Resolvably Decomposing Complete Equipartite Graphs Minus a One-Factor into Cycles of Uniform Even Length

Dean G Hoffman
Auburn University
Department of Mathematics and Statistics
133-C Allison Lab
Auburn AL 36849

Sarah H Holliday
Southern Polytechnic State University
Mathematics Department
1100 S Marietta Pkwy
Marietta GA 30060
shollida@spsu.edu

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Abstract

We seek a decomposition of a complete equipartite graph minus a one-factor into parallel classes each consisting of cycles of length k. In this paper, we address the problem of resolvably decomposing complete multipartite graphs with r parts each of size a with a one-factor removed into k-cycles. We find the necessary conditions, and give solutions for even cycle lengths.

1 Introduction

A problem was posed by Gerhard Ringel in 1967 at a graph theory conference in Oberwolfach, Germany; the most casual way to state this problem is to ask if some number of people can be seated at a specified number of round tables for a series of meals, with these restrictions: Each person is assigned one seat for each meal, all the people are friends (but not very good friends) and wish to be seated next to each other, so every person must have as a neighbor every other person (but just once). (A neighbor at a dinner table is defined to be only the person either to the immediate right or the immediate left, not any of the people across the table.) The original statement of the problem is as follows: At a gathering, there are n mathematicians, and we wish to seat them at s round tables of seating capacities $k_1, k_2, k_3, \ldots, k_s$ ($3 \leq k_i$, and $\sum_{i=1}^{s} k_i = n$) for x meals, so that every mathematician sits next to every other mathematician exactly once, and every mathematician is seated exactly once for each meal. In graph theory terminology, we are decomposing K_n into x parallel classes, each consisting of cycles of lengths $k_1, k_2, k_3, \ldots, k_s$ (so $\sum_{i=1}^{s} k_i = n$, $x = (n-1)/2$, so n must be odd). The question of decomposing K_n into cycles of uniform length has been well-studied. In 2000 and 2003, Jiuqiang Liu considered the resolvable decomposition of the complete equipartite graph into uniform cycles. This work is an extention of Liu's work [3, 4], for different parity of number of partite sets and size of partite sets.

2 Definitions and Notation

$G = (V, E)$ is a simple graph with no loops or multiple edges. In this work, we shall designate a specific case or family of cases as $RD(a, r, k)^*$, in which a indicates the size of each partite set, r the number of partite sets in the equipartite graph, and k the cycle length used in the decomposition with the star to indicate that a one-factor has been removed; thus, $RD(a, r, k)^*$ is resolvably decomposing the complete equipartite graph with r parts each of size a, with a one-factor removed, into k-cycles.

3 Obvious Necessary Conditions

The number of vertices in a cycle must divide the total number of vertices in the graph. Thus we derive the condition $k|ar$. We note that the degree of each vertex in the graph must be even, so $2|a(r-1)-1$, telling us $a(r-1)$ must be odd, thus a must be odd and r must be even. We state here another obvious necessary condition, that bipartite graphs cannot contain odd cycles. Thus, if $r = 2$ then $k \equiv 0 \pmod 2$.

4 Preliminary Results

Our first result is on Hamilton cycles in the complete equipartite graph.

Theorem 4.1. *$RD(a, r, ar)^*$ can be constructed.*

Proof. In a complete graph K_n, $(n = ar)$ we can assign a label to each vertex. If we let the vertex labels be the elements of $\mathbb{Z}_n$, then we can say that the edge between vertices labeled i and j $(i < j)$has distance $|j - i|$ (mod n). In the complete graph K_{ar}, there are the following distances: $\{1, 2, 3, \ldots, ar/2\}$. Let the set of edges of distance $ar/2$ be the one-factor, and let the edges of distances $r, 2r, \ldots, (a-1)r/2$ be the edges removed from between vertices in the same partite set. The remaining distances are $\{1, 2, 3, \ldots, r-1, r+1, r+2, \ldots, 2r-1, 2r+1, \ldots ar/2 - 1\}$.

