Combinatorial Aspects of Mixed Arrangements

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Abstract

In this paper, we consider mixed arrangements which are composed of the hyperplanes (or subspaces) and the spheres. We research the posets of their intersection sets and calculate the Möbius functions of the mixed arrangements through the hyperplane (or subspace) arrangements' Möbius functions. Moreover, by the method of deletion and restriction, we calculate the recurse formulas of the triples of these mixed arrangements.

1 Introduction

A hyperplane arrangement $\mathscr{A}$ is a finite collection of codimension one affine subspaces in a finite dimensional vector space. Because hyperplane arrangements have many important properties in the aspects of combinatorics and topology, many mathematicians are interested in studying them. Now many good results have been gotten in studying the regions of $\mathscr{A}$, the Möbius function, the characteristic polynomial of the intersection set $L(\mathscr{A})$ of the elements of $\mathscr{A}$, and the topology structure of the complement of $\mathscr{A}$. The main results about hyperplane arrangement can be found in [3].

Yi Hu considered another kind of arrangement called mixed arrangement $\mathscr{M}$ in [2], and calculated the homology of the complement of affine subspace arrangement and mixed arrangement by induction and the Mayer-Vietoris sequences.

Definition 1.1. [2] A mixed arrangement is a finite set $\mathscr{M} = \{M_0, \dots, M_k\}$ of closed subspaces (with induce topology) of $\mathbb{R}^n$, which satisfies

(1) every M_i is a copy of a differentiable ball or sphere of some dimension (that is, of a differentiable copy of $\mathbb{R}^k$ of S^k for some $k < n$);

(2) every two subspaces meet transversally.

In this paper, we are also interesting in the mixed arrangements, and we'll consider the poset $L(\mathscr{M})$ of the mixed arrangement $\mathscr{M}$ from the view of combinatorial point, but here we drop the second condition in the above conditions, i.e. we consider more universal cases.

We organized this paper as follows. In section 2, we give some basic definitions. In section 3, we calculate the Möbius functions of the poset of mixed arrangements. In section 4, we get the deletion-restriction formulas for the characteristic polynomial and the number of regions.

2 Preliminary

Definition 2.1. A finite hyperplane arrangement $\mathscr{A}_n$ is a finite set of affine hyperplanes in some vector space K^n, where K is a field. If $T = \cap_{H \in \mathscr{A}_n} H \neq \emptyset$, we call $\mathscr{A}_n$ centered with center T. If $\mathscr{A}_n$ is centered, then coordinates may be chosen so that each hyperplane contains the origin. In this case we call $\mathscr{A}_n$ central.

Definition 2.2. If $K = \mathbb{R}$, set $\mathscr{M}_n = \mathscr{A}_n \cup \{S^{n-1}\}$, where S^{n-1} is a sphere of dimension $n-1$ in $\mathbb{R}^n$. Let $L(\mathscr{A}_n)$ be the partial order set of all nonempty intersections of elements of $\mathscr{A}_n$, and $L(\mathscr{M}_n)$ be the partial order set of $\mathscr{M}_n$. Both the partial orders are defined by reverse inclusion, i.e.

$$X \leq Y \Leftrightarrow X \supseteq Y.$$

Thus $\mathbb{R}^n = \hat{0}$ is the minimum element in $L(\mathscr{A}_n)$ and $L(\mathscr{M}_n)$.

Definition 2.3. For any $X, Y \in L(\mathscr{A}_n)$, define the meet by $X \vee Y = X \cap Y$, and the join by $X \wedge Y = \bigcap\{Z | Z \in L(\mathscr{A}_n)\}, X \cup Y \subseteq Z\}$. And we define the meet $\wedge$ and the join $\vee$ by the same way in $L(\mathscr{M}_n)$.

Definition 2.4. If every pair of elements of a poset L has a meet, we say that L is a meet-semilattice. A poset L is locally finite if every interval $[x,y] \subseteq L$ is finite. Define the Möbius function μ of locally finite poset L by the following conditions:

$$\begin{aligned} \mu(x,x) &= 1, \forall x \in L \\ \mu(x,y) &= -\sum_{x \leq z < y} \mu(x,y), \text{ for all } x < y \text{ in } P. \end{aligned}$$

If L has a $\hat{0}$, than we write $\mu(x) = \mu(\hat{0}, x)$.

3 The Möbius Function

3.1 The Mixed Arrangement with One Sphere

Here we suppose that $\mathscr{A}_n = \{H_1, \cdots, H_m\}$ is a finite hyperplane arrangement in $\mathbb{R}^n$. Let $\mathscr{M}_n = \mathscr{A}_n \cup \{S^{n-1}\}$ be the corresponding mixed arrangement in $\mathbb{R}^n$.

Now using X^k denotes some $X \in L(\mathscr{M}_n)$ with $\dim X = k$, and $\mu_{\mathscr{A}_n}$ denotes the Möbius function in $L(\mathscr{A}_n)$.

