Distinct Rado Numbers for $x_1 + x_2 + c = x_3$

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Abstract

For every integer c, let $n = R_d(c)$ be the least integer such that for every coloring $\Delta : \{1, 2, \ldots, n\} \to \{0, 1\}$ there exists a solution (x_1, x_2, x_3) to

$$x_1 + x_2 + c = x_3$$

such that $x_i \neq x_j$ when $i \neq j$

and $\Delta(x_1) = \Delta(x_2) = \Delta(x_3)$.

In this paper it is shown that for every integer c,

$$R_d(c) = \begin{cases} 4c+8 & \text{if } c \geq 1 \\ 8 & \text{if } -3 \leq c \leq -6 \\ 9 & \text{if } c = 0, -2, -7, -8 \\ 10 & \text{if } c = -1, -9 \\ |c| - \left\lfloor \frac{|c|-4}{5} \right\rfloor & \text{if } c \leq -10. \end{cases}$$

Note: Part of the research for this paper occurred when the second author was an undergraduate student at South Dakota State University.

Introduction

Let $\mathbb{N}$ represent the set of natural numbers and let $[a, b]$ denote the set $\{n \in \mathbb{N} | \, a \leq n \leq b\}$. A function $\Delta : [1, n] \to [0, t-1]$ is referred to as a *t-coloring* of the set $[1, n]$. Given a t-coloring Δ and a system L of linear equations or inequalities in m variables, a solution $(x_1, x_2, \ldots, x_m)$ to the system L is *monochromatic* if and only if

$$\Delta(x_1) = \Delta(x_2) = \cdots = \Delta(x_m).$$

In 1916, I. Schur [19] proved that for every $t \geq 2$, there exists a least integer $n = S(t)$ such that for every t-coloring of the set $[1, n]$, there exists a monochromatic solution to

$$x_1 + x_2 = x_3.$$

The integers $S(t)$ are called *Schur numbers*. It is known that $S(2) = 5$, $S(3) = 14$ and $S(4) = 45$, but no other Schur numbers are known [21]. In 1933, R. Rado generalized the concept of Schur numbers to arbitrary systems of linear equations. Rado found necessary and sufficient conditions to determine if an arbitrary system of linear homogeneous equations admits a monochromatic solution under every t-coloring of the natural numbers [4, 12, 13, 14]. For a given system of linear equations L, the least integer n, provided that it exists, such that for every t-coloring of the set $[1, n]$ there exists a monochromatic solution to L is called the t-color *Rado number* (or t-color *generalized Schur number*) for the system L. If such an integer n does not exist, then the t-color Rado number for the system L is defined to be infinite. In recent years the exact Rado numbers for several families of equations and inequalities have been found, but almost entirely for 2-colorings [1, 7, 8, 9, 10, 11, 17, 18].

Recently several variations of the classical Rado numbers have been considered [5, 6, 16]. One possible variation is to add to the system of equations the condition that all variables in the solution be distinct. We will refer to the Rado numbers for any system that requires the integers in the solution to be distinct as *distinct Rado numbers*. This variation has the following motivation. The study of Schur numbers and Rado numbers is closely linked to a similar area of study in graph theory. Let K_n represent a complete graph on n vertices. In 1930, F. Ramsey [15] proved that for every integer $t \geq 2$ and all integers $s_1, s_2, \ldots, s_t$ there exists a least integer $n = R(s_1, s_2, \ldots, s_t)$ such that if the edges of a K_n are colored with t colors then there exists an $i \in [1, t]$ such that a K_{s_i} with all edges colored the ith color can be found in the K_n. The integers $R(s_1, s_2, \ldots, s_t)$ are called *Ramsey numbers* and are known for only a few small values of t and s_i. Since any graph with n vertices can be found in a K_n, it is clear that for every $t \geq 2$ and for all graphs $G_1, G_2, \ldots, G_t$, there must also exist a least integer $n = R(G_1, G_2, \ldots, G_t)$ such that if the edges of a K_n are colored with t colors then there exists an $i \in [1, t]$ such that a G_i with all edges colored the ith color can be found in the K_n. This integer is called the *Ramsey number* for the set of graphs $\{G_1, G_2, \ldots, G_t\}$. The study of Schur and Rado numbers as well as Ramsey numbers is considered a part of Ramsey Theory, although ironically Schur's theorem was published fourteen years before Ramsey's theorem.