The total number of distances is $\frac{ar}{2}$. We removed one distance for the one-factor, leaving $\frac{ar}{2} - 1$ distances. We then removed the distances that are multiples of r, leaving $\frac{ar}{2} - 1 - \frac{a-1}{2} = \frac{1}{2}(ar - 2 - (a-1)) = \frac{1}{2}(ar - a - 1) = \frac{1}{2}(a(r-1) - 1)$ distances.

If $\frac{1}{2}(a(r-1)-1)$ is even, we order the remaining distances into the following pairs: $\{(1,2), (3,4), \ldots, (r-1, r+1), (r+2, r+3), \ldots\}$. If $\frac{1}{2}(a(r-1)-1)$ is odd, we order the remaining distances as follows: $\{(1), (2,3), (4,5), \ldots, (r-2, r-1), (r+1, r+2), \ldots\}$, where (1) is the Hamilton cycle determined by edges of distance 1. Note that each of these pairs of distances are relatively prime, and therefore can be organized into Hamilton cycles using techniques described in the theorem of Bermond, Favaron, Maheo [1]. □

We present some techniques for combining decompositions of smaller graphs to generate decompositions of larger graphs using some results from Liu [4], notated as $RD(a, r, k)$ without star: Resolvable decompositions of equipartite graphs having r parts of size a into k-cycles.

Lemma 4.2. *If $RD(a, r, k)^*$ and $RD(ar, m, k)$ can be constructed, then an $RD(a, mr, k)^*$ can be constructed.*

Proof. We begin with m copies of a $RD(a, r, k)^*$. These will be treated as the m partite sets of the $RD(ar, m, k)$. We use the $RD(ar, m, k)$ to cover the edges between partite sets.

Clearly, this produces a $RD(ar, m, k)$. □

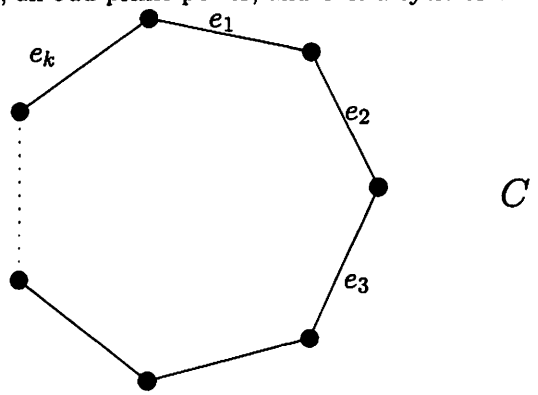
Lemma 4.3. *An $RD(a, r, k)^*$ can be used to create an $RD(a, mr, k)^*$.*

Proof. We know that a is odd, r is even and $k|ar$. We need an $RD(ar, m, k)$. The necessary conditions for existence are satisfied, with one exception: $RD(6, 2, 6)$. This can occur two ways, $a = 1, r = 6, m = 2, k = 6$ or $a = 3, r = 3, m = 2, k = 6$. In the first case, we refer to [2], which offers a solution, and in the latter case, we refer to figure 1. □

Let T be a tournament on V. If $e = vw$ is an arc 

of a digraph D on V, we say either e is positive, and $sgn(e) = +1$ if vw is an arc of T, or e is negative, and $sgn(e) = -1$ if wv is an arc of T.

Let G be a graph on V, $f : E(G) \to F$, where F is a finite field of order q, an odd prime power, and C is a cycle of G:



We say f is consistent on C if the following holds: let D be any one of the two orientations of C with indegrees = outdegrees =1, at all vertices. Then, $\sum_{e \in E(D)} sgn(e)f(e) = 0$.

We begin with a $(qRD(a,r,k))^*$ (henceforth known as a SD or SH[3] Design) on $qK_r^a - I$. If $f : E(qK_r^a) \to F$, we say f is consistent on SD if
1. f is consistent on each cycle of the SD, and
2. For each $vw \in E(K_r^a)$, $\{f(e) | e \in qK_r^a, e \text{ has ends } vw\} = F$.