Proposition 3.1. Let $L(S^{n-1}) = \{X \cap S^{n-1} : X \in L(\mathscr{A}_n)\} \setminus \{\emptyset\}$, then $L(\mathscr{M}_n) = L(\mathscr{A}_n) \cup L(S^{n-1})$. $L(\mathscr{M}_n)$ and $L(S^{n-1})$ both are meet-semilattices.

Proof. For $\mathscr{M}_n = \mathscr{M} \cup \{S^{n-1}\}$, so $\forall X \in L(\mathscr{M}_n)$, X is either an affine subspace or a sphere.

(1) If X is an affine subspace, then $X = \bigcap_i H_i$, for some $H_i \in \mathscr{A}_n$, so $X \in L(\mathscr{M}_n)$.

(2) If X is a sphere, let $\dim X = k$, $X^k = S^k$, so there exists a unique affine subspace $X^{k+1} \in L(\mathscr{A}_n)$ such that S^m is the sphere (not always unit sphere) which is got by the meet of S^{n-1} and X^{k+1}. Thus $L(\mathscr{M}_n) = L(\mathscr{A}_n) \cup L(S^{n-1})$.

Now in $L(S^{n-1})$, we define the order by the reverse inclusion as in definition 1.1. Consider $S^i \cap S^j \neq \emptyset$, define $S^i \vee S^j = S^i \cap S^j$, and $S^i \wedge S^j = \bigcap\{Z | Z \in L(S^{n-1}), S^i \cup S^j \subseteq Z\}$.

For $\forall S^i, S^j \in L(S^{n-1})$,

$$
\begin{aligned}
S^i \wedge S^j &= \bigcap\{S \in L(S^{n-1}) | S^i \cup S^j \subseteq S\} \\
&= \bigcap\{X^d \cap S^{n-1} | (X^{d_i} \cap S^{n-1}) \cup (X^{d_j} \cap S^{n-1}) \subseteq X^d \cap S^{n-1}\} \\
&= (\bigcap\{X^d \in L(\mathscr{A}_n) | X^{d_i} \cup X^{d_j} \subseteq X^d\}) \cap S^{n-1} \\
&= (X^{d_i} \wedge X^{d_j}) \cap S^{n-1},
\end{aligned}
$$

where if X^d intersects with S^{n-1}, then $d = \dim S + 1$, if X^d is tangental to S^{n-1}, then $d = \min\{t | X^t \cap S^{n-1} = S\}$, the same to X^{d_i} and X^{d_j}. So $S^i \wedge S^j \in L(S^{n-1})$. Therefore $L(S^{n-1})$ is a meet-semilattice.

Now for any $X^i \in L(\mathscr{A}_n)$, and $S^j = X^{d_j} \cap S^{n-1}$. Let $X^i \wedge S^j = \bigcap\{Z \in L(\mathscr{M}) | X^i \cup S^j \subseteq Z\} = X_1$, in fact, $\forall S^i \in L(S^{n-1})$, there are just two cases:

(i) There is no sphere S^k such that the affine subspace $X^i \subseteq S^k$, so $X_1 \in L(\mathscr{A}_n)$ is an affine subspace. Let $X^i \wedge X^{d_j} = \bigcap\{Z \in L(\mathscr{A}) | X^i \cup X^{d_j} \subseteq Z\} = X_2$. Thus from $S^j \subseteq X_1$, it can be seen that $X^{d_j} \subseteq X_1$, so $X_1 \supseteq X_2$. On the other hand, $X^i \cup S^j \subseteq X^i \cup X^{d_j} \subseteq X_2$, so $X_1 \subseteq X_2$, i.e. $X^i \wedge S^j = X^i \wedge X^{d_j}$.

(ii) X^i must be a point. If $X^i \subseteq S^j$, then $X^i \wedge S^j = S^j$, else just the same as (i), let $X^i \wedge X^{d_j} = \bigcap\{Z \in L(\mathscr{A}) | X^i \cup X^{d_j} \subseteq Z\}$, we have $X^i \wedge S^j = X^i \wedge X^{d_j}$.

So $L(\mathscr{M}_n)$ is a meet-semilattice. ∎

If $\bigcap\{X | X \in \mathscr{M}_n\} = \emptyset$, and we let $\bar{L}(\mathscr{M}_n) = L(\mathscr{M}_n) \cup \{\emptyset\}$, then it's clearly $\emptyset$ is the maximum element $\hat{1}$ in $\bar{L}(\mathscr{M}_n)$, and $\bar{L}(\mathscr{M}_n)$ is an atomic lattice.

Let O and r be respectively the center and radius of S^{n-1}, and the norm $||X - Y||$ be the Euclidean distance of $X, Y \in \mathbb{R}^n$. For the set N of maximal elements in $L(\mathscr{A}_n)$, we partition them into the following three parts:

(1) $N_1 = \{X^k \in N | X^k \cap S^{n-1} = \emptyset \text{ for } k > 0, \text{ or } ||X^0 - O|| > r\}$,

(2) $N_2 = \{X^k \in N | X^k \text{ intersect with } S^{n-1} \text{ for } k > 0, \text{ or } ||X^0 - O|| < r\}$,

(3) $N_3 = \{X^k \in N | X^k \text{ is tangental to } S^{n-1} \text{ for } k > 0, \text{ or } ||X^0 - O|| = r\}$.