The similarities between investigating the Rado numbers for specific equations and the Ramsey numbers for specific graphs are very strong. In the first case we are looking for a sufficiently large set of natural numbers that when arbitrarily colored will assure a monochromatic set of numbers that also forms a solution to a specified equation. In the second case we are looking for a sufficiently large complete graph such that when the edges are arbitrarily colored there will exist a monochromatic set of edges that also forms a specified graph. There are some major differences however. Probably the most significant difference is that while the Ramsey number for any set of graphs is finite, the

Rado numbers for many equations are infinite. Another difference is that while a given integer may be used more than one time in the solution to an equation, no single edge is ever used more than once in a given graph. In the case of distinct Rado numbers this difference is eliminated, so the problems are more closely linked to the problem of finding Ramsey numbers for graphs.

W. Sierpinski [20] considered the problem of finding the distinct Schur numbers and was able to show that the 2-color distinct Schur number is 9. Also, in [2] the 2-color Rado numbers we determined for the two systems

$$l(m) : x_1 + x_2 + \cdots + x_{m-1} < x_m$$

and

$$\begin{aligned} l_d(m) &: x_1 + x_2 + \cdots + x_{m-1} < x_m \\ & x_i \neq x_j \text{ when } i \neq j. \end{aligned}$$

Note that the subscript on $l_d(m)$ refers to the fact that the integers in a solution must be distinct. Let $r(m)$ and $r_d(m)$ represent the 2-color Rado numbers for the systems $l(m)$ and $l_d(m)$ respectively. While it is trivial to show that for every $m \geq 3$, $r(m) = m^2 - m + 1$, it is nontrivial to show that for every $m \geq 3$

$$r_d(m) = \begin{cases} \frac{9}{16}m^3 - m^2 + m + 1 & m \equiv 0 (\text{mod } 4) \\ \frac{9}{16}m^3 - m^2 + \frac{13}{16}m + \frac{13}{8} & m \equiv 1 (\text{mod } 4) \\ \frac{9}{16}m^3 - m^2 + m + \frac{1}{2} & m \equiv 2 (\text{mod } 4) \\ \frac{9}{16}m^3 - m^2 + \frac{17}{16}m + \frac{5}{8} & m \equiv 3 (\text{mod } 4). \end{cases}$$

As illustrated in the above example, adding to a system the requirement that the integers in a solution must be distinct may make the determination of the distinct Rado numbers for this new system considerably more challenging than determining the Rado numbers for the original system.

We continue the investigation of distinct Rado numbers by considering the following variation of the Schur equation. For every integer c, let $L(c)$ represent the system

$$L(c) : x_1 + x_2 + c = x_3$$

and let $R(c)$ represent the 2-color Rado number for this system. It was determined by Burr and Loo [3] that

$$R(c) = \begin{cases} 4c + 5 & \text{if } c \geq 0 \\ |c| - \left\lfloor \frac{|c| - 1}{5} \right\rfloor & \text{if } c < 0. \end{cases}$$

In this paper we consider the distinct Rado numbers for this system. For every integer c, let $L_d(c)$ represent the system

$$\begin{aligned} L_d(c) &: x_1 + x_2 + c = x_3 \\ & x_i \neq x_j \text{ when } i \neq j \end{aligned}$$

and let $R_d(c)$ represent the distinct 2-color Rado number for $L_d(c)$. In this paper we determine $R_d(c)$ for every integer c. Note that when $c = 0$ this is the above mentioned result of Sierpinski.

