An improved upper bound on the adjacent vertex distinguishing edge chromatic number of a simple graph

Tao Wang[a], Deming Li[b],
[a]Department of Foundation, North China Institute of Science and Technology, Hebei 065201, P. R. China
E-mail: wangtgy@ncist.edu.cn
[b]Department of Mathematics, Capital Normal University, Beijing 100048, P. R. China
E-mail: lidm@mail.cnu.edu.cn

Abstract: An adjacent vertex distinguishing edge coloring or an avd-coloring of a simple graph G is a proper edge coloring of G such that no two adjacent vertices are incident with the same set of colors that are used by them. H. Hatami showed that every simple graph G with no isolated edges and maximum degree Δ has an avd-coloring with at most $\Delta + 300$ colors, provided that $\Delta > 10^{20}$. We improve this bound as what follows: if $\Delta > 10^{15}$, then the avd-chromatic number of G is at most $\Delta + 180$, where Δ is the maximum degree of G.

Keywords: Chernoff bound; adjacent vertex distinguishing edge coloring; vertex distinguishing edge coloring; adjacent strong edge coloring; strong edge coloring

1. Introduction

Let $G(V, E)$ be a graph with vertex set V and edge set E, respectively. For a vertex x of G, let $E_G(x)$ be the set of edges which are incident with x and let $N_G(x)$ be the set of its neighbors in G and $d_G(x) = |N_G(x)|$ be the degree of x in G if G is a simple graph, otherwise $d_G(x)$ be the number of edges with one end x, that is, $d_G(x) = |E_G(x)|$. $\Delta(G)$, or simply, Δ denotes the maximum value of $d_G(x)$ for each $x \in V(G)$. A k-edge coloring of G is a mapping $f : E \to \{1, 2, ..., k\}$ such that adjacent edges are assigned different colors. A partial k-edge coloring of G is a mapping f from a subset F of E to the color set $\{1, 2, ..., k\}$ such that f is a k-edge coloring of the edge-induced subgraph by F. For a vertex u and a partial k-edge coloring f of G, let $C_f(u)$ denote the set of colors of edges in $E_G(u)$. If $C_f(u) \neq C_f(v)$ for any $v \in N(u)$, we say u is adjacent-vertex distinguishable or distinguishable under the coloring f. If all vertices in G are distinguishable under a k-edge coloring f, f is called a k-adjacent-vertex distinguishable edge coloring or avd-coloring of G. The minimum such a k is called the adjacent-vertex distinguishable edge chromatic number, denoted by $\chi_{avd}(G)$.

The adjacent vertex distinguishable edge colorings are investigated by many researchers under different names, and there is a few directions related with the avd-coloring of graphs, [1-7, 9, 13, 14]. The following conjecture was introduced in 2002. Since then, there are a lots of results related with this conjecture. Many of these results are on special class of graphs. The properties of their structures help us calculate the value of $\chi_{avd}(G)$. A breakthrough is made recently by Hatami [11], since an upper bound is for nearly all graphs and the bound is neat. For notations not defined here, we follow [8, 10, 12].

Conjecture 1.1 *(Zhang et al. [13]). The avd-chromatic number of every simple connected graph G except the cycle of length five and isolated edges is at most $\Delta + 2$, where Δ is the maximum degree of G.*

Conjecture 1.1 is interesting, since we think it is not easy to use the induction method. Given a graph G and one of its avd-coloring f, if we remove an edge uv, which is colored by α, we may not obtain an avd-coloring of the resulting graph $G - uv$. For example, we choose the path with three vertices.

Theorem 1.2 *(Balister et al. [2, 3]). If G is a graph with no isolated edges, then the avd-chromatic number of G is at most $\Delta + O(log\chi)$, where χ is the chromatic number of G.*

Theorem 1.3. *(H. Hatami [11]). If G is a graph with no isolated edges and maximum degree Δ, then the avd-chromatic number of G is at most $\Delta + 300$, if $\Delta > 10^{20}$.*

Here, we use the same probabilistic modal and the same steps to construct an avd-coloring of G as those in [11], but we use a strong form of the Chernoff Bound to evaluate the upper bound of the avd-chromatic number. The tools we used here is of independent interests, and it may be used in other problems. Our upper bound for avd-coloring is as what follows.

Theorem 1.4. *Let G be a connected simple graph. Then the avd-chromatic number is at most $\Delta + 180$, where Δ is the maximum degree of G and $\Delta > 10^{15}$.*

In the proof of Theorem 1.4, Lovasz's local lemma and Talagrand's inequality will be used several times. And we list them here.