Theorem 3.2. The Möbius function $\mu_{\mathscr{M}_n}(x)$ of $L(\mathscr{M}_n)$ is following:

(1) For any $X^l \in L(\mathscr{A}_n)$ for $l > 0$ and $X^0 \in N_1 \cup N_2$, $\mu_{\mathscr{M}_n}(X^l) = \mu_{\mathscr{A}_n}(X^l)$.

(2) If $Y^k \in N_2$ for $k > 0$, or Y^1 is covered by $X^0 \in N_2$ in $L(\mathscr{A}_n)$. Let $S^{k-1} = Y^k \cap S^{n-1}$, S^{k-1} is the maximal element in $L(\mathscr{M}_n)$, and for any $X^l \in [0, S^{k-1}] \cap L(S^{n-1})$, $\mu_{\mathscr{M}_n}(X^l) = -\mu_{\mathscr{A}_n}(\bar{X}^{l+1})$, where $\bar{X}^{l+1} \in \mathscr{A}_n$ and $X^l = \bar{X}^{l+1} \cap S^{n-1}$.

(3) If $Y^k \in N_3$, Suppose $Y^k = \bigcap_{j=1}^{t} H_{i_j}$, where $H_{i_j} \in \mathscr{A}_n$, for $1 \le j \le t$, and all the hyperplanes containing Y^k are here. Let $Z^0 = Y^k \cap S^{n-1}$, for any $X^l \in [0, Z^0) \cap L(S^{n-1})$, $\mu_{\mathscr{M}_n}(X^l) = -\mu_{\mathscr{A}_n}(\bar{X}^{l+1})$, where $X^l = \bar{X}^{l+1} \cap S^{n-1}$, and

(i) if there is no hyperplane tangental to S^{n-1},

$$\mu_{\mathscr{M}_n}(Z^0) = -\sum_{\substack{X^l \in L(\mathscr{A}_n) \\ Z^0 \text{ cover } X^l}} \mu_{\mathscr{A}_n}(X^l),$$

(ii) if there exists a hyperplane, suppose H_{i_1}, such that $Z^0 \in H_{i_1}$ and H_{i_1} is tangental to S^{n-1}, then

$$\mu_{\mathscr{M}_n}(Z^0) = -\sum_{X \in [H_{i_1}, Z^0) \cap L(\mathscr{A}_n)} \mu_{\mathscr{A}_n}(X).$$

Proof. $\forall X^k \in L(\mathscr{A}_n)$ for $k > 0$ and $X^0 \in N_1 \cup N_2$, the interval $[0, X^l]$ in $L(\mathscr{A}_n)$ is equal to the interval $[0, X^l]$ in $L(\mathscr{M}_n)$, because the affine space X^l can't be included in any sphere. Therefore (1) is clearly.

Now we'll prove (2). First let M be the set of all subspaces intersecting with S^{n-1}, and whose dimension is larger than 0. Then we can give the following map f:

$$\begin{aligned} f: \quad M &\longrightarrow L(S^{n-1}) \\ \bar{X}^{t+1} &\longmapsto S^{n-1} \cap \bar{X}^{t+1} = S^t \end{aligned}$$

and use $f|_A$ to denote the restriction of f on some set A.

Let $\mathscr{A}$ be the set of all hyperplanes in $\mathscr{M}$. $\forall X^l \in L(S^{n-1})$, there exists a unique subspace $\bar{X}^{l+1} \in L(\mathscr{A})$ such that $X^l = \bar{X}^{l+1} \cap S^{n-1}$. Assume that $\bar{X}_1^{l+2}, \cdots, \bar{X}_r^{l+2}$

are all the $l+2$ dimensional subspaces containing X^{l+1} in $L(\mathscr{M}')$. Let $S_j^{l+1} = \bar{X}_j^{l+2} \cap S^{n-1}$ for $1 \le j \le r$, then

$$S^l = \bar{X}^{l+1} \cap S^{n-1} = (\bar{X}^{l+1} \cap \bar{X}_j^{l+2}) \cap S^{n-1} = \bar{X}^{l+1} \cap S_j^l, 1 \le j \le k.$$

We'll prove $\{T^{l+1} < S^l | T^{l+1} \in L(\mathscr{M}')\} = \{\bar{X}^{l+1}, S_1^{l+1}, \cdots, S_r^{l+1}\}$.

First "$\supseteq$" holds according to above.