Main Results

Theorem: For every integer c,

$$R_d(c) = \begin{cases} 4c+8 & \text{if } c \geq 1 \\ 8 & \text{if } -3 \leq c \leq -6 \\ 9 & \text{if } c = 0,\ -2,\ -7,\ -8 \\ 10 & \text{if } c = -1,\ -9 \\ |c| - \left\lfloor \frac{|c|-4}{5} \right\rfloor & \text{if } c \leq -10. \end{cases}$$

Proof: The proof will be accomplished in three major parts. In Part 1, we will show that $R_d(c) = 4c + 9$ for all integers $c \geq 1$. In Part 2, we will show that $R_d(c) = |c| - \left\lfloor \frac{|c|-4}{5} \right\rfloor$ for all integers $c \leq -14$. Finally, in Part 3, we will give a proof that is representative of the individual cases which must be considered for each integer c such that $-13 \leq c \leq 0$.

Part 1: First we shall show that

$$R_d(c) = 4c + 8$$

for every positive integer c. For every positive integer c, it is easy to verify that the coloring $\Delta : [1, 4c + 7] \to [0, 1]$ defined by

$$\Delta(x) = \begin{cases} 0 & \text{if } 1 \leq x \leq c+2 \\ 1 & \text{if } c+3 \leq x \leq 3c+6 \\ 0 & \text{if } 3c+7 \leq x \leq 4c+7 \end{cases}$$

avoids a monochromatic solution to $L_d(c)$. It follows that

$$R_d(c) \geq 4c + 8$$

whenever c is a positive integer. We shall now show that

$$R_d(c) \leq 4c + 8$$

for every positive integer c. Let a positive integer c be given and let $\Delta : [1, 4c + 8] \to [0, 1]$ be an arbitrary 2-coloring of the set $[1, 4c + 8]$.

We must show that Δ contains a monochromatic solution to $L_d(c)$. Without loss of generality we may assume that

$$\Delta(1) = 0.$$

We shall consider four cases based on the four possible ways to color the integers 3 and $c + 3$.

Case 1. Assume that $\Delta(3) = 0$ and $\Delta(c+3) = 0$. If $\Delta(c+4) = 0$, then $(1, 3, c+4)$ is a monochromatic solution to $L_d(c)$ and we are done, so we may assume that

$$\Delta(c+4) = 1.$$

If $\Delta(2c+6) = 0$, then $(3, c+3, 2c+6)$ is a monochromatic solution, so we may assume that

$$\Delta(2c+6) = 1.$$

Now, if $\Delta(2) = 0$, then $(1, 2, c+3)$ is a monochromatic solution and if $\Delta(2) = 1$, then $(2, c+4, 2c+6)$ is a monochromatic solution.

Case 2. Assume that $\Delta(3) = 0$ and $\Delta(c+3) = 1$. If $\Delta(c+4) = 0$, then $(1, 3, c+4)$ is a monochromatic solution, so we may assume that

$$\Delta(c+4) = 1.$$

If $\Delta(3c+7) = 1$, then $(c+3, c+4, 3c+7)$ is a monochromatic solution, so we may assume that

$$\Delta(3c+7) = 0.$$

If $\Delta(2c+4) = 0$, then $(3, 2c+4, 3c+7)$ is a monochromatic solution, so we may assume that

$$\Delta(2c+4) = 1.$$

Now, if $\Delta(4c+8) = 0$, then $(1, 3c+7, 4c+8)$ is a monochromatic solution and if $\Delta(4c+8) = 1$, then $(c+4, 2c+4, 4c+8)$ is a monochromatic solution.