Lemma 1.5 *(The Symmetric Lovsz Local Lemma). Let $A_1, A_2, ..., A_n$ be events in a probability space. For all i, $Pr(A_i) = p$ and the event A_i is mutually independent of all events but at most d events. If $4pd < 1$, then $Pr(\cap_{i=1}^{n} \overline{A_i}) > 0$.*

Lemma 1.6 *(Talagrand's Inequality). Let X be a non-negative random variable, not identically 0, which is determined by n independent trials $T_1, T_2, ..., T_n$, and satisfying the following for some $c, r > 0$,*

1. Changing the outcome of any one trial can affect X by at most c, and

2. For any s and any outcome of trials, if $X \geq s$, then there is a set of at most rs trials whose outcomes certify that $X \geq s$, then for any $0 \leq t \leq E[x]$,

$$Pr[|X - E(X)| > t + 60c\sqrt{rE[X]}] \leq 4e^{-\frac{t^2}{8c^2rE[X]}}.$$

2. The strong form of the Chernoff Bound

The Chernoff Bound is an important tool for studying random graphs. This bound is often used in the following form.

Lemma 2.1 *(The Chernoff Bound) Suppose that $BIN(n,p)$ is the sum of n independent Bernoulli variables, each equal to 1 with probability p and 0 otherwise. For any $0 \leq t \leq np$,*

$$Pr[|BIN(n,p) - np| > t] < 2e^{-t^2/(3np)}.$$

The following theorem is the main result in this section. We use this theorem to improve Hatami's upper bound. This theorem is interesting since if we consider one problem by Lemma 2.1 to get a bound, we may use Theorem 2.2 to try the work. Maybe a better bound is obtained.

Theorem 2.2 *Suppose that $BIN(n, \lambda/n)$ is the sum of n independent Bernoulli variables, each equal to 1 with probability λ/n and 0 otherwise,* *and $(k-\lambda)^2 \geq k$, where λ and k are non-negative integers and $\lambda < n$. Then we have the following:*

1. *If $k < \lambda$, then $Pr[BIN(n, \lambda/n) \leq k] \leq \frac{\lambda^{k+1} e^{-\lambda}}{(\lambda-k)\cdot k!}$.*
2. *If $k > \lambda$, then $Pr[BIN(n, \lambda/n) \geq k] \leq \frac{(k+1)\lambda^{k} e^{-\lambda}}{(k+1-\lambda)\cdot k!}$.*

Before proving Theorem 2.2, we have the following two lemmas.

Lemma 2.3 *Let λ be a positive integer and k be a non-negative integer such that $(k-\lambda)^2 \geq k$. For $x \geq k$ and $x > \lambda \geq 0$, then we have*

$$F(x) = \frac{x+1-\lambda}{x+1-k} \cdot \left(\frac{x}{x+1}\right)^{x} \cdot \left(\frac{x+1-\lambda}{x-\lambda}\right)^{(x-k)} \geq 1.$$

Proof. Let $A(x) = (x+1-\lambda)/(x+1-k)$, $B(x) = (\frac{x}{x+1})^x$ and $C(x) = (\frac{x+1-\lambda}{x-\lambda})^{(x-k)}$.

And then

$$A'(x) = \left(1 + \frac{k-\lambda}{x+1-k}\right)' = \frac{\lambda - k}{(x+1-k)^2},$$

$$B'(x) = B(x) \cdot \left\{\ln \frac{x}{1+x} + \frac{1}{1+x}\right\},$$

and

$$C'(x) = C(x) \cdot \left\{\ln \frac{x+1-\lambda}{x-\lambda} - \frac{x-k}{x+1-\lambda} \cdot \frac{1}{x-\lambda}\right\}.$$

So, we have the following.

$$\begin{aligned} F'(x) = \; & B(x)C(x)\{A'(x) + A(x)\cdot \\ & \left\{\ln \tfrac{x}{1+x} + \tfrac{1}{1+x} + \ln(1 + \tfrac{1}{x-\lambda}) - \tfrac{x-k}{(x+1-\lambda)(x-\lambda)}\right\}\} \\ = \; & B(x)C(x)\{A'(x) + A(x)\cdot \\ & \left\{\ln(1 + \tfrac{\lambda}{(x+1)(x-\lambda)}) + \tfrac{1}{x+1} - \tfrac{x-k}{(x+1-\lambda)(x-\lambda)}\right\}\} \end{aligned}$$

by $\ln \frac{x}{1+x} + \ln(1 + \frac{1}{x-\lambda}) = \ln(1 + \frac{\lambda}{(x+1)(x-\lambda)})$.

