A note on the minus edge domination number in graphs

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Abstract

The closed neighborhood $N_G[e]$ of an edge e in a graph G is the set consisting of e and of all edges having a common end-vertex with e. Let f be a function on $E(G)$, the edge set of G, into the set $\{-1,0,1\}$. If $\sum_{x\in N_G[e]} f(x) \geq 1$ for each $e \in E(G)$, then f is called a minus edge dominating function of G. The minimum of the values $\sum_{e\in E(G)} f(e)$, taken over all minus edge dominating function f of G, is called the minus edge domination number of G and is denoted by $\gamma'_m(G)$. It has been conjectured that $\gamma'_m(G) \geq n-m$ for every graph G of order n and size m. In this paper we prove that this conjecture is true and then classify all graphs G with $\gamma'_m(G) = n-m$.

Keyword: Minus edge dominating function; Minus edge domination number

1 Introduction

Let G be a simple graph with no isolates, vertex set $V(G)$ and edge set $E(G)$. We use [4] for terminology and notation which are not defined here. Two edges e_1, e_2 of G are called *adjacent* if they are distinct and have a common end-vertex. The *open neighborhood* $N_G(e)$ of an edge $e \in E(G)$ is the set of all edges adjacent to e. Its *closed neighborhood* is $N_G[e] = N_G(e) \cup \{e\}$. For a function $f : E(G) \longrightarrow \{-1,0,1\}$ and a subset S of $E(G)$ we define $f(S) = \sum_{e\in S} f(e)$. If $S = N_G[e]$ for some $e \in E$,

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then we denote $f(S)$ by $f[e]$. For each vertex $v \in V(G)$ we also define $f(v) = \sum_{e \in E(v)} f(e)$, where $E(v)$ is the set of all edges incident to vertex v. A function $f : E(G) \longrightarrow \{-1, 0, 1\}$ is called a *minus edge dominating function* (MEDF) of G, if $f[e] \geq 1$ for each edge $e \in E(G)$. The minimum of the values $f(E(G))$, taken over all minus edge dominating functions f of G, is called *minus edge domination number* of G. The minus edge domination number was introduced by Xu and Zhou in [9] and denoted by $\gamma'_m(G)$. The minus edge dominating function f of G with $f(E(G)) = \gamma'_m(G)$ is called $\gamma'_m(G)$*-function.*

A function $f : E(G) \longrightarrow \{-1, 1\}$ is called a *signed edge dominating function* (SEDF) of G, if $f[e] \geq 1$ for each edge $e \in E(G)$. The minimum of the values $f(E(G))$, taken over all signed edge dominating functions f of G, is called *signed edge domination number* of G. The signed edge domination number was introduced by Xu in [5] and denoted by $\gamma'_s(G)$. The signed edge dominating function f of G with $f(E(G)) = \gamma'_s(G)$ is called $\gamma'_s(G)$*-function.* This parameter has been studied by several authors [1, 2, 3, 5, 6, 7, 8, 10, 11].

In 2007, it was conjectured [9] that for all graphs G of order n, size m and with no isolates, $\gamma'_m(T) \geq n - m$. In Section 2, we first prove that this conjecture is true. Then we characterize all graphs for which $\gamma'_m(T) = n - m$.

When G is not connected, let $G_1, \cdots, G_k$ be its components. Then $\gamma'_m(G)$ and $\gamma'_s(G)$ exist if each G_i has order at least 2, and $\gamma'_m(G) = \sum_{i=1}^{k} \gamma'_m(G_i)$. Hence, it is sufficient we prove the conjecture for connected graphs. Throughout this paper $\ell(v)$ denotes the number of pendant edges incident to vertex v.

Here are some well-known results on $\gamma'_m(G)$.

Theorem A. (See [9]) For any graph G of order n and maximum degree Δ,

$$\gamma'_m(G) \geq \frac{(4-n)\Delta}{4}.$$

Theorem B. (See [9]) For any graph G of order n and size m,

$$\gamma'_m(G) \geq \frac{4m - (\Delta - \delta)n^2}{4(2\Delta - 1)}.$$

We will use the following observation and properties.

Observation 1. For any simple graph G of order $n \geq 2$ and with no isolates,

$$\gamma'_m(G) \leq \gamma'_s(G).$$

Theorem C. (See [1]) For any tree T of order $n \geq 2$, $\gamma'_s(T) \geq 1$ with equality if and only if T has no vertex of even degree and $\ell(v) \geq \lfloor(\deg(v) - 1)/2\rfloor$ for every vertex v. In addition, if $\gamma'_s(T) = 1$ and f is a $\gamma'_s(T)$-function, then

1. $f(e) = 1$ for each non-pendant edge $e \in E(T)$;
2. $f(v) = 1$ for each vertex v of degree greater than 1;
3. $f[e] = 1$ for each edge $e \in E(T)$.

Theorem D. (See [7]) Let G be a graph with $\delta(G) \geq 1$. Then $\gamma'_s(G) \geq |V(G)| - |E(G)|$ and this bound is sharp.

2 A proof of the conjecture

In 2007, Xu and Zhou [9] conjectured that $\gamma'_m(G) \geq n - m$ for every graph G of order n, size m and with no isolates. In this section we prove that this conjecture is true. We also characterize all graphs G for which $\gamma'_m(G) = n - m$.

Theorem 2. For every simple connected graph G of order $n \geq 2$ and size m, $\gamma'_m(G) \geq n - m$.

Proof. The proof is by induction on m. Obviously the statement is true for $m = 1, 2$. Assume the statement is true for all simple connected graphs of size less than m, where $m \geq 3$. Let G be a simple connected graph of size m and f a $\gamma'_m(G)$-function. We may assume $Z = \{e \in E(G) \mid f(e) = 0\} \neq \emptyset$ for