Case 3. Assume that $\Delta(3) = 1$ and $\Delta(c+3) = 0$. If $\Delta(2c+4) = 0$, then $(1, c+3, 2c+4)$ is a monochromatic solution, so we may assume that

$$\Delta(2c+4) = 1.$$

If $\Delta(3c+7) = 1$, then $(3, 2c+4, 3c+7)$ is a monochromatic solution, so we may assume that

$$\Delta(3c+7) = 0.$$

If $\Delta(2) = 0$, then $(1, 2, c+3)$ is a monochromatic solution, so we may assume that

$$\Delta(2) = 1.$$

Now, if $c = 1$, then $c+3 = 4$, so $\Delta(4) = 0$. If $\Delta(6) = 0$, then $(1, 4, 6)$ is a monochromatic solution and if $\Delta(6) = 1$, then $(2, 3, 6)$ is a monochromatic solution. So the case were $c = 1$ is finished and we may continue with the assumption that $c \geq 2$.

If $\Delta(c+5) = 1$, then $(2, 3, c+5)$ is a monochromatic solution, so we may assume that

$$\Delta(c+5) = 0.$$

If $\Delta(4) = 0$, then $(1, 4, c+5)$ is a monochromatic solution, so we may assume that

$$\Delta(4) = 1.$$

Now, if $\Delta(3c+8) = 0$, then $(c+3, c+5, 3c+8)$ is a monochromatic solution and if $\Delta(3c+8) = 1$, then $(4, 2c+4, 3c+8)$ is a monochromatic solution.

Case 4. Assume that $\Delta(3) = 1$ and $\Delta(c+3) = 1$. If $\Delta(2c+6) = 1$, then $(3, c+3, 2c+6)$ is a monochromatic solution, so we may assume that

$$\Delta(2c+6) = 0.$$

If $\Delta(c+5) = 0$, then $(1, c+5, 2c+6)$ is a monochromatic solution, so we may assume that

$$\Delta(c+5) = 1.$$

If $\Delta(2) = 1$, then $(2, 3, c+5)$ is a monochromatic solution, so we may assume that

$$\Delta(2) = 0.$$

Now, if $\Delta(3c+8) = 0$, then $(2, 2c+6, 3c+8)$ is a monochromatic solution and if $\Delta(3c+8) = 1$, then $(c+3, c+5, 3c+8)$ is a monochromatic solution.

Since we found in Δ a monochromatic solution to $L_d(c)$ in every case we may conclude that $R_d(c) \leq 4c+8$ and since it was previously shown that $R_d(c) \geq 4c+8$, we conclude that

$$R_d(c) = 4c+8$$

for every positive integer c.

Part 2: We shall now consider the case where $c \leq -14$. Let an integer $c \leq -14$ be given. First we shall show that

$$R_d(c) \geq |c| - \left\lfloor \frac{|c|-4}{5} \right\rfloor$$

by exhibiting a coloring Δ_c: $\left[1, |c| - \left\lfloor \frac{|c|-4}{5} \right\rfloor - 1\right] \to [0,1]$ that avoids a monochromatic solution to $L_d(c)$. Note that c can be uniquely expressed as

$$c = -5t - a.$$

where $t \geq 2$ and $a \in [4, 8]$. Since t is positive, it follows that

$$R_d(t) = 4t + 8.$$

Hence, there exists a coloring $\Delta_t : [1, 4t+7] \to [0,1]$ that avoids a monochromatic solution to $L_d(t)$. Now, since

$$|c| - \left\lfloor \frac{|c|-4}{5} \right\rfloor - 1 = |-5t-a| - \left\lfloor \frac{|-5t-a|-4}{5} \right\rfloor - 1$$
$$= (5t+a) - \left\lfloor \frac{5t+a-4}{5} \right\rfloor - 1$$
$$= (5t+a) - t + \left\lfloor \frac{a-4}{5} \right\rfloor - 1$$
$$= (5t+a) - t - 1$$
$$= 4t + a - 1,$$

our desired coloring Δ_c: $\left[1, |c| - \left\lfloor \frac{|c|-4}{5} \right\rfloor - 1\right] \to [0,1]$ may be expressed as $\Delta_c : [1, 4t+a-1] \to [0,1]$. Define $\Delta_c : [1, 4t+a-1] \to [0,1]$ by