Let

$$\begin{aligned} G(x) = \; & (x+1-k)^2 \cdot \tfrac{F'(x)}{B(x)C(x)} \\ = \; & (x+1-k)^2 \cdot \{A'(x) + A(x)\cdot \\ & \left\{\ln(1 + \tfrac{\lambda}{(x+1)(x-\lambda)}) + \tfrac{1}{x+1} - \tfrac{x-k}{(x+1-\lambda)(x-\lambda)}\right\}\}. \end{aligned}$$

By $x \geq 0$ and $\ln(1+x) \leq x$, we have

$$\begin{aligned} G(x) \leq \; & (x+1-k)^2 \cdot \left\{ \tfrac{\lambda-k}{(x+1-k)^2} + \tfrac{x+1-\lambda}{x+1-k} \cdot \right. \\ & \left. \left\{ \tfrac{\lambda}{(x+1)(x-\lambda)} + \tfrac{1}{x+1} - \tfrac{x-k}{(x+1-\lambda)(x-\lambda)} \right\} \right\} \\ = \; & (x+1-k)^2 \cdot \left\{ \tfrac{\lambda-k}{(x+1-k)^2} + \tfrac{x+1-\lambda}{x+1-k} \cdot \right. \\ & \left. \left\{ \tfrac{x}{(x-\lambda)(x-\lambda)} - \tfrac{x-k}{(x+1-\lambda)(x-\lambda)} \right\} \right\} \\ = \; & \tfrac{(\lambda-k)(x+1)(x-\lambda)}{(x+1)(x-\lambda)} + \tfrac{x(x+1-k)(x+1-\lambda)}{(x+1)(x-\lambda)} - \tfrac{(x-k)(x+1-k)(x+1)}{(x-\lambda)(x+1)} \\ = \; & \tfrac{(k-(k-\lambda)^2)x+(\lambda+1)k-k^2-\lambda^2}{(x+1)(x-\lambda)}. \end{aligned}$$

By $k \leq (\lambda - k)^2$, we have the following.

$$(\lambda+1)k - k^2 - \lambda^2 \leq (\lambda+1)k - (k + 2k\lambda) = -k\lambda \leq 0.$$

For $0 < \lambda \leq x$, we have

$$(k - (k-\lambda)^2)x + (\lambda+1)k - k^2 - \lambda^2 \leq 0.$$

So $F'(x) \leq 0$.

Note that $\lim_{x\to+\infty} A(x) = 1$, $\lim_{x\to+\infty} B(x) = e^{-1}$ and $\lim_{x\to+\infty} C(x) = e^1$, and then $\lim_{x\to+\infty} F(x) = 1$, so we have $F(x) \geq 1$. □

Lemma 2.4 *Let λ be a positive integer and k be a non-negative integer such that $(k-\lambda)^2 \geq k$. Then we have*

$$BIN(k; n, \lambda/n) \leq \frac{\lambda^k e^{-\lambda}}{k!},$$

where $BIN(k; n, \lambda/n) = Pr[BIN(n, \lambda/n) = k]$.

Proof. By Lemma 2.3, we have

$$\begin{aligned} \tfrac{BIN(k;n+1,\lambda/(n+1))}{BIN(k;n,\lambda/n)} = \; & \tfrac{\binom{n+1}{k}}{\binom{n}{k}} \cdot \left(\tfrac{\lambda/(n+1)}{\lambda/n}\right)^k \cdot \tfrac{(1-\lambda/(n+1))^{n+1-k}}{(1-\lambda/n)^{n-k}} \\ = \; & \tfrac{n+1-\lambda}{n+1-k} \cdot \left(\tfrac{n}{n+1}\right)^n \cdot \left(\tfrac{n+1-\lambda}{n-\lambda}\right)^{n-k} \\ = \; & F(n) \geq 1. \end{aligned}$$

Note that $\lim_{n\to+\infty} BIN(k; n, \lambda/n) = \frac{\lambda^k e^{-\lambda}}{k!}$. And then we have the conclusion. □

Now we are ready for proving Theorem 2.2.