For $\forall T^{l+1}$ belongs to the left, if $T^{l+1} \in L(\mathscr{A})$, thus $T^{l+1} = \bar{X}^{l+1}$; if $T^{l+1} \in L(S^{n-1})$, $T^{l+1} = \bar{X}^{l+2} \cap S^{n-1} \supset S^l$, so $\bar{X}^{l+2} \cap S^{n-1} \supset \bar{X}^{l+1} \cap S^{l-1}$, we have $\bar{X}^{l+2} \supset \bar{X}^{l+1}$. But $\bar{X}_1^{l+2}, \cdots, \bar{X}_r^{l+2}$ are all the $l+2$ dimensional subspaces containing X^{l+1} in $L(\mathscr{M})$, so $\exists q$, such that $\bar{X}^{l+2} = \bar{X}_q^{l+2}$, thus $T^{l+1} = S_q^{l+1}$.

Now $\forall X^l \in L(S^{n-1})$, we calculate $\mu_{\mathscr{M}_n}(X^l)$ by induction.

When $h = n-1$, $S^{n-1} = \mathbb{R}^n \cap S^{n-1}$, so $\mu_{\mathscr{M}_n}(S^{n-1}) = -1 = \mu_{\mathscr{A}_n}(\mathbb{R}^n)$.

Suppose when $h > l$, it holds.

When $h = l$,

$$\begin{aligned}
\mu_{\mathscr{M}_n}(X^l) &= -\sum_{Z<X^l} \mu_{\mathscr{M}_n}(Z) \\
&= -\mu_{\mathscr{M}_n}(\bar{X}^{l+1}) + \left(\sum_{j=1}^{r} \mu_{\mathscr{M}_n}(S_j^{l+1}) + \sum_{j=1}^{r} \mu_{\mathscr{M}_n}(\bar{X}_j^{l+2})\right) \\
&\quad + \cdots + \left(\sum_{S\in I_\alpha} \mu_{\mathscr{M}_n}(S) + \sum_{X\in J_{\alpha+1}} \mu_{\mathscr{M}_n}(X)\right) + \cdots \\
&\quad + \left(\mu_{\mathscr{M}_n}(S^{n-1}) + \mu_{\mathscr{M}_n}(0)\right),
\end{aligned}$$

where $I_\alpha = \{S^\alpha \in [0, X^l] \cap L(S^{n-1})\}$, $J_\alpha = \{X^\alpha \in [0, X^l] \cap L(\mathscr{A}_n)\}$. I_α and J_α are finite set, $f(J_\alpha) = I_{\alpha-1}$ and $f|_{J_\alpha}$ is a bijection. So by the hypothesis of induction, the sum of every parentheses is 0. Hence

$$\mu_{\mathscr{M}_n}(X^l) = -\mu_{\mathscr{M}_n}(\bar{X}^{l+1}) = -\mu_{\mathscr{A}_n}(\bar{X}^{l+1}).$$

Last let's prove (3). By the same process of the proof of (2), $\forall X^l \in [0, Z^0) \cap L(S^{n-1})$, we have $\mu_{\mathscr{M}_n}(X^l) = -\mu_{\mathscr{A}_n}(\bar{X}^{l+1})$.

(a) If there is no hyperplane tangental to S^{n-1}.

First assume $Z^0 \in L(\mathscr{A}_n)$, i.e. $Y^k = Z^0$, then

$$\begin{aligned}
\mu_{\mathscr{M}_n}(Z^0) &= -\sum_{Z<Z^0} \mu_{\mathscr{M}_n}(Z) \\
&= -\sum_{\substack{X^l \in L(\mathscr{A}_n) \\ Z^0 \text{ cover } X^l}} \mu_{\mathscr{M}_n}(X^l) + \sum_{\alpha=k}^{n-1} \left(\sum_{S\in I_\alpha} \mu_{\mathscr{M}_n}(S) + \sum_{X\in J_{\alpha+1}} \mu_{\mathscr{M}_n}(X)\right) \\
&= -\sum_{\substack{Z^k \in L(\mathscr{A}_n) \\ Z^0 \text{ cover } X^l}} \mu_{\mathscr{A}_n}(X^l);
\end{aligned}$$

If $Z^0 \notin L(\mathscr{A}_n)$, then Y^k is the unique subspace covered by Z^0,

$$\mu_{\mathscr{M}_n}(Z^t) = -\mu_{\mathscr{A}_n}(Y^k),$$

So (i) hold.

(b) If there exists such a hyperplane H_{i_1} tangental to S^{n-1}. We consider $\mathscr{A}' = \{H_{i_1}, \cdots, H_{i_t}\}$, and $\mathscr{M}' = \mathscr{A}' \cup S^{n-1}$. It can be seen $L(\mathscr{M}')$ is a lattice and Z^0 is the maximum element $\hat{1}$. Let $\mathscr{A}'' = \{H_{i_2}, \cdots, H_{i_t}\}$ and $\mathscr{M}'' = \mathscr{A}'' \cup S^{n-1}$. Compare $L(\mathscr{M}'')$ and $L(\mathscr{M}')$, we'll find, in $L(\mathscr{M}')$, the abundant elements are all subspaces tangental to S^{n-1} at point Z^0, so it's apparent

$$\mu_{\mathscr{M}_n}(Z^0) = - \sum_{X \in [H_{i_1}, Z^0) \subseteq L(\mathscr{A}_n)} \mu_{\mathscr{A}_n}(X).$$

