On the Laplacian-Energy-Like of graphs

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Abstract: The Laplacian-energy-like graph invariant of a graph G, is defined as $LEL(G) = \sum_{i=1}^{n} \sqrt{\mu_i}$, where μ_i are the Laplacian eigenvalues of graph G. In this paper, we study the maximum LEL among graphs with given vertices and matching number. Some results on $LEL(G)$ and $LEL(\overline{G})$ are obtained.

Key words: Laplacian spectrum; Laplacian-energy-like; Matching; Complement graph

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*Corresponding author. This work was supported by the NNSF of China (No. 11226283, 11071088) and the Zhujiang Technology New Star Foundation of Guangzhou (2011J2200090). E-mail address: youzhf@hotmail.com (Z. You).

1. Introduction

Let G be a simple graph with n vertices and m edges. The Laplacian matrix of G is defined as $L = D - A$, where A is the adjacency matrix of G and $D = diag(d_1, d_2, \ldots, d_n)$ the diagonal matrix of vertex degrees. The spectrum of G is the spectrum of its adjacency matrix, and consists of the values $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The Laplacian spectrum of G is the spectrum of its Laplacian matrix, and consists of the values $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$.

The energy of a graph G is defined as

$$E(G) = \sum_{i=1}^{n} |\lambda_i|.$$

This quantity, introduced by I. Gutman in 1978 [5], has a lot of chemical applications. The mathematical properties of graph energy can be found in the review [6].

The Laplacian energy of a graph is defined as [9]

$$LE(G) = \sum_{i=1}^{n} |\mu_i - \frac{2m}{n}|.$$

Similar to graph energy and Laplacian energy, the Laplacian-energy-like invariant of G, denoted by $LEL(G)$, has recently been defined as [11]

$$LEL(G) = \sum_{i=1}^{n} \sqrt{\mu_i}.$$

J. Liu and the second author[11] showed that the Laplacian-energy-like shares a number of properties with the usual graph energy. In [15], D. Stevanović et al. proved that: LEL is as good as the Randić index (a connectivity index) and better than the Wiener index (a distance based index). D. Stevanović [14] exhibited further similarities between them by showing that among the n-vertex trees, the star S_n has minimal Laplacian-energy-like and the path P_n has maximal LEL value.

In 1993, Klein and Randić [10] introduced resistance distance based on the electrical network theory. The Kirchhoff index [1] is defined $Kf(G) = \sum_{i<j} r_{ij}$, where r_{ij} is the resistance distance between vertices v_i and v_j as computed with Ohm's law where all the edges of G are considered to the unit resistors. The Kirchhoff index is an important structure descriptor. For a connected graph G with $n \geq 2$ vertices, it has been proven [8,20] that $Kf(G) = n\sum_{i=1}^{n-1} \frac{1}{\mu_i}$. It is interesting that when $Kf(G)$ attains the minimum value among some graphs and $LEL(G)$ has the maximum value. For example, B. Zhou et al. [17] proved that: For a connected graph G with $n \geq 2$ vertices and connectivity κ, $Kf(G) \geq Kf((K_1 \cup K_{n-k-1}) \vee K_\kappa)$, and the extremal graph with the minimum Kf is determined among connected graphs with n vertices and chromatic number χ. In [19], B. Zhu showed that the above extremal graphs attain the maximum LEL among the respective graphs.

In [13], Nordhaus and Gaddum obtained bounds for the sum of the chromatic numbers of a graph and its complement. Let $\overline{G}$ be the complement of the graph G and $I(G)$ be some invariant of G. Then the relations on $I(G)+I(\overline{G})$ are said to of Nordhaus-Gaddum-type.

The matching number of the graph G, denoted $\beta(G)$, is the number of edges in a maximum matching.

In this paper, we study the maximum LEL among graphs with n vertices and matching number. Nordhaus-Gaddum type bounds for LEL are obtained. Moreover, results on the difference and comparison of $LEL(G)$ and $LEL(\overline{G})$ are presented.

2. The maximum Laplacian-energy-like with given vertices and matching number

Lemma 2.1 [10] *Let G be a non-complete connected graph and $G^* = G + e$. Then $Kf(G^*) < Kf(G)$.*

Lemma 2.2 [16] *For a non-complete graph G and $G^* = G + e$, we have $LEL(G^*) > LEL(G)$.*

Noting that the inverse effect of Lemmas 2.1 and 2.2, by using the similar way in the proof of Proposition 2 of [18], we obtain the inverse extremal graphs about Kf and LEL among graphs with given vertices and matching number.