Lemma 4.4. *If f is consistent on the $(qRD(a,r,k))^*$ SD, then there is a $RD(qa,r,k)^*$.*

Proof. Let $A \times V$ be the set on which the $(qRD(a,r,k))^*$ SD is defined, on groups $A \times \{v\}, v \in V$. We define a $RD(qa,r,k)^*$ on $F \times A \times V$, using difference methods on the additive group $(F,+)$, where, for each group $G \subseteq A \times V$ of the $(qRD(a,r,k))^*$, $F \times G$ is a group of the $RD(qa,r,k)^*$.

For each (α,s,v), $(\beta,t,w) \in F \times A \times V$, with $v \neq w$, the edge joining (α,s,v) and (β,t,w) is said to be an edge of difference $\beta - \alpha$ if $wv \in T$, and difference $\alpha - \beta$ if $vw \in T$.

For each cycle $C = ((a_1,v_1),(a_2,v_2),\ldots,(a_k,v_k))$ of the $(q(a,r,k))^*$ SD, we may find a cycle $C^+ = ((\alpha_1,a_1,v_1),\ldots,(\alpha_k,a_k,v_k))$ in such a way that each edge of C^+ is of difference the f-value of the corresponding edge in C. By adding the q elements of F to the first coordinates of the vertices of C^+, we obtain q of the cycles of our sequence $RD(qa,r,k)^*$. Moreover, each parallel class of cycles of our $(qRD(a,r,k))^*$ SD yields a parallel class of cycles of the $RD(qa,r,k)^*$. Each edge of the one-factor in the $(qRD(a,r,k))^*$ was left behind to become exactly the one-factor of the $RD(qa,r,k)^*$.

□

A $RD(a,r,k)''$ is a pair consisting of an $RD(a,r,k)^*$ and a $2RD(a,r,k)$.

Theorem 4.5. *If there is a $RD(a,r,k)''$, then there is a $RD(qa,r,k)''$, provided $q \geq 5$ is an odd prime power.*

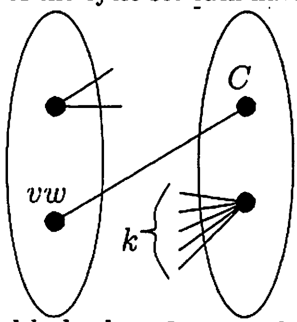
Proof. First, we construct a $RD(qa,r,k)^*$. Our $RD(a,r,k)^*$, together with $\frac{q-1}{2}$ copies of our $2RD(a,r,k)$, gives rise to a $(qRD(a,r,k))^*$. We need to find an f. Define $f \equiv 0$ on the edges of the $RD(a,r,k)^*$ (and the one-factor).

Claim 1 The edges of the $2K_r^a$ can be labeled with elements of R and N so that:

1. Every doubled edge will have one edge labeled R and the other labeled N.
2. No cycle of the $2RD(a,r,k)$ has labels all R, or all N.

Proof of Claim 1

We form a bipartite graph with partite sets: doubled edges, and cycles of the $2RD(a,r,k)$. We add edges (vw, C) if ab is an edge of C, a cycle of the $2RD(a,r,k)$. The vertices of the doubled edge set each have degree 2, and edges of the cycle set each have degree k.



Doubled edges Cycles of the $2(a,r,k)$

Since all degrees are even, there is an Euler tour on each component. Also, each component has an even number of edges. We label the edges of each Euler tour with elements of N and R alternately as we traverse the tour. □

Claim 2 If $q \geq 5$, then $D = R - N = F^*$

Proof of Claim 2

Certainly, $D = R - N \subseteq F^*$, since $R \cap N = \emptyset$. If $\alpha \in R$, then $\alpha D = D$, so $D \in \{R, N, F^*\}$.

If $q \equiv 1 \pmod 4$, let $\nu \in N$, so $\nu R = N$, $\nu N = R$. So $\nu D = N - R = -D = D$, since $q \equiv 1 \pmod 4$. Thus $D = F^*$.