Therefore, we get (ii). ∎

Now the main result of [6] can be regarded as the corollary of theorem 3.2

Corollary 3.3. [6] If $\mathscr{A}_n$ is central. For $X^l \in L(\mathscr{M}_n)$, if $X^l \in L(\mathscr{A}^{n-1})$, then $\mu_{\mathscr{M}_n}(X^l) = \mu_{\mathscr{A}_n}(X^l)$; if $X^l \in L(S^{n-1})$, then there exists the unique $X^{l+1} \in L(\mathscr{A}_n)$, such that $X^l = S^l = S^{l-1} \cap X^{l+1}$ and $\mu_{\mathscr{M}_n}(X^l) = \mu_{\mathscr{A}_n}(X^{l+1})$.

3.2 The Mixed Arrangement with Subspaces and Multiple Spheres

Now we consider more general case, the mixed arrangement with subspaces and n-spheres.

Definition 3.4. [1] A subspace arrangement (or affine subspace arrangement) is a finite set

$$\mathscr{A} = \{K_1, \cdots, K_h\}$$

of affine proper subspaces K_i in real Euclidean space $\mathbb{R}^n$.

A subspace arrangement $\mathscr{A}_n$ will be called

(i) central, if all K_i are linear subspaces, i.e., if $0 \in K_i$,

(ii) simple, if $K_i \subseteq K_j$ implies $i = j$ for all $1 \le i, j \le h$,

(iii) pure, if $\dim(K_i) = \dim(K_j)$, for all $1 \le i, j \le h$,

(iv) d-dimensional, if $d = \max_{1 \le i \le h} \dim(K_i)$.

Here we let $\mathscr{S} = \{S_1, \cdots, S_p\}$ be the set of p different spheres of $n-1$ dimension and the intersection of any two spheres is $\emptyset$. Let $\mathscr{M} = \mathscr{A} \cup \mathscr{S}$ be the corresponding mixed arrangement.

The same as before, let $L(\mathscr{A})$ be the partial order set of all nonempty intersections of elements of $\mathscr{A}$, and $L(\mathscr{M})$ be the partial order set of $\mathscr{M}$. We use X^k to denote some $X \in L(\mathscr{M})$ of k dimension, and $\mu_{\mathscr{A}}$ to denote the Möbius function in $L(\mathscr{A})$.

Theorem 3.5. Let $L(\mathscr{S}) = \{X \cap S : X \in L(\mathscr{A}), S \in \mathscr{S}\} \setminus \{\emptyset\}$, then $L(\mathscr{M}) = L(\mathscr{A}) \cup L(\mathscr{S})$. $L(\mathscr{M})$ and $L(\mathscr{S})$ both are meet-semilattices.

Let O_i and r_i are the center and radius of S_i respectively, $1 \leq i \leq p$. For the set N of maximal elements in $L(\mathscr{A})$, we put them into the following parts:

(1) $N_1 = \{X^k \in N | \forall 1 \leq i \leq p, X^k \cap S_i = \emptyset$ for $k > 0$, or $||X^0 - O_i|| > r_i\}$,

(2) $N_2 = \{X^k \in N | \exists 1 \leq i \leq p$, such that X^k intersect with S_i for $k > 0$, or $0 \leq ||X^0 - O_i|| < r_i\}$,

(3) $N_3 = \{X^k \in N | \exists 1 \leq i \leq p$, such that X^k is tangental to S_i for $k > 0$, or $||X^0 - O_i|| = r_i\}$.

Note $N_2 \cap N_3$ may be not empty set.

Theorem 3.6. The Möbius function $\mu_{\mathscr{M}}(x)$ of $L(\mathscr{M})$ is following:

(1) For any $X^l \in L(\mathscr{A})$ for $l > 0$ and $X^0 \in N_1 \cup N_2$, $\mu_{\mathscr{M}}(X^l) = \mu_{\mathscr{A}}(X^l)$.

(2) If $Y^k \in N_2$ for $k > 0$, or Y^1 is covered by $X^0 \in N_2$ in $L(\mathscr{A}_n)$. Let $S^{k-1} = Y^k \cap S_i$, S^{k-1} is the maximal element in $L(\mathscr{M}_n)$, and for any $X^l \in [0, S^{k-1}] \cap L(S_i)$, $\mu_{\mathscr{M}}(X^l) = -\mu_{\mathscr{A}}(\bar{X}^{l+1})$, where $\bar{X}^{l+1} \in \mathscr{A}$ and $X^l = \bar{X}^{l+1} \cap S^{n-1}$.