$G \cong S_n$ or $G \cong K_3$ when $\beta = 1$ for a connected graph with $n \geq 2$ vertices.

Proposition 2.3 *Let G be a connected graph with n vertices and matching number β, $2 \leq \beta \leq \lfloor \frac{n}{2} \rfloor$.*

(1) if $\beta = \lfloor \frac{n}{2} \rfloor$, then $LEL(G) \leq (n-1)\sqrt{n}$ with equality if and only if $G \cong K_n$.

(2) if $2 \le \beta \le \lfloor \frac{n}{2} \rfloor - 1$, then $LEL(G) \le s\sqrt{n} + \sum_{i=1}^{t}(n_i - 1)\sqrt{s+n_i} + (t-1)\sqrt{s}$ with equality if and only if $G \cong K_s \vee (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_t})$, where s is the order of subgraph X of G, t is the number of odd components of $G - X$ and n_i is the order of respective components.

Proposition 2.4 *Let G be a connected graph with n vertices and matching number β ($2 \le \beta \le \lfloor \frac{n}{2} \rfloor$), then $LEL(G) \le s\sqrt{n} + (2\beta - 2s)\sqrt{2\beta - s + 1} + (n - 2\beta + s - 1)\sqrt{s}$, where the equality holds if and only if $G \cong K_s \vee (\overline{K_{n-2\beta+s-1}} \cup K_{2\beta-2s+1})$ and $s \le \beta$.*

Proof. Suppose G is the extremal graph in (2) of Proposition 2.3. Then $LEL(G) = s\sqrt{n} + \sum_{i=1}^{t}(n_i - 1)\sqrt{n_i + s} + (t-1)\sqrt{s}$.

Let $f(x) = (x-1)\sqrt{s+x} + (m-x-1)\sqrt{s+m-x}$ for $x < m - x$. Consider the first derivative of $f(x)$, $f'(x) = \sqrt{s+x} - \sqrt{s+m-x} + \frac{1}{2}(\frac{x-1}{\sqrt{s+x}} - \frac{m-x-1}{\sqrt{s+m-x}})$.

Let $g(x) = \frac{x-1}{\sqrt{s+x}} - \frac{m-x-1}{\sqrt{s+m-x}}$. Then $g'(x) = \frac{1}{\sqrt{s+x}} - \frac{x-1}{2\sqrt{s+x}(s+x)} + \frac{1}{\sqrt{s+m-x}} - \frac{m-x-1}{2\sqrt{s+m-x}(s+m-x)}$. Obviously, $g'(x) > 0$. Then $g(x) \le g(\frac{m}{2}) = \frac{\frac{m}{2}-1}{\sqrt{s+\frac{m}{2}}} - \frac{m-\frac{m}{2}-1}{\sqrt{s+\frac{m}{2}}} = 0$. Then $f'(x) < 0$ and $f(x)$ is a decreasing function on x. Hence $f(n_i) < f(n_i - 2)$, i.e.,

$$(n_i - 1)\sqrt{s+n_i} + (m - n_i - 1)\sqrt{s+m-n_i}$$
$$< (n_i - 3)\sqrt{s+n_i-2} + (m - n_i + 1)\sqrt{s+m-n_i+2}.$$

Since $x < m - x$, we take $m = n_i + n_j$. The above inequality can be transformed to

$$(n_i - 1)\sqrt{s+n_i} + (n_j - 1)\sqrt{s+n_j} < (n_i - 3)\sqrt{s+n_i-2} + (n_j + 1)\sqrt{s+n_j+2}.$$

Thus by replacing any pair (n_i, n_j) with $3 \le n_i \le n_j - 2$ by the pair $(n_i - 2, n_j + 2)$ in the sum $\sum_{i=1}^{t}(n_i - 1)\sqrt{n_i + s}$, we increase the sum. By repeating this process, we find $\sum_{i=1}^{t}(n_i - 1)\sqrt{n_i + s}$ with $\sum_{i=1}^{t} = n-s$ and $1 \le n_1 \le \cdots \le n_t$ is maximum if and only if $n_1 = \cdots = n_{t-1} = 1$ and $n_t = n - s - t + 1 = 2\beta - 2s + 1$. Then $G \cong K_s \vee (\overline{K_{n-2\beta+s-1}} \cup K_{2\beta-2s+1})$ and $LEL(G) = s\sqrt{n} + (2\beta - 2s)\sqrt{2\beta - s + 1} + (n - 2\beta + s - 1)\sqrt{s}$.