So, $q \equiv 3 \pmod 4$, $D = R + R$. If $D = R$, then $R \cup \{0\}$ is a subgroup of $(F, +)$, a contradiction. By [5], for some $r \in R$, $r + 1 \in R$. So, $r + 1 \in R + R = D$, so $D \cap R \neq \emptyset$, so $D \neq N$. Thus, $D = F^*$. □

Claim 3 $S = R + N = \begin{cases} F^* & \text{if } q \equiv 1 \pmod 4 \\ F & \text{if } q \equiv 3 \pmod 4 \end{cases}$

Proof of Claim 3

Certainly $0 \in R + N$ if and only if $q \equiv 3 \pmod 4$.

Also, if $q \equiv 1 \pmod 4$, $R + N = R - N = F^*$. So, assume $q \equiv 3$ (mod 4). Again, $S \in \{R \cup \{0\}, N \cup \{0\}, F\}$. But $-S = S$, so $S = F$. □

We return to the proof of the theorem by showing that a $(a, r, k)''$ will give us a $(qa, r, k)''$.

We label the edges of each doubled cycle so that every doubled edge will have one edge labeled R and the other labeled N, and such that for each cycle C in the decomposition, find a vertex v on C such that the two incident edges to v in C have different labels, R or N. Define $g : E(qK_r^a) \to F^*$ on all edges of C arbitrarily according to $g(e) \in N$ if e is labeled N, $g(e) \in R$ if e is labeled R, except the two edges incident to v. We need to find $\rho \in R$ and $\nu \in N$, the labels on the edges incident to v. We need $\rho \pm \nu =$ the signed sum of the values of the remaining edges of the cycle. (If for some reason, this sum is 0, then we change any label to a different value of the same set. It is possible to find such ρ and ν because by claim 2 and claim 3, and the fact that $q \geq 5$, there must be at least 2 elements in each N and R which is enough to consistently label ρ and ν.)

For the "zero blob", we already defined $f \equiv 0$. For each $\alpha \in R$, define f on a copy of the $2RD(a, r, k)$ by $f(e) = \alpha g(e)$, (so there will be $\frac{q-1}{2}$ copies of the $2RD(a, r, k)$).

Now, f is consistent on the $qRD(a, r, k)^*$. We break it apart into the $RD(qa, r, k)^*$ using Lemma 4.4.

We now need to make a $2RD(qa, r, k)$. We refer to the above construction and repeat it with a few minor changes. We use a $2RD(a, r, k)$ instead of a $RD(a, r, k)^*$ to form the "zero blob", but leave everything else constructed as before. We use two copies of all non-"zero blob" cycles, and one copy of the $2RD(a, r, k)$ "zero blob" and observe that we have covered all necessary edges to make a $2RD(qa, r, k)$, in the manner shown above.

□

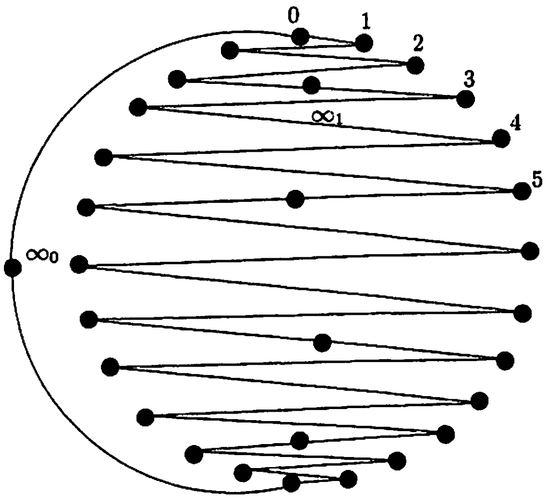
Theorem 4.6. *There is a $RD(3a, r, ar)''$, a odd, r even, $ar \neq 4$.*

Proof. Previously, we used an arbitrary tournament T on V, so now we will specify our tournament. Let $V = \{\infty_0, \ldots, \infty_{a-1}\} \cup \mathbb{Z}_{a(r-1)}$ and T be the tournament on V created in the following manner:

1. For all i, j, with $1 \leq j \leq \frac{a(r-1)-1}{2}$, orient an edge from i to $i + j$.
2. For all $i \in \mathbb{Z}_{a(r-1)}$, and $1 \leq j \leq a$, orient an edge from ∞_j to i.
3. Orient edges on the ∞'s arbitrarily.