(3) If $Y^k \in N_3$, suppose $Y^k = \bigcap_{j=1}^{t} H_{i_j}$, where $H_{i_j} \in \mathscr{A}$, for $1 \leq j \leq t$, and all the hyperplanes containing Y^k are here. Let $Z^0 = Y^k \cap S_i$, $\forall X^l \in [0, Z^0) \cap L(S_i)$, $\mu_{\mathscr{M}}(X^l) = -\mu_{\mathscr{A}}(\bar{X}^{l+1})$, where $X^l = \bar{X}^{l+1} \cap S_i$,

   (a) if there is no subspace belonging to $\mathscr{A}$ which is tangental to S_i,

$$\mu_{\mathscr{M}}(Z^0) = -\sum_{\substack{X^l \in L(\mathscr{A}) \\ Z^0 \text{ cover } X^l}} \mu_{\mathscr{A}}(X^l),$$

   (b) if there exists such a subspace, suppose H_{i_1}, such that $Z^0 \in H_{i_1}$ and H_{i_1} is tangental to S_i, then

$$\mu_{\mathscr{M}}(Z^0) = -\sum_{X \in [H_{i_1}, Z^0) \cap L(\mathscr{A}_n)} \mu_{\mathscr{A}}(X).$$

Proof. $\forall 1 \leq i < j \leq p$, $L(S_i) \cap L(S_j) = \emptyset$ and $L(\mathscr{M})$ doesn't change the Möbius function of the subspaces except for points sometimes, so we can calculate the Möbius function of $L(S_i)$ alone, just like we are dealing with the mixed arrangements with a single sphere. So by theorem 1.1, we know this theorem holds. ∎

4 The Deletion-Restriction Forumla

In this section, we let $\mathscr{M}_n = \mathscr{A}_n \cup \mathscr{S}$, where $\mathscr{A}_n$ is a hyperplane arrangement and $\mathscr{S}$ is the set of p different spheres of $n-1$ dimension with the intersection of any two spheres empty.

4.1 Characteristic Polynomial

Definition 4.1. Define the characteristic polynomial of $\mathscr{M}_n$ to be

$$\chi(\mathscr{M}_n, t) = \sum_{X \in L(\mathscr{M}_n)} \mu(X) t^{\dim(X)}.$$

Definition 4.2. If $X \in L(\mathscr{M}_n)$, define $\mathscr{M}_X \subseteq \mathscr{M}_n$ by

$$\mathscr{M}_X = \{Y \in \mathscr{M}_n | X \subseteq Y\}.$$

Also define $\mathscr{M}^X \subseteq \mathscr{M}_n$ by

$$\mathscr{M}^X = \{Y \cap X \neq \emptyset | Y \in \mathscr{M} \setminus \mathscr{M}_X\}.$$

Choose $X \in \mathscr{M}_n$, let $\mathscr{M}' = \mathscr{M}_n \setminus \{X\}, \mathscr{M}'' = \mathscr{M}^X$, and we call $(\mathscr{M}_n, \mathscr{M}', \mathscr{M}'')$ is a triple of the mixed arrangement $\mathscr{M}_n$.

Lemma 4.3. [4] Let L be a finite lattice. Let X be a subset of L such that $\hat{0} \notin X$, and such that if $y \in L$, $y \neq \hat{0}$, then some $x \in X$ satisfies $x \leq y$. Let N_k be the number of k-element subsets of X with join $\hat{1}$. Then

$$\mu_L(\hat{0}, \hat{1}) = N_0 - N_1 + N_2 - \cdots.$$

Lemma 4.4. Let $\mathscr{M}_n$ be an arrangement in $\mathbb{R}^n$, then

$$\chi_{\mathscr{M}_n}(t) = \sum_{\substack{\mathscr{B} \subseteq \mathscr{M}_n \\ L(\mathscr{B}) \text{ has } \hat{1}_{\mathscr{B}}}} (-1)^{|\mathscr{B}|} t^{\dim(\hat{1}_{\mathscr{B}})}, \tag{4.1}$$

where $\hat{1}_{\mathscr{B}}$ is the minimum element of $L(\tilde{\mathscr{B}})$.

Proof. $\forall X \in \mathscr{M}_n$, by lemma 4.3, we have

$$\mu_{\mathscr{M}_n}(X) = \sum_k (-1)^k N_k(X),$$

where $N_k(X)$ is the number of k-subsets of $\mathscr{M}_n \setminus \mathscr{M}_X$ with join X. In other words,

$$\mu_{\mathscr{M}_n}(X) = \sum_{\substack{\mathscr{B} \subseteq \mathscr{M}_n \setminus \mathscr{M}_X \\ X = \bigcap_{Y \in \tilde{\mathscr{B}}} Y}} (-1)^{|\tilde{\mathscr{B}}|}.$$

Note X is the minimum element $\hat{1}_{\tilde{\mathscr{B}}}$ of $\tilde{\mathscr{B}}$, now multiply both sides by $t^{\dim(X)}$ and sum over X to obtain the equation. ∎