□

Remark 2.5 In Theorem 2.4, we have not determined the maximum value about s. The monotonicity of function $f(s) := s\sqrt{n} + (2\beta - 2s)\sqrt{2\beta - s + 1} + (n - 2\beta + s - 1)\sqrt{s}$ depends on not only s but β. Consider $f(s + 1) - f(s) = \sqrt{n} + (2\beta - 2s - 2)(\sqrt{2\beta - s} - \sqrt{2\beta - s + 1}) - 2\sqrt{2\beta - s + 1} + (n + s - 2\beta - 1)(\sqrt{s + 1} - \sqrt{s}) + \sqrt{s + 1}$.

For example, let $s = 1$ and $\beta = \frac{n}{4}$, then

$$\begin{aligned} f(2) - f(1) &= \sqrt{n} + (\tfrac{n}{2} - 4)(\sqrt{\tfrac{n}{2} - 1} - \sqrt{\tfrac{n}{2}}) - 2\sqrt{\tfrac{n}{2}} + \tfrac{n}{2}(\sqrt{2} \\ &\quad -1) + \sqrt{2} \\ &\ge -(\sqrt{2}-1)\sqrt{n} - (\tfrac{n}{2}-4)\frac{1}{2\sqrt{\frac{n}{2}-1}} + \tfrac{n}{2}(\sqrt{2}-1) + \sqrt{2} \\ &\ge \tfrac{\sqrt{2}-1}{2}n - \tfrac{5\sqrt{2}-4}{4}\sqrt{n} + \tfrac{3}{2}\frac{1}{\sqrt{\frac{n}{2}-1}} + \sqrt{2}. \end{aligned}$$

Then $f(2) - f(1) > 0$ for $n \ge 9$.

Let $s = 1$ and $\beta = \frac{n}{2} - 2$. Then $f(2) - f(1) = \sqrt{n} - 2\sqrt{n - 4} + \sqrt{2} + \frac{4}{\sqrt{2}+1} - \frac{n-8}{\sqrt{n-5}+\sqrt{n-4}}$. Hence $f(2) - f(1) < 0$ for $n \ge 9$.

For example, let $\beta = \frac{n}{2} - 4$ and $s = \beta - 1$. Then

$$\begin{aligned} f(s+1) - f(s) &= \sqrt{n} - 2\sqrt{\beta + 2} + (n - \beta - 2)(\sqrt{\beta} - \sqrt{\beta - 1}) + \sqrt{\beta} \\ &= \sqrt{n} - 2\sqrt{\tfrac{n}{2} - 2} + (\tfrac{n}{2} + 2)(\sqrt{\tfrac{n}{2} - 4} - \sqrt{\tfrac{n}{2} - 5}) \\ &\quad + \sqrt{\tfrac{n}{2} - 4} \end{aligned}$$

$$\geq \sqrt{n} - \sqrt{2n-8} + (\tfrac{n}{2}+2)(\tfrac{1}{2\sqrt{\frac{n}{2}-4}}) + \sqrt{\tfrac{n}{2}-4}$$
$$= \sqrt{n} - \sqrt{2n-8} + \tfrac{3}{2}\sqrt{\tfrac{n}{2}-4} + \tfrac{3}{\sqrt{\frac{n}{2}-4}} > 0.$$

Let $\beta = \frac{n}{2} - 4$ and $s = 2$. Then

$$f(s+1) - f(s) = \sqrt{n} + (n-14)(\sqrt{n-10} - \sqrt{n-9}) - 2\sqrt{n-9} + 9(\sqrt{3}-\sqrt{2}) + \sqrt{3}$$
$$\leq \sqrt{n} - \tfrac{n-14}{\sqrt{n-9}+\sqrt{n-10}} - 2\sqrt{n-9} + 5 < 0 \text{ for } n \geq 47.$$

3. Nordhaus-Gaddum-Type bounds for Laplacian-energy-like

Lemma 3.1 [12] *Let G be a graph with order n and μ_i $(i = 1, \ldots, n)$ be the Laplacian eigenvalues. Then the Laplacian eigenvalues of $\overline{G}$ are $n - \mu_{n-1}$, $n - \mu_{n-2}$, $\ldots$, $n - \mu_1$, 0.*

Theorem 3.2 *Let G be a graph with n vertices. Then $LEL(G) + LEL(\overline{G}) \geq (n-1)\sqrt{n}$, where the equality holds if and only if $G \cong K_n$ or $G \cong \overline{K_n}$.*