This tournament admits $\mathbb{Z}_{a(r-1)}$ as an automorphism, so that when we find our base blocks, we can simply "click" them (mod $a(r-1)$).

We now define the method of using cracked easter eggs: We form a path of length $a(r-1)$ on $\mathbb{Z}_{a(r-1)}$ whose vertices are $(0, 1, -1, 2, -2, \ldots, \frac{a(r-1)-1}{2}, -\frac{a(r-1)-1}{2}$. We then subdivide all r of the edges of difference $\equiv 0 \bmod a$ and label the new vertices $\infty_1, \infty_2, \ldots, \infty_{a-1}$. The vertex ∞_0 is added, and edges connected to the vertices labeled 0 and $(a(r-1)-1)/2$ to form a cycle of length ar. To form the remaining cycles, we "click" the cycle over $a(r-1)$



To find the $RD(3a, r, ar)''$, we use a similar method to Theorem 5.2. We begin by labeling a $RD(a, r, ar)^*$ as our "zero blob" (we can find a $RD(a, r, ar)^*$ using Theorem 4.1). We then create the $2RD(a, r, ar)$ using a non-zero consistent labeling f over $\mathbb{Z}_3$ by cracked easter eggs. Once the base blocks are labeled, we can "click" the cycles over $3a$ and be done with it.

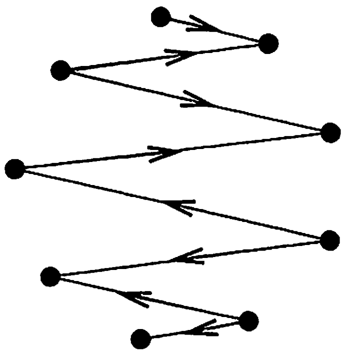
Here is how we find our non-zero f on our base block: For each $1 \leq i \leq \frac{a(r-1)-1}{2}$ with $a \nmid i$, there are two edges of the base block with difference i; the f-values on these will be x_i and $-x_i$ for some $x_i \in \{1, 2\} = \mathbb{Z}_3 \backslash \{0\}$. For each $0 \leq j \leq a-1$, there are two edges of the base block at ∞_j; the f-values on these will be y_j and $-y_j$ for some $y_j \in \{1, 2\}$. In the required consistency equation, the two occurrences of each variable might or might not cancel out. After simplification, we will have an equation of the form $\pm z_1 \pm z_2 \pm \ldots \pm z_t \equiv 0 \pmod 3$, where the z's are the t variables that did not cancel out. We require a non-zero solution. Obviously, this is possible if and only if $t \neq 1$.

Referring to the tournament, we see that the a variables $y_0, \ldots, y_{a-1}$ do not cancel out, so we are done unless $a = 1$, so we shall now analyze the cases when $a = 1$.

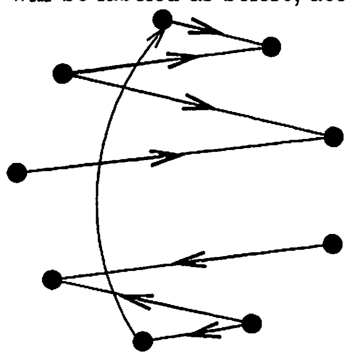
Note that the variable x_i cancels out if and only if the tournament orientation on the two edges of difference i is consistent as the cycle is traversed.

Unfortunately, this is always the case in our base block, so we must alter it.

Case 1: $n \equiv 1$ mod 4 By the previously stated method for orienting edges, we discover that all of the differences cancel out and our equation is the impossible $y_0 = 0$, which has no non-zero solution.