Proof of Theorem 2.2. 1. If $\lambda > k \geq m$, then

$$\frac{BIN(m-1;n,\lambda/n)}{BIN(m;n,\lambda/n)} = \frac{(n-\lambda)m}{(n-m+1)\lambda} \leq \frac{m}{\lambda} \leq \frac{k}{\lambda}.$$

By Lemma 2.4, for $0 \leq i \leq k$,

$$BIN(i;n,\lambda/n) \leq \left(\frac{k}{\lambda}\right)^{k-i} \cdot BIN(k;n,\lambda/n),$$

and

$$1 + \frac{k}{\lambda} \cdots + \left(\frac{k}{\lambda}\right)^k = \frac{1-(k/\lambda)^{k+1}}{1-k/\lambda} < \frac{\lambda}{\lambda - k}.$$

And so,

$$Pr[BIN(n,\lambda/n) \leq k] \leq \frac{\lambda}{\lambda - k} BIN(k;n,\lambda/n) \leq \frac{\lambda^{k+1}e^{-\lambda}}{(\lambda - k)\cdot k!}.$$

2. If $\lambda < k \leq m$, then

$$\frac{BIN(m+1;n,\lambda/n)}{BIN(m;n,\lambda/n)} = \frac{(n-m)\lambda}{(n-\lambda)(m+1)} \leq \frac{\lambda}{m+1} \leq \frac{\lambda}{k+1}.$$

By Lemma 2.4, for $k \leq i \leq n$,

$$BIN(i;n,\lambda/n) \leq \left(\frac{\lambda}{k+1}\right)^{i-k} \cdot BIN(k;n,\lambda/n),$$

and

$$\begin{aligned} & 1 + \tfrac{\lambda}{k+1} \quad \cdots + \left(\tfrac{\lambda}{k+1}\right)^{n-k} \\ = & \quad \tfrac{1-(\lambda/(k+1))^{n-k+1}}{1-\lambda/(k+1)} < \tfrac{k+1}{k+1-\lambda}. \end{aligned}$$

And then,

$$Pr[BIN(n,\lambda/n) \geq k] \leq \frac{k+1}{k+1-\lambda} BIN(k;n,\lambda/n) \leq \frac{(k+1)\lambda^k e^{-\lambda}}{(k+1-\lambda)\cdot k!}.$$

3 Proof of Theorem 1.4.

Although our proof is based on Hatami's in [11], for self-contained reason, we restate the proof in details. The proof includes three steps. At the last step, an avd-coloring is constructed. The numbers used in the following paragraphs meet the requirements of Theorem 2.2.

Step 1. Let G be a simple graph and x be a vertex of G. Then x is called a light vertex if $d_G(x) < \Delta/3$ otherwise a heavy vertex. Let xy be an edge of G. Contracting edge xy means that first removing edge xy from G and then identifying x and y. We denote the resulting graph as G/xy. Let H be the subgraph of G induced by light vertices. Let F be the set of isolate edges in H. Let $G' = G/F$ and $H' = H/F$, which are obtained by contracting the set of edges in F. Note that $\Delta(G) = \Delta(G')$ and the multiplicity of G' is at most two.

If G' is p-avd-colorable, then G is also p-avd-colorable. Suppose that G' has a p-avd-coloring σ', we can obtain a p-avd-coloring σ of G as what follows: for an edge e in G' with color $\sigma'(e)$, we let $\sigma^*(e) = \sigma'(e)$ for the corresponding edge, then we have a partial p-avd-coloring of G. And then we can obtain the p-avd-coloring σ by extending the coloring σ^* to all edges in G since for an edge xy in F, $d_G(x) + d_G(y) < \Delta$.

Step 2. This step contains two phases. Let f be an edge coloring of G' with at most $\Delta + 2$ colors.

Phase one: 1.1 Uncolor each edge of $E(G') - E(H')$ with probability $114/\Delta$.

After this step, each vertex x has a set of uncolored edges $U(x)$, which is a subset of $E_{G'}(x) - E(H')$.

1.2 For a vertex v in G', if $|U(v)| > 170$, recover the colors of all uncolored edges in $U(v)$, and call v a recovered vertex.

Let f' be the partial coloring obtained after performing Phase one. Let $U_{f'}(v)$ be the set of uncolored edges in $E_{G'}(v) - E(H')$ and let R be the set of recovered vertices, Q be the set of all vertices v such that $|U(v)| < 20$, T be the set of vertices in $V(G') - V(H')$ such that there is an edge $vw \in U(v)$ and $w \in R$, and L be the set of vertices v in $V(G') - V(H')$ such that $|U_{f'}(v)| < 20$.

Clearly $L \subseteq R \cup Q \cup T$.

Lemma 3.1 *Let $v \in V(G') - V(H')$. We have*
3.1.1 $Pr[v \in R] \leq 1/1000$.
3.1.2 $Pr[v \in Q] \leq 1/1000$.
3.1.3 $Pr[v \in T] \leq 1/1000$.