Proof. By Lemma 3.1,

$$LEL(G) + LEL(\overline{G}) = \sum_{i=1}^{n-1} \sqrt{\mu_i} + \sum_{i=1}^{n-1} \sqrt{n-\mu_i}$$
$$= \sqrt{\mu_1} + \sqrt{n-\mu_1} + \cdots + \sqrt{\mu_{n-1}} + \sqrt{n-\mu_{n-1}}$$
$$\geq \sqrt{\mu_1 + n - \mu_1} + \cdots + \sqrt{\mu_{n-1} + n - \mu_{n-1}}$$
$$= \sqrt{n} + \cdots + \sqrt{n} = (n-1)\sqrt{n}.$$

Equality holds if and only if $\mu_i = 0$ or $n - \mu_i = 0$ for some $i \in \{1, \ldots, n-1\}$.

Note that there is at least a connected graph between G and $\overline{G}$.

If G is connected, then $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} > 0$. Thus

$n-\mu_1=\cdots=n-\mu_{n-1}=0$, i.e., $\mu_1=\cdots=\mu_{n-1}=n$, $\mu_n=0$. Then $G\cong K_n$.

If G is disconnected, then $\overline{G}$ is connected. By similar arguments, we have $\overline{G}\cong K_n$. Then $G\cong\overline{K_n}$. □

Lemma 3.3 [4] *Let G be a graph with at least one edge and maximum vertex degree Δ. Then $\mu_1\geq 1+\Delta$ with equality for connected graph if and only if $\Delta=n-1$.*

Lemma 3.4 [16] *Let G be a connected graph with $n\geq 2$. Then $\mu_2=\cdots=\mu_{n-1}$ and $\mu_1=1+\Delta$ if and only if $G\cong K_n$ or $G\cong K_{1,n-1}$.*

Theorem 3.5 *Let G be a graph with n vertices and m edges. Denote Δ by the maximum vertex degree of G. Then*

$$LEL(G)+LEL(\overline{G})\leq\sqrt{1+\Delta}+\sqrt{n-2}\sqrt{2m-1-\Delta}+$$
$$\sqrt{n-1-\Delta}+\sqrt{n-2}\sqrt{n(n-2)-2m+1+\Delta}$$
$$\leq\sqrt{n}+\sqrt{n-2}(\sqrt{2m-n}+\sqrt{n(n-1)-2m})$$

with equalities hold if and only if $G\cong K_{1,n-1}$, or $G\cong K_n$.

Proof. The function $y=-x^{\frac{1}{2}}$ is a strictly convex function.

Then $\sum_{i=2}^{n-1}\frac{1}{n-2}\mu_i^{\frac{1}{2}}\leq(\sum_{i=2}^{n-1}\frac{1}{n-2}\mu_i)^{\frac{1}{2}}$ and

$LEL(G)-\sqrt{\mu_1}\leq\sqrt{n-2}\sqrt{2m-\mu_1}$.

Thus $LEL(G)\leq\sqrt{\mu_1}+\sqrt{n-2}\sqrt{2m-\mu_1}$. (1)

Similarly, we have $\sum_{i=2}^{n-1}\frac{1}{n-2}(n-\mu_i)^{\frac{1}{2}}\leq\left[\sum_{i=2}^{n-1}\frac{1}{n-2}(n-\mu_i)\right]^{\frac{1}{2}}$.

And $LEL(\overline{G})\leq\sqrt{n-\mu_1}+\sqrt{n-2}\sqrt{n(n-2)-2m+\mu_1}$.(2)

By (1) and (2),

$LEL(G)+LEL(\overline{G})\leq\sqrt{\mu_1}+\sqrt{n-2}\sqrt{2m-\mu_1}+\sqrt{n-\mu_1}+$

$$\sqrt{n-2}\sqrt{n(n-2)-2m+\mu_1}.$$

Let $f(x)=\sqrt{x}+\sqrt{n-2}\sqrt{2m-x}+\sqrt{n-x}+$
$$\sqrt{n-2}\sqrt{n(n-2)-2m+x}.$$

Then $f'(x)=\frac{1}{2}\left(\sqrt{\frac{1}{x}}-\sqrt{\frac{n-2}{2m-x}}\right)+\frac{1}{2}\left(-\sqrt{\frac{1}{n-x}}+\sqrt{\frac{n-2}{n(n-2)-2m+x}}\right)$.