Theorem 4.5. Let $(\mathscr{M}_n, \mathscr{M}', \mathscr{M}'')$ is a triple of $\mathscr{M}_n$, then

$$\chi_{\mathscr{M}_n}(t) = \chi_{\mathscr{M}'}(t) - \chi_{\mathscr{M}''}(t)$$

Proof. Let $X \in \mathscr{M}_n$ define the triple $(\mathscr{M}_n, \mathscr{M}', \mathscr{M}'')$. The equation (4.1) can be regarded as the sum of the following two parts,

$$\sum_{\substack{X \notin \tilde{\mathscr{B}} \subseteq \mathscr{M}_n \\ L(\tilde{\mathscr{B}}) \text{ has } \hat{1}_{\tilde{\mathscr{B}}}}} (-1)^{|\tilde{\mathscr{B}}|} t^{\dim(\hat{1}_{\tilde{\mathscr{B}}})}, \tag{4.2}$$

and

$$\sum_{\substack{X \in \tilde{\mathscr{B}} \subseteq \mathscr{M}_n \\ L(\tilde{\mathscr{B}}) \text{ has } \hat{1}_{\tilde{\mathscr{B}}}}} (-1)^{|\tilde{\mathscr{B}}|} t^{\dim(\hat{1}_{\tilde{\mathscr{B}}})}, \tag{4.3}$$

It's clear the formula (4.2) is just $\chi_{\mathscr{M}'}(t)$. In formula (4.3), let $\tilde{\mathscr{C}} = (\tilde{\mathscr{B}} \setminus \{X\})^X$, so $\dim(\hat{1}_{\tilde{\mathscr{C}}}) = \dim(\hat{1}_{\tilde{\mathscr{B}}})$ and the formula (4.3) is equal to

$$\sum_{\substack{\tilde{\mathscr{C}} \subseteq \mathscr{M}^X \\ L(\tilde{\mathscr{C}}) \text{ has } \hat{1}_{\tilde{\mathscr{C}}}}} (-1)^{|\tilde{\mathscr{C}}|+1} t^{\dim(\hat{1}_{\tilde{\mathscr{C}}})} = -\chi_{\mathscr{M}''}(t).$$

∎

4.2 Regions

Definition 4.6. A region R of a mixed arrangement $\mathscr{M}_n$ is a connected component of the complement X of the hyperplanes and spheres:

$$X = \mathbb{R}^n - \bigcup_{H \in \mathscr{A}_n} H - \bigcup_{S^{n-1} \in \mathscr{S}} S^{n-1}.$$

Let $\mathscr{R}(\mathscr{M}_n)$ denote the set of regions of $\mathscr{M}_n$, and let

$$r(\mathscr{M}_n) = |\mathscr{R}(\mathscr{M}_n)|$$

be the number of regions.

Definition 4.7. Let $(\mathscr{M}_n, \mathscr{M}', \mathscr{M}'')$ is a triple of $\mathscr{M}_n$, where $X = S_i \in \mathscr{S}$, for some $1 \le i \le p$, define

$$\mathscr{R}(\mathscr{M}'') = \mathbb{S}^{n-1} - \bigcup_{Y \in \mathscr{M}''} Y.$$

Now let $n > 2$. We consider the mixed arrangement $\mathscr{M}$ composed of subspace arrangement $\mathscr{A}$ and spheres $\mathscr{S}$, where $\mathscr{S} = \{S_1, \cdots, S_p\}$ is the set of p different spheres of $n-1$ dimension and the intersection of any two spheres is $\emptyset$.

Since the subspaces whose dimensions are less than $n-1$ don't create regions in the n-dimensional space, we only need to consider the mixed arrangement $\mathscr{M}_n = \mathscr{A}_n \cup \mathscr{S}$, where $\mathscr{A}_n$ is the hyperplane arrangement composed of the hyperplanes in $\mathscr{A}$.

Theorem 4.8. Assume $n > 2$ and $(\mathscr{M}_n, \mathscr{M}', \mathscr{M}'')$ is a triple of $\mathscr{M}_n$, then

$$r(\mathscr{M}_n) = r(\mathscr{M}') + r(\mathscr{M}'').$$

Proof. Note that $r(\mathscr{M}_n)$ equals $r(\mathscr{M}')$ plus the number of regions of $\mathscr{M}_n$ cut into two regions by X (which is not always right for $n \leq 2$). Let R' be such a region of $\mathscr{M}'$, then $R' \cap X \in \mathscr{R}(\mathscr{M}'')$. Conversely, if $R'' \in \mathscr{R}(\mathscr{M}_n)$, then points near R'' on either side of X belong to the same region $R' \in \mathscr{R}(\mathscr{M}')$, since any $Y \in \mathscr{M}'$ separating them would intersect R''. Thus R' is cut into two parts by X. We have got a bijection between regions of $\mathscr{M}'$ cut into two parts by X and regions of $\mathscr{M}''$. We get the equation. ∎

The above theorem isn't always right for the case $n \leq 2$. For example, we consider $\mathscr{M}_2 = \{A, S\}$, where A is a line, which intersects with the circle S in a plane. Then $r(\mathscr{M}_2) = 4$, but $r(\mathscr{M}') = 2$ and $r(\mathscr{M}'') = 3$ if we let $X = A$. and the counter-example for $n = 1$ is easy to be found. But we have the two following conclusions for $n = 1, 2$.