$$\Delta_c(x) = \Delta_t(4t + a - x).$$

It will be shown that Δ_c defined in this way avoids a monochromatic solution to $L_d(c)$. Assume that (x_1, x_2, x_3) is a solution to $L_d(c) = L_d(-5t-a)$. Let

$$y_i = 4t + a - x_i \quad \text{for every } i \in [1,3].$$

Therefore,

$$\Delta_t(y_i) = \Delta_t(4t + a - x_i) = \Delta_c(x_i) \quad \text{for every } \; i \in [1,3]$$

and the triple (x_1, x_2, x_3) is monochromatic in Δ_c if and only if the triple (y_1, y_2, y_3) is monochromatic in Δ_t. Also, since $x_i \neq x_j$ when $i \neq j$, it follows that $y_i \neq y_j$ when $i \neq j$. Now,

$$y_1 + y_2 + t = (4t + a - x_1) + (4t + a - x_2) + t$$
$$= 4t + a - (x_1 + x_2 + (-5t - a))$$
$$= 4t + a - (x_1 + x_2 + c)$$
$$= 4t + a - x_3$$
$$= y_3.$$

Therefore, (y_1, y_2, y_3) is a solution to $L_d(t)$. Since the coloring Δ_t avoids monochromatic solutions to $L_d(t)$, the triple (y_1, y_2, y_3) is not monochromatic in Δ_t so the triple (x_1, x_2, x_3) is not monochromatic in Δ_c. Thus, the coloring $\Delta_c : \left[1, |c| - \left\lfloor \frac{|c|-4}{5} \right\rfloor - 1\right] \to [0,1]$ avoids a monochromatic solution to $L(c)$ and

$$R_d(c) \geq |c| - \left\lfloor \frac{|c|-4}{5} \right\rfloor.$$

Now we shall show that

$$R_d(c) \leq |c| - \left\lfloor \frac{|c|-4}{5} \right\rfloor.$$

Let an arbitrary coloring $\Delta_c : \left[1, |c| - \left\lfloor \frac{|c|-4}{5} \right\rfloor\right] \rightarrow [0,1]$ be given. We must show that Δ_c contains a monochromatic solution to $L_d(c)$. Note that c can be uniquely expressed as

$$c = -5s - b$$

where s is a positive integer and $b \in [9, 13]$. Since

$$\begin{aligned} |c| - \left\lfloor \frac{|c|-4}{5} \right\rfloor &= |-5s - b| - \left\lfloor \frac{|-5s-b|-4}{5} \right\rfloor \\ &= (5s + b) - s - \left\lfloor \frac{b-4}{5} \right\rfloor \\ &= (5s + b) - s - 1 \\ &= 4s + b - 1, \end{aligned}$$

Δ_c can be represented as $\Delta_c : [1, 4s + b - 1] \rightarrow [0, 1]$. Let

$\Delta_s : [1, 4s + 8] \rightarrow [0, 1]$ be defined by

$$\Delta_s(x) = \Delta_c(4s + b - x).$$

Since s is positive, $R_d(s) = 4s + 8$, so Δ_s contains a monochromatic solution to $L_d(s)$. Let (y_1, y_2, y_3) be this solution. Let

$$x_i = 4s + b - y_i \quad \text{for every } i \in [1, 3].$$

Therefore, $\Delta_c(x_i) = \Delta_c(4s + b - y_i) = \Delta_s(y_i)$ for every $i \in [1, 3]$. Since the triple (y_1, y_2, y_3) is monochromatic in Δ_s, it follows that the triple (x_1, x_2, x_3) is monochromatic in Δ_c. Also, since $y_i \neq y_j$ when $i \neq j$, it follows that $x_i \neq x_j$ when $i \neq j$. Finally,

$$\begin{aligned} x_1 + x_2 + c &= (4s + b - y_1) + (4s + b - y_2) + (-5s - b) \\ &= 4s + b - (y_1 + y_2 + s) \\ &= 4s + b - y_3 \\ &= x_3. \end{aligned}$$