Note that $n\Delta\geq 2m$. By Lemma 3.3, then $\mu_1\geq 1+\Delta\geq 1+\dfrac{2m}{n}=\dfrac{2m+n}{n}\geq\dfrac{2m}{n-1}$. It is easily verify that $\sqrt{\dfrac{1}{x}}-\sqrt{\dfrac{n-2}{2m-x}}\leq 0$ and $-\sqrt{\dfrac{1}{n-x}}+\sqrt{\dfrac{n-2}{n(n-2)-2m+x}}\leq 0$, i.e., $f'(x)\leq 0$ for $x\geq\dfrac{2m}{n-1}$. Thus $f(x)$ is a decreasing function for $x\geq\dfrac{2m}{n-1}$.

The equalities (1) and (2) hold if and only if $\mu_2=\cdots=\mu_{n-1}$ and $\mu_1=\Delta+1$, by Lemma 3.4, i.e., $G\cong K_{1,n-1}$, or $G\cong K_n$.

The theorem follows. □

4. The Laplacian-energy-like of G and $\overline{G}$

By Lemma 2.2, we know that $LEL(G-e)<LEL(G)$. Then $LEL(\overline{G-e})>LEL(\overline{G})$ holds. In this section, we study some special graphs with $LEL(G)=LEL(\overline{G})$.

Lemma 4.1 [11] *Let G be a simple graph with n vertices and m edges. Then $LEL(G)\leq\sqrt{2}m$.*

Lemma 4.2 [7] *Let G be a simple graph with n vertices and m edges. Then $LEL(G)\geq\frac{2m}{\sqrt{n}}$ with equality if and only if $G\cong K_n$ or $G\cong K_2$.*

Theorem 4.3 *Let G be a simple graph with n vertices and m edges. If $m<\frac{n(n-1)}{2+\sqrt{2n}})$, then $LEL(G)<LEL(\overline{G})$.*

Proof. Let $\overline{m}$ denote the number of edges for $\overline{G}$. By Lemmas 4.1 and 4.2, if $\frac{2\overline{m}}{\sqrt{n}} > \sqrt{2}m$, then $LEL(\overline{G}) > LEL(G)$. This inequality can be transformed to $\frac{n(n-1)-2m}{\sqrt{n}} > \sqrt{2}m$. Then $m < \frac{n(n-1)}{2+\sqrt{2n}}$. □

By Theorem 4.3, $m = n-1 < \frac{n(n-1)}{2+\sqrt{2n}}$ holds for $n \geq 6$.

Corollary 4.4 *Let T be a tree with $n \geq 6$ vertices. Then $LEL(T) < LEL(\overline{T})$.*

Corollary 4.5 *Let T be a tree with n vertices. Then $LEL(T) = LEL(\overline{T})$ if and only if $T \cong P_4$.*

Proof. By Corollary 4.4, we only need to calculate the LEL-value of trees with $n \leq 5$.

(1) $n = 2$. $LEL(P_2) = \sqrt{2} > 0 = LEL(\overline{P_2})$.

(2) $n = 3$. $LEL(P_3) = \sqrt{3} + 1 > \sqrt{2} = LEL(\overline{P_3})$.

(3) $n = 4$. There are two cases. Since $P_4 \cong \overline{P_4}$, $LEL(P_4) = LEL(\overline{P_4})$. And $LEL(S_4) = 4 > 2\sqrt{3} = LEL(\overline{S_4})$.

(4) $n = 5$. There are three cases. Note that $LEL(S_5) = \sqrt{5} + 3 < 6 = LEL(\overline{S_5})$, and $LEL(P_5) \doteq 6.77 > 5.314 \doteq LEL(\overline{P_5})$. We have $LEL(T^*) \doteq 6.668 > 5.282 \doteq LEL(\overline{T^*})$, where T^* is the tree obtained by attaching an isolated vertex to one of pendent vertices of S_4.

By all cases exhausted, the proof is completed. □

Note that $m = n < \frac{n(n-1)}{2+\sqrt{2n}}$ holds for $n \geq 7$.

Corollary 4.6 *Let T be a unicyclic graph with $n \geq 7$ vertices. Then $LEL(U) < LEL(\overline{U})$.*

Corollary 4.7 *Let U be a unicyclic graph with n vertices. Then*

$LEL(U) = LEL(\overline{U})$ *if and only if* $U \cong U_i$ $(i = 1, 2)$ (*Figure 1*).

Proof. By Corollary 4.6, we only need to calculate the LEL-value of unicyclic graphs with $n \leq 6$.

(1) $n = 3, 4, 5$. By [2] (Table 1) and direct calculations, only two unicyclic graphs with $LEL(U_i) = LEL(\overline{U_i})$ $(i = 1, 2)$.