Theorem 4.9. Suppose $n = 2$, if X satisfies that

(a) $X \in \mathscr{A}_2$,

(b) let $\mathscr{S}' \subseteq \mathscr{S}$ be the set of the spheres touching with X, and m spheres in $\mathscr{S}'$ only touch with X, where $1 \leq m \leq |\mathscr{S}'|$,

then

$$r(\mathscr{M}_2) = r(\mathscr{M}') + r(\mathscr{M}'') - m,$$

else

$$r(\mathscr{M}_2) = r(\mathscr{M}') + r(\mathscr{M}'').$$

Proof. Let $|\mathscr{S}'| = k$ and $\mathscr{S}' = \{S_{q_1}, \ldots, S_{q_k}\}$. Let $\mathscr{S}^{(1)} \subseteq \mathscr{S}'$ be the set of spheres which intersect with X, and $\mathscr{S}^{(1)} = \mathscr{S}' \setminus \mathscr{S}^{(1)}$ be the spheres which is tangental to X. If X satisfies (a) and (b), without loss of generality, assume these m spheres are $S_{q_1}, \ldots, S_{q_m}$.

(i)If for some $1 \leq i \leq m$, $S_{q_i} \in \mathscr{S}^{(1)}$, then let $S_{q_i} \cap X = \{x_i, y_i\} \subseteq \mathscr{M}''$. $\forall X' \in \mathscr{M}_2$, let $X' \cap X = a$. $a_i = \max(\{a | a < x_i, a = X' \cap X, X' \in \mathscr{A}_2 \cup \mathscr{S}^{(1)}\} \cup \{-\infty\})$, $b_i = \min(\{a | a > y_i, a = X' \cap X, X' \in \mathscr{A}_2 \cup \mathscr{S}^{(1)}\} \cup \{+\infty\})$. When $a_i = -\infty$ ($b_j = +\infty$), it means there are no lines intersecting with X at the left (right) region of x_i (y_i). So when we travel along X from negative infinity to positive infinity, we'll find X doesn't create a new region in $\mathscr{M}$ when through the line segment $[a_i, x_i]$, but it does when through $[b_i, y_i]$, i.e., $[a_i, x_i]$ and $[y_i, b_i]$ cut a region of $\mathscr{M}'$ into two parts together.

(ii)If for some $1 \leq j \leq m$, $S_{q_j} \in \mathscr{S}^{(2)}$, then let $S_{q_j} \cap X = \{x_j\} \subseteq \mathscr{M}''$. Set $c_j = \max(\{a | a < x_j, a = X' \cap X, X' \in \mathscr{M} \setminus \{S_{q_j}\}\} \cup \{-\infty\})$ and $d_j = \min(\{a | a > x_j, a = X' \cap X, X' \in \mathscr{M} \setminus \{S_{q_j}\}\} \cup \{+\infty\})$. Then it's obvious x_j cuts $[c_j, d_j]$ into two parts, i.e., x_j creates a new region in $\mathcal{M}''$, but $[c_j, d_j]$ doesn't create a new one in $\mathcal{M}$, because it can be contained in some $[a_i, x_i]$ or $[x_i, b_i]$ for some $S_{q_i} \in \mathcal{S}^{(1)}$. According to theorem 4.8, we can establish a bijection between regions of $\mathcal{M}'$ cut into two parts by X and $\mathcal{R}(\mathcal{M}'') \setminus (\{[a_i, x_i] | S_{q_i} \in \mathcal{S}^{(1)}\} \cup \{[c_j, x_j] | S_{q_j} \in \mathcal{S}^{(2)}\})$, so

$$r(\mathcal{M}_2) = r(\mathcal{M}') + r(\mathcal{M}'') - |\mathcal{S}^{(1)}| - |\mathcal{S}^{(2)}|,$$

i.e.

$$r(\mathcal{M}_2) = r(\mathcal{M}') + r(\mathcal{M}'') - m,$$

If X doesn't satisfy the two conditions, i.e. X doesn't touch any spheres, or $m = 0$, or $X \in \mathcal{S}$, then we can prove

$$r(\mathcal{M}_2) = r(\mathcal{M}') + r(\mathcal{M}''),$$

just the same as we prove Theorem 4.8.

∎

The following theorem is obvious.

Theorem 4.10. When $n = 1$, for any $X \in \mathcal{M}_1$, and some i

(i) If $X = S_i$,

$$r(\mathcal{M}_1) = r(\mathcal{M}') + 2 - |\mathcal{M}''|,$$

(ii) If $X = A_i$,

$$r(\mathcal{M}_1) = r(\mathcal{M}') + 1 - |\mathcal{M}''|.$$

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