Hence, the triple (x_1, x_2, x_3) is both monochromatic in Δ_c and a solution to $L_d(c)$. Therefore, we have shown that the arbitrary coloring $\Delta_c : \left[1, |c| - \left\lfloor \frac{|c|-4}{5} \right\rfloor\right] \rightarrow [0,1]$ contains a monochromatic solution to $L_d(c)$ and that

$$R_d(c) \leq |c| - \left\lfloor \frac{|c|-4}{5} \right\rfloor.$$

Since it was previously shown that $R_d(c) \geq |c| - \left\lfloor \frac{|c|-4}{5} \right\rfloor$, we have that

$$R_d(c) = |c| - \left\lfloor \frac{|c|-4}{5} \right\rfloor \quad \text{for every } c \leq -14.$$

Part 3: Finally, we shall consider the cases where $c \in [-13, 0]$. Each of these fourteen values of c must be considered separately. Note that even though the formula for $R_d(c)$ is the same for $c \leq -14$ and for $c \in [-13, -10]$, the proof for $c \leq -14$ does not generalize to $c \in [-13, -10]$. For each of the fourteen values of c in the set $[-13, 0]$, the proof of the upper bound for $R_d(c)$ is straightforward case analysis. However, each proof requires several cases. Also, the lengths of the strings to be colored are small enough that the values of $R_d(c)$ can easily be determined with the help of a computer. For these reasons, we will demonstrate the proof technique for these special cases by proving one case and we will provide the results of computer experiments for the other cases. Since the case for $c = 0$ was previously known, we will show that $R_d(-1) = 10$.

First note that the coloring $\Delta : [1, 9] \to [0, 1]$ defined by

$$\Delta(x) = \begin{cases} 0 & \text{if } \; x = 1, 2, 3, 5, 9 \\ 1 & \text{if } \; x = 4, 6, 7, 8 \end{cases}$$

avoids a monochromatic solution and thus

$$R_d(-1) \geq 10.$$

To prove the upper bound for $R_d(-1)$, let an arbitrary coloring $\Delta : [1, 10] \to [0, 1]$ be given. We will show that Δ contains a monochromatic solution to $L_d(-1)$. As an interesting note, the coloring of the integer 1 does not affect the possibility of a monochromatic solution to the equation $x_1 + x_2 - 1 = x_3$ because no set of three distinct natural numbers that includes 1 could provide a solution to the equation. Therefore, $\Delta(1)$ can be either color. Without loss of generality, we may assume that

$$\Delta(2) = 0$$

and we shall consider four cases based on the four possible ways to color the integers 3 and 5.

Case 1. Assume that $\Delta(3) = 0$ and $\Delta(5) = 0$.

If $\Delta(4) = 0$, then $(2, 3, 4)$ is a monochromatic solution to $L_d(-1)$, so we may assume that $\Delta(4) = 1$. If $\Delta(7) = 0$, then $(3, 5, 7)$ is a monochromatic solution to $L_d(-1)$, so we may assume that $\Delta(7) = 1$. If $\Delta(6) = 0$, then $(2, 5, 6)$ is a monochromatic solution to $L_d(-1)$, so we may assume that $\Delta(6) = 1$. If $\Delta(9) = 1$, then $(4, 6, 9)$ is a monochromatic solution to $L_d(-1)$, so we may assume that $\Delta(9) = 0$. Now, if $\Delta(10) = 1$, then $(4, 7, 10)$ is a monochromatic solution to $L_d(-1)$, and if $\Delta(10) = 0$, then $(2, 9, 10)$ is a monochromatic solution to $L_d(-1)$.