(2) $n = 6$. By [3] (Table 1), there are 13 unicyclic graphs and $LEL(U) \neq LEL(\overline{U})$.

The result follows. □

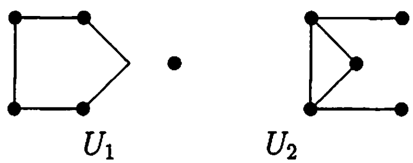

Figure 1. Unicyclic graphs with $LEL(U) = LEL(\overline{U})$

5. Difference of the Laplacian-energy-like between G and $\overline{G}$

In Section 3, the sum of $LEL(G)$ and $LEL(\overline{G})$ has been studied. In this section, we obtain some results on the difference of $LEL(G)$ and $LEL(\overline{G})$.

Theorem 5.1 *Let* G *be a connected graph with* n *vertices. Then* $LEL(G) - LEL(\overline{G}) \leq (n-1)\sqrt{n}$ *with equality holds if and only if* $G \cong K_n$.

Proof. Among connected graphs with n vertices, K_n attains the maximum LEL-value. And $\overline{K_n}$ is the n-vertex graph with minimum LEL-value. Then $LEL(G) - LEL(\overline{G}) \leq LEL(K_n) - LEL(\overline{K_n}) = (n-1)\sqrt{n}$. Obviously, the equality holds if and only if $LEL(\overline{K_n}) = 0$, i.e., $G \cong K_n$. □

Theorem 5.2 *Let $G \ncong K_n$ be a connected graph with n vertices. Then $LEL(G) - LEL(\overline{G}) \leq (n-2)\sqrt{n} + \sqrt{2}$ with equality holds if and only if $G \cong K_n - e$.*

Proof. Among connected graphs with n vertices, $K_n - e$ attains the second maximum LEL-value. And $\overline{K_n - e}$ has the second minimum LEL-value. Then $LEL(G) - LEL(\overline{G}) \leq LEL(K_n - e) - LEL(\overline{K_n - e}) = (n-2)\sqrt{n} - \sqrt{2}$.

Obviously, if $G \cong K_n - e$, then $LEL(G) - LEL(\overline{G}) = (n-2)\sqrt{n} - \sqrt{2}$.

Since $G \ncong K_n$, $\overline{G}$ has at least an edge. Then $LEL(\overline{G}) \geq LEL(K_2) = \sqrt{2}$ and $LEL(G) - LEL(\overline{G}) = (n-2)\sqrt{n} - \sqrt{2}$. Thus $LEL(G) \geq (n-2)\sqrt{n}$. By Lemma 2.2, for $G \ncong K_n$ and $G \ncong K_n - e$, $LEL(G) < (n-2)\sqrt{n}$. Then $G \cong K_n - e$. □

Theorem 5.3 *Let G be a connected graph with n vertices and $\mu_1 = n$. Then $LEL(G) - LEL(\overline{G}) \geq \sqrt{n} + (n-2)(1 - \sqrt{n-1})$ with equality holds if and only if $G \cong S_n$.*

Proof. For a connected graph G with $\mu_1 = n$, $\overline{G}$ is disconnected. Suppose $\overline{G}$ has k components and ith component with order n_i. Note that $n_i \leq n-1$ and $n_i \geq 2$. By Lemma 2.2,

$$\begin{aligned} LEL(\overline{G}) &\leq LEL(K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_k}) \\ &= (n_1 - 1)\sqrt{n_1} + (n_2 - 1)\sqrt{n_2} + \cdots + (n_k - 1)\sqrt{n_k} \\ &\leq (n_1 - 1 + \cdots + n_k - 1)\sqrt{n-1} \\ &= (n-k)\sqrt{n-1} \\ &\leq (n-2)\sqrt{n-1}. \end{aligned}$$

The equality holds if and only if $k = 2$ and $n_1 = n - 1$, i.e., $\overline{G} \cong K_{n-1} \cup K_1$. Then $G \cong \overline{K_{n-1} \cup K_1} \cong S_n$. Note that S_n has the minimum LEL-value among connected graphs with n vertices.

Then $$LEL(G) - LEL(\overline{G}) \geq LEL(S_n) - LEL(\overline{S_n}) = \sqrt{n} + (n-2)(1 - \sqrt{n-1}). \quad \square$$

Acknowledgements

The authors are very grateful to the anonymous referees for their valuable comments and suggestions, which have improved the presentation of the paper.

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