Proof. 3.1.1 Since $U(v)$ has Bernoulli distribution, and we know that $\Delta/3 \leq d_{G'}(v) \leq \Delta$, Theorem 2.2 implies that

$$Pr[v \in R] = Pr[|U(v)| > 170] \leq \frac{172 \cdot 114^{171} \cdot e^{-114}}{58 \cdot 171!} \leq 1/1000,$$

where $k = 171$, $\lambda = 114$ and $k + 1 - \lambda = 58$, $(k - \lambda)^2 > k$ holds.

3.1.2 The worst case is when $d_{G'}(v) = \Delta/3$. Then Theorem 2.2 implies that

$$Pr[v \in Q] = Pr[|U(v)| < 20] \le \frac{38^{20} \cdot e^{-38}}{19 \cdot 19!} \le 1/1000,$$

where $k = 19$ and $\lambda = 38$, $(k - \lambda)^2 > k$ holds.

3.1.3 Consider an edge vw, where $v \in V(G') - V(H')$ and $w \in V(G')$. First, we prove an upper bound for $Pr[vw \in U(v) \text{ and } w \in R]$. Then Theorem 2.2 implies that

$$Pr[w \in R | vw \in U(v)] \le \frac{171 \cdot 114^{170} \cdot e^{-114}}{57 \cdot 170!} \le 1/10^6,$$

where $k = 171$ and $\lambda = 114$, $(k - \lambda)^2 > k$ holds.

Hence,

$$Pr[vw \in U(v)\, and\, w \in R] \le \frac{1}{10^6} \cdot \frac{114}{\Delta} \le \frac{1}{10^3\Delta}.$$

Since v has at most Δ neighbors, $Pr[v \in T] \le \frac{1}{10^3\Delta} \cdot \Delta \le \frac{1}{1000}$. □

By Talagrand's Inequality, for $\Delta > 10^{15}$, similar to the proof of Lemma 2 in [10], we have the following.

Lemma 3.2 *For every vertex $v \in V(G') - V(H')$,*

$$Pr[|N_{G'}(v)| \cap L| > \frac{\Delta}{100}] \le \frac{1}{\Delta^7}.$$

Proof. Note that $|N(v) \cap L| \le |N(v) \cap R| + |N(v) \cap T| + |N(v) \cap Q|$. If $|N_{G'}(v) \cap L| > \Delta/100$, then at least one of $|N_{G'}(v) \cap R| > \Delta/300$, $|N_{G'}(v) \cap Q| > \Delta/300$ and $|N_{G'}(v) \cap T| > \Delta/300$ holds. By Lemma 3.1, $E[|N(v) \cap R|] \le \Delta/1000$, $E[|N(v) \cap Q|] \le \Delta/1000$ and $E[|N(v) \cap T|] \le \Delta/1000$. For $uv \in E(G') - E(H')$, consider the independent Bernoulli trials T_{uv}, where the outcome of T_{uv} determines whether uv is uncolored in Phase 1.1 or not.

Claim. $Pr[|N_{G'}(v) \cap R| > \Delta/300] \le 1/(3\Delta^7)$, $Pr[|N_{G'}(v) \cap T| > \Delta/300] \le 1/(3\Delta^7)$ and $Pr[|N_{G'}(v) \cap Q| > \Delta/300] \le 1/(3\Delta^7)$ holds.

Proof of Claim. Changing the outcome of each trial affects $|N_{G'}(v) \cap R|$ by at most two, and every assignment to trials that results $|N_{G'}(v) \cap R| \ge$

k can be certified by the outcome of $170k$ trials. Talagrand's inequality implies that

$$Pr\left[|N_{G'}(v) \cap R| > \tfrac{\Delta}{1000} + t + 60 \times 2\sqrt{170 \cdot \tfrac{\Delta}{1000}}\right] \\ \leq 4e^{-1000t^2/8\times 4\times 170\Delta}.$$

Since changing the outcome of each trial may add or remove at most two vertices from R, at most 2×170 vertices may be added or removed from T. So it affects $|N_{G'}(v) \cap T|$ by at most 2×170. And every assignment to trials that results $|N_{G'}(v) \cap T| \geq k$ can be certified by the outcome of $170k$ trials. Talagrand's inequality implies that

$$Pr\left[|N_{G'}(v) \cap T| > \tfrac{\Delta}{1000} + t + 60 \times 2 \times 170\sqrt{170 \cdot \tfrac{\Delta}{1000}}\right] \\ \leq 4e^{-1000t^2/8\times 4\times 170^3\Delta}.$$