Case 2. Assume that $\Delta(3) = 0$ and $\Delta(5) = 1$.

If $\Delta(4) = 0$, then $(2, 3, 4)$ is a monochromatic solution to $L_d(-1)$, so we may assume that $\Delta(4) = 1$. If $\Delta(8) = 1$, then $(4, 5, 8)$ is a monochromatic solution to $L_d(-1)$, so we may assume that $\Delta(8) = 0$. If $\Delta(9) = 0$, then $(2,8,9)$ is a monochromatic solution to $L_d(-1)$, so we may assume that $\Delta(9) = 1$. Now, if $\Delta(6) = 1$, then $(4,6,9)$ is a monochromatic solution to $L_d(-1)$, and if $\Delta(6) = 0$, then $(3,6,8)$ is a monochromatic solution to $L_d(-1)$.

Case 3. Assume that $\Delta(3) = 1$ and $\Delta(5) = 0$.

If $\Delta(6) = 0$, then $(2,5,6)$ is a monochromatic solution to $L_d(-1)$, so we may assume that $\Delta(6) = 1$. If $\Delta(8) = 1$, then $(3,6,8)$ is a monochromatic solution to $L_d(-1)$, so we may assume that $\Delta(8) = 0$. If $\Delta(9) = 0$, then $(2,8,9)$ is a monochromatic solution to $L_d(-1)$, so we may assume that $\Delta(9) = 1$. Now, if $\Delta(4) = 0$, then $(2,4,5)$ is a monochromatic solution to $L_d(-1)$, and if $\Delta(4) = 1$, then $(4,6,9)$ is a monochromatic solution to $L_d(-1)$.

Case 4. Assume that $\Delta(3) = 1$ and $\Delta(5) = 1$.

If $\Delta(7) = 1$, then $(3,5,7)$ is a monochromatic solution to $L_d(-1)$, so we may assume that $\Delta(7) = 0$. If $\Delta(8) = 0$, then $(2,7,8)$ is a monochromatic solution to $L_d(-1)$, so we may assume that $\Delta(8) = 1$. If $\Delta(10) = 1$, then $(3,8,10)$ is a monochromatic solution to $L_d(-1)$, so we may assume that $\Delta(10) = 0$. Now, if $\Delta(4) = 1$, then $(4,5,8)$ is a monochromatic solution to $L_d(-1)$, and if $\Delta(4) = 0$, then $(4,7,10)$ is a monochromatic solution to $L_d(-1)$.

Since in all four cases we found a monochromatic solution to $L_d(-1)$, we may conclude that

$$R_d(-1) \leq 10$$

and the case for $c = -1$ is complete. □

The following table was made with the help of a computer. For all values of $c \in [-13, 0]$, this table lists all of the longest colorings (up to a permutation of the colors) that avoid a monochromatic solution to $L_d(c)$ and thus establish the lower bounds. These colorings are expressed as strings. For example, the string 00101110 represents the coloring $\Delta : [1,8] \rightarrow [0,1]$ defined by

$$\Delta(x) = \begin{cases} 0 & \text{if } x = 1,2,4,8 \\ 1 & \text{if } x = 3,5,6,7. \end{cases}$$

c	$R_d(c)$	maximal strings
0	9	00101110
− 1	10	000101110
		011010001
− 2	9	01010110
− 3	8	0011101
		0001101
− 4	8	0110001
		0111001
− 5	8	0100011
		0100111
− 6	8	0010101
		0101001
		0101010
		0101011
		0101100
		0101110
		0111010
− 7	9	01101010
− 8	9	01111001
		01111000
		01101001
		01101000
		00010111
		00010110
− 9	10	011101001
	10	011101000
− 10	9	00000111
		00111000
		00111010
		00111100
		01000111
		01111000
		01111100
− 11	10	000000111
		001000111
		001111000
		001111100
		010000111
		011000111
		011111000
− 12	11	0011111000
		0111111000
− 13	12	00111111000

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