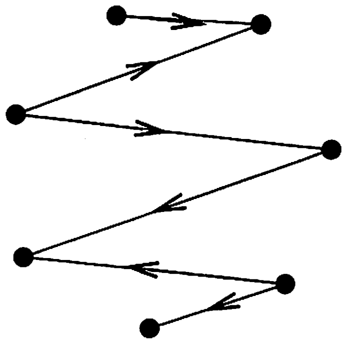


So we will remove an edge of maximum length, and replace the edge between the former beginning and end of the Hamilton path. This new edge will be labeled as before, according to the tournament.

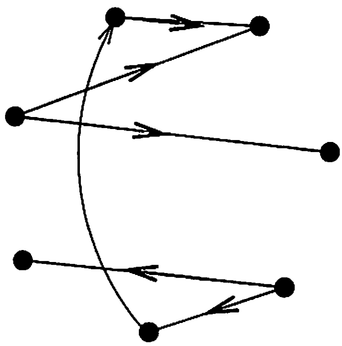


We find that this creates a Hamilton path whose equation now has more than one variable, so there is a non-zero solution.

Case 2: $n \equiv 3$ mod 4 Again, by the previously stated method for orienting edges, we discover that all of the differences cancel out and our equation is again $z_0 = 0$.

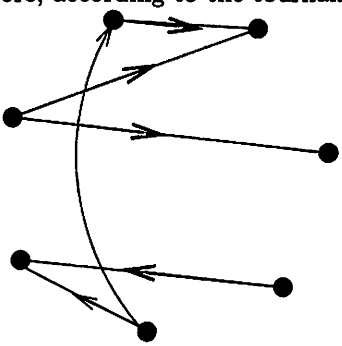


So, again, we will remove an edge of maximal length, and replace the edge between the former beginning and end of the Hamilton path. This new edge will be labeled as before, according to the tournament.



Unfortunately, all of the differences still cancel out and our equation still is $z_0 = 0$.

So, we will "flip" the bottom half of the design, and relabel the edges as before, according to the tournament.



To create the $2RD(3a, r, ar)$, we use the same method as in Theorem 5.2 to find the $2RD(qa, r, k)$.

5 Main result

Theorem 5.1. *A complete multipartite graph with r parts each of size a, minus a one-factor, can be resolvably decomposed into k-cycles if a is odd, r is even, $k|ra$, and k is even.*

Proof. To form a $RD(a,r,k)^*$, we need a to be odd, r,k to be even, and $k|ar$. Let $k = k_1k_2$, so $a = a'k_1$, $r = r'k_2$.

1. Make a $RD(k_1,k_2,k)''$.
   If $a' \equiv \pm 3 \pmod{18}$, use Theorem 5.3 to make our $RD(3k_1,k_2,k)''$.

2. Use Theorem 5.2 to blow up the $RD(k_1,k_2,k)''$ (or our $RD(3k_1,k_2,k)''$) into a $RD(a'k_1,k_2,k)''$.

3. Use Lemma 5.2 to blow up the $RD(a'k_1,k_2,k)^*$ into a $RD(a'k_1,r'k_2,k)^*$ (a $RD(a,r,k)^*$).

For the first step, we'll need a $RD(k_1,k_2,k_1k_2)''$, with k_1 odd, k_2 even, and $(k_1,k_2) \neq (1,2)$. We can create a $RD(k_1,k_2,k_1k_2)^*$ by Theorem 4.1. We will create the $2RD(k_1,k_2,k_1k_2)$ using cracked easter eggs as before. □

References

[1] J.-C. Bermond, O. Favaron, and M. Maheo, Hamilton Decomposition of Cayley Graphs of Degree 4, *Journal of Combinatorial Theory, Ser. B* 46(1989), 142-153.

[2] D.G. Hoffman and P.J. Schellenburg, The existence of C_k-factorizations of $K_{2n} - F$, *Discrete Mathematics*, 97(1991), 243-250.

[3] J. Liu, A Generalization of the Oberwolfach Problem and C_t-factorizations of Complete Equipartite Graphs, Journal of Combinatorial Designs 8(2000), 42-49.

