t,t+1})\}$$

$$\times \{ \prod_{\substack{i \text{ odd} \\ 1 \le i \le n-2 \\ i+2 \le j \le n}} (x_{i,j}\ x_{j,i}\ x_{i+1,j}\ x_{j,i+1})(y_{i,j}\ y_{j,i}\ y_{i+1,j}\ y_{j,i+1})\} \times (x_{nn})(y_{nn})$$

We can verify the conclusion in a similar way.

Corollary 1. *For each almost self-complementary graph G such that $\chi(G) = n$ and $|V(G)| = 2n^2$, G contains n "cocktail party" subgraphs, $K_{2n} - nK_2$, on $2n$ vertices, where those subgraphs are mutually disjoint.*

Proof. There are at $2n^2$ vertices and n colors. So there are at least $2n$ vertices equipped with the same color. Similar to the proof of Theorem 2, if there are $2n+1$ vertices with the same color, we obtain $\chi(G) \geq n+1$, a contradiction.

Hence each color assigns to exact $2n$ vertices which is an independent set in G. Because $\chi(AC_M(G)) = \chi(G) = n$, the subgraph on the $2n$ vertices for each color must be "cocktail party graph": $K_{2n} - nK_2$. We have $AC_M(G)$ contains n $(K_{2n} - nK_2)$, which are mutually disjoint. The conclusion follows since $G \cong AC_M(G)$. □

References

[1] B. Alspach, J. Morris, V. Vilfred, Self-complementary circulant graphs, *Ars Combinatoria*, **53**(1999) 187-191.

[2] C. Y. Chao and G. Whitehead, Chromaticity of self-complementary graphs, *Arch. Math.*, **32**(1979) 295-304.

[3] E. Dobson and M. Šajna, Almost self-complementary circulant graphs, *Discrete Math.*, **278**(2004) 23-44.

[4] E. A. Nordhaus and J. W. Gaddum, On complementary graphs, *American Math. Monthly*, **63**(1956) 175-177.

[5] P. Potočnik, M. Šajna, On almost self-complementary graphs, *Discrete Math.*, **306**(2006) 107-123.

[6] G. Ringel, Selbstkomplementäre Graphen, *Arch. Math.*, **14**(1963) 354-358.