Instead of considering the random variable $|N_{G'}(v) \cap Q|$, we apply Talagrand's inequality to $X_v = d_{G'}(v) - |N_{G'}(v) \cap Q|$. Changing the outcome of each trial affects X_v by at most two, and every assignment to trials that results $X_v \geq k$ can be certified by the outcome of $20k$ trials. For $X_v = d_{G'}(v) - |N_{G'}(v) \cap Q|$, and $E[X_v] \geq d_{G'}(v) - \Delta/1000$. Talagrand's inequality implies that

$$Pr\left[X_v < d_{G'}(v) - \tfrac{\Delta}{1000} - t - 60 \times 2\sqrt{20 \cdot \Delta}\right] \\ \leq 4e^{-t^2/8\times 4\times 20\times (d_{G'}(v)-\Delta)/1000} \leq 4e^{-t^2/1000\Delta}.$$

Substituting $t = \Delta/1000$ in the above inequalities. We have $Pr[|N_{G'}(v) \cap R| > \Delta/300] \leq 1/(3\Delta^7)$, $Pr[|N_{G'}(v) \cap T| > \Delta/300] \leq 1/(3\Delta^7)$ and

$$Pr[|N_{G'}(v) \cap Q| > \Delta/300] = Pr[X_v < d_{G'}(v) - \Delta/300] \\ \leq Pr\left[X_v < d_{G'}(v) - \tfrac{\Delta}{1000} - t - 60 \times 2\sqrt{20 \cdot \Delta}\right] \leq \tfrac{1}{3\Delta^7}.$$

For $\Delta > 10^{11}$, similar to the proof of Lemma 3 in [11], we have the following.

Lemma 3.3 *For every two adjacent vertices $u, v \in V(G') - V(H')$, where $d_{G'}(u) = d_{G'}(v)$.*

$$Pr[u \notin L\, and\ |C_{f'}(u) \Delta C_{f'}(v)| < 10] < \frac{1}{8\Delta^6}$$

Proof. It is sufficient to prove that $Pr[|C_{f'}(u)\Delta C_{f'}(v)| < 10 | u \notin L] < \frac{1}{8\Delta^6}$. We assume that $C_f(u) = C_f(v)$. Suppose that $|U_{f'}(u)| = k \geq 20$. If

$|C_{f'}(u)\Delta C_{f'}(v)| < 10$, then there are at least $k-10$ colors in $C_f(u)-C_{f'}(u)$ which are also in $C_f(v)-C_{f'}(v)$. Since there are at most two edges between u and v, the probability of this occuring is at most

$$\binom{k}{k-10} \cdot \frac{114}{\Delta}^{k-12} < \frac{1}{8\Delta^6}$$

So the lemma follows. □

By Lemmas 3.2 and 3.3 and the local lemma, we have

Lemma 3.4 *If we apply Phase one to the edge coloring f, and obtain a partial edge coloring f', then with positive probability, we have*

3.4.1 For every vertex $v \in V(G') - V(H')$, we have $|N_{G'}(v) \cap L| \leq \Delta/100$.

3.4.2 For every two adjacent vertices $u, v \in V(G') - V(H')$, where $d_{G'}(u) = d_{G'}(v)$ and $u \notin L$. We have $|C_{f'}(u)\Delta C_{f'}(v)| \geq 10$.

Proof. We define the following bad events. For $v \in V(G') - V(H')$, let A_v be the event that $|N(v) \cap L| > \Delta/100$. For each edge uv such that $u, v \in V(G') - V(H')$, $d_{G'}(u) = d_{G'}(v)$, let A_{uv} be the event that $u \notin L$ and $|C_{f'}(u)\Delta C_{f'}(v)| < 10$.

Clearly, each event A_X is mutually independent of all events A_Y as long as all vertices in X are in a distance of at least six from all vertices in Y. Hence each event is independent of all events but at most $2 \times \Delta^6$ events, and by Lemmas 3.2 and 3.3, each event occurs with a probability of at most $1/(8\Delta^6)$. So the Local Lemma implies this lemma. □

Phase two: For every vertex $u \in L$, choose five colored edges uv_i under f' for $1 \leq i \leq 5$ and $v_i \in V(G') - L$ uniformly at random and uncolor them.

After performing Phase two, we obtain a partial edge coloring $\bar{f}$ of $E(G') - E(H')$. Let $U'(v)$ be the set of uncolored edges by performing Phase two.

By the local lemma, we have the following.

Lemma 3.5 *Suppose that f' is a partial edge coloring of which satisfies Properties (3.4.1)and (3.4.2) in Lemma 3.4. If we apply Phase two to f', and obtain a partial coloring $\bar{f}$, then with positive probability, we have*

(3.5.1) For every vertex $v \in V(G') - V(H') - L$, we have $|U'(v)| \leq 4$.

(3.5.2) For every two adjacent vertices u, $v \in V(G') - V(H')$, where $d_{G'}(u) = d_{G'}(v)$, we have $C_{\bar{f}}(u) \neq C_{\bar{f}}(v)$.

Proof. Suppose that $u, v \in V(G') - V(H')$ are two adjacent vertices such that $d_{G'}(u) = d_{G'}(v)$ and $u \notin L$. By Lemma 3.4.2, it is sufficient to prove the following instead of (3.5.2).

(3.5.2)$'$ *For every two adjacent vertices $u, v \in L$, where $d_{G'}(u) = d_{G'}(v)$, we have $C_{\bar{f}}(u) \neq C_{\bar{f}}(v)$.*

We define the following two types of bad events. For every vertex $u \in V(G') - V(H') - L$, and every five edges uv_i for $1 \leq i \leq 5$ such that $v_i \in L$ and uv_i is a colored edge under f', let $A_{u,\{v_1,\dots,v_5\}}$ denote the event that $uv_i \in U'(u)$ for $1 \leq i \leq 5$. For every two adjacent vertices $u, v \in L$ where $d_{G'}(u) = d_{G'}(v)$, let A_{uv} denote the event that $C_{\bar{f}}(u) = C_{\bar{f}}(v)$.

We estimate $Pr[A_{u,\{v_1,\dots,v_5\}}]$ and $Pr[A_{uv}]$ firstly. For every $v \in V(G') - V(H')$, we have $d_{G'}(v) \geq \Delta/3$, $|N(v) \cap L| \leq \Delta/100$, and the uncolored degree of v is at most 170. So every edge uv is uncolored in Phase two with a probability of at most $5/(d_{G'}(v) - \Delta/100 - 170) \leq 5/(\Delta/3 - \Delta/100 - 170) < 20/\Delta$, and also two parallel edges are uncolored with a probability of at most $\frac{5}{\Delta/4} \times \frac{4}{(\Delta/4-1)} < (20/\Delta)^2$. Since $u \in L$, this implies

$$Pr[A_{u,\{v_1,\dots,v_5\}}] \leq (\frac{20}{\Delta})^5.$$

For two adjacent vertices $u, v \in L$ with $d_{G'}(v) = d_{G'}(u)$ and $C_{\bar{f}}(u) = C_{\bar{f}}(v)$, since $C_{\bar{f}}(u) = C_{\bar{f}}(v)$, all five edges in $U'(v)$ are determined by the five edges in $U'(u)$, and the edges in $U'(u)$ are chosen independently from the edges in $U'(u)$. Then we have $Pr[A_{uv}] \leq 1/\binom{\Delta/4}{5} \leq (\frac{20}{\Delta})^5$.

Secondly, we construct a graph D, whose vertices are all the events of two types $A_{u,\{v_1,\dots,v_5\}}$ and A_{wv}, two vertices of $A_{u,\{v_1,\dots,v_5\}}$, $A_{w,\{v'_1,\dots,v'_5\}}$, A_{wv} and $A_{w'v'}$ are adjacent if and only if one of $\{v_1, \dots, v_5\} \cap \{v'_1, \dots, v'_5\} \neq \emptyset$, $\{v_1, \dots, v_5\} \cap \{w, v\} \neq \emptyset$, $\{w, v\} \cap \{w', v'\} \neq \emptyset$ holds. For a vertex $w \in L$, since there are at most $\Delta\binom{\Delta/100}{4}$ events of $A_{u,\{v_1,\dots,v_5\}}$ and at most $\Delta/100$ events of the form A_{wv}, the maximum degree of D is at most $5(\Delta \cdot \binom{\Delta/100}{4} + \Delta/100) < 6\Delta\binom{\Delta/100}{4} < 6\Delta^5/10^8$. By the local lemma, the conclusion follows. □

Step 3: *From a partial edge coloring $\bar{f}$ of $E(G') - E(H')$ to obtain an avd-edge coloring G'.*

We only recolor the edges of H'. Let $f_1 = \bar{f}$. We repeated modify f_1 to eventually obtain a $(\Delta + 180)$-avd-coloring of G'. Given f_k for $k \geq 1$, we suppose that for an edge $uv \in H'$ with $C_{f_k}(u) = C_{f_k}(v)$. Since H' does not have isolated edges, we may assume $d_{H'}(u) \geq 2$, and $v_1, v_2, \dots, v_r$ are the neighbors of u in H', where $r = d_{H'}(u)$. Uncolor all edges uv_i for $1 \leq r \leq r$ to obtain a partial coloring f'_k.