[4] J. Liu, A Complete Solution to the Generalized Oberwolfach Problem with Uniform Table Sizes, J. Combin. Theory Ser. A. 101 (2003), no. 1, 20–34.

[5] T. Storer, *Cyclotomy and Difference Sets*, Markham Publishing Company, Chicago, IL, 1967.

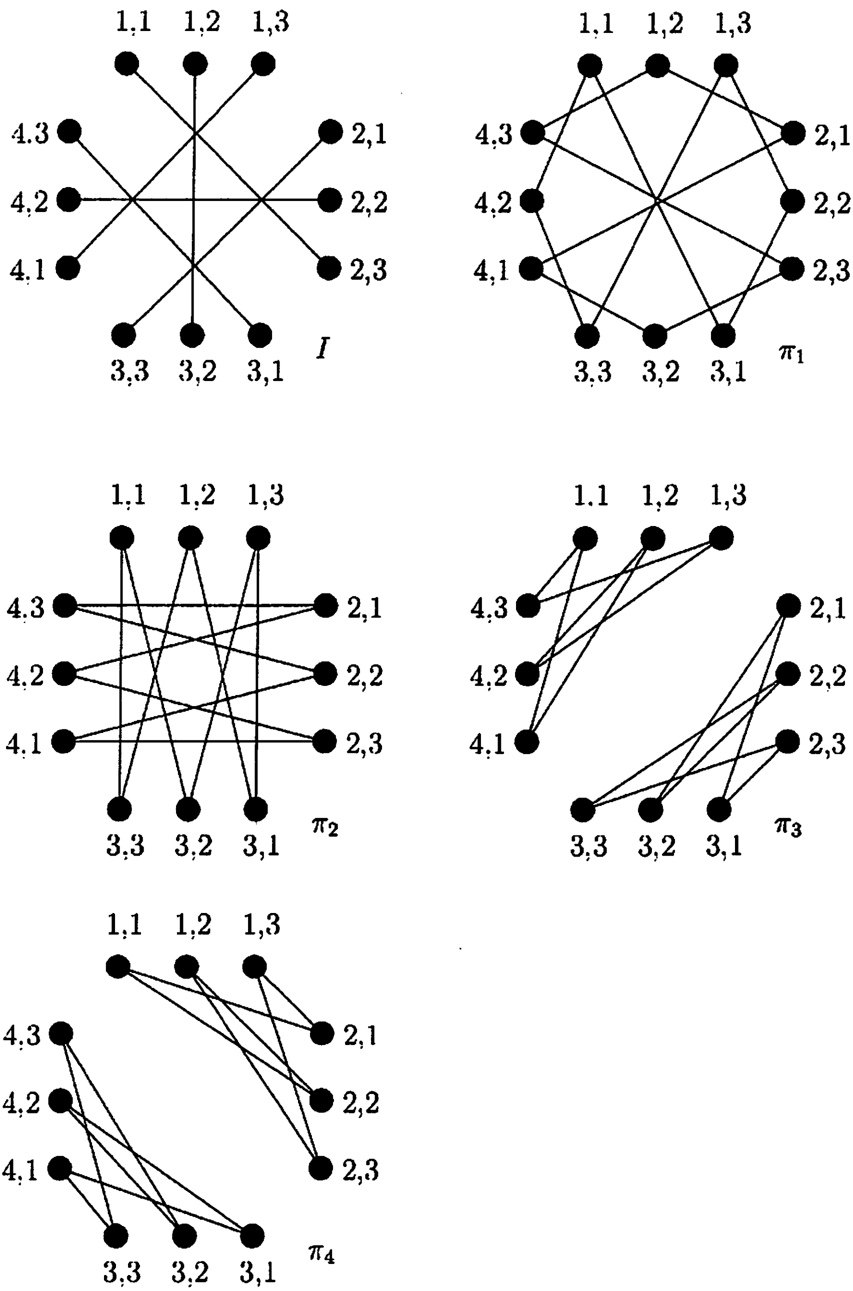


Figure 1: $RD(3, 4, 6)^*$