Similar to the proof of Lemma 6 in [11], we have the following.

Lemma 3.6 *There exist sets $L(uv_i) \subseteq \{1, ..., \Delta + 180\}$ for $1 \leq i \leq r$, all of size $r + 180$ such that*

(3.5.1) In the partial edge coloring f'_k all the colors in $L(uv_i)$ are available for the edge uv_i.

(3.5.2) If $v' \neq u$ is adjacent to v_i and $C_{f'_k}(v') - C_{f'_k}(v_i)$ has only one element x, then $x \notin L(uv_i)$.

Proof. Since $d_{G'}(u), d_{G'}(v) \leq \Delta/3$, there are at least $\Delta/3 + r + 180$ available colors for uv_i. And there are at most $\Delta/3$ colors for uv_i dissatisfying (3.5.2). □

Consider all possible proper coloring f'_k such that the color of uv_i is chosen from $L(uv_i)$. There are at least $(r + 180)(r + 179) \cdots 181$ colorings. Choose one from them as $\bar{f}_k$. There are at most $(r + 180)(r - 1)!$ different possible choices of $\bar{f}_k$ such that $C_{\bar{f}_k}(u) = C_{\bar{f}_k}(v_i)$. So we have

$$\begin{aligned} \Pr[\cup_{1 \leq i \leq r} C_{\bar{f}_k}(u) = C_{\bar{f}_k}(v_i)] &\leq \textstyle\sum_{i=1}^{r} Pr[C_{\bar{f}_k}(u) = C_{\bar{f}_k}(v_i)] \\ &\leq \textstyle\sum_{i=1}^{r} \frac{(r-1)!}{(r+179)\cdots 181} < 1. \end{aligned}$$

This implies that there is a proper edge coloring f_{k+1} as $\bar{f}_k$ such that u is distinguishable. Hence by repeatedly applying this procedure we will obtain a $(\Delta + 180)$-avd-coloring of G'.

Acknowledgments

This work is support by NSFC (10201022, 10971144), BNSF (1102015) and the Fundamental Research Funds for the Central Universities (2011B019). We thank the referees for their helpful suggestion.

References

[1] S. Akbari, H. Bidkhori, N. Nosrati, r-strong edge colorings of graphs, Discrete Mathematics, 306(2006) 3005-3010.

[2] P. Balister, E. Györi, J. Lehel and R. Schelp, Adjacent vertex distinguishing edge colorings. SIAM J. on Discrete Mathematics 21:2(2007) 237-250.

[3] P. Balister, O. Riordan, R. Schelp, Vertex-distinguishing edge colorings of graphs, J. Graph Theory 42 (2003) 95-109.

[4] C. Bazgan, A. Harkat-Benhamdine, Li Hao, M. Woźniak. On the vertex-Distinguishing Proper Edge-Coloring of Graphs. J. Combin. Theory Ser. B 75(2)(1999) 288-301.

[5] A.C. Burris, R.H. Schelp, Vertex distinguishing proper edge colorings, J. Graph Theory 26 (2) (1997) 73-82.

[6] D. W. Cranston, Strong edge-coloring of graphs with maximum degree 4 using 22 colors Discrete Mathematics, 306: 21(2006) 2772-2778.

[7] M. Ghandehari, H. Hatami, Two upper bounds for the strong edge chromatic number,

[8] Gary Chartrand, Ping Zhang, Chromatic graph theory, CRC Press, 2008.

[9] M. Karonski, T. Luczak, A. Thomason, Edge weights and vertex colours, J. Combin.Theory Ser. B 91 (2004)151-157.

[10] M. Molloy, B. Reed, Graph Colouring and the Probabilistic Method, Springer, Berlin, 2002.

[11] H. Hatami, $\Delta + 300$ is a bound on the adjacent vertex distinguishing edge chromatic number, J. of Combin. Theory,Ser. B 95(2005) 246-256.

[12] D.B. West, Introduction to Graph Theory, 2ed., Prentice-Hall, Englewood Cliffs, NJ, USA, 2001.

[13] Z. Zhang, L. Liu, J. Wang, Adjacent strong edge coloring of graphs, Appl. Math. Lett. 15 (2002) 623-626.

[14] Z. Zhang, D. Woodall, B. Yao, J. Li, X. Chen and L. Bian, Adjacent strong edge colorings and total colorings of regular graphs, Science in China Series A: Mathematics, 52 (2009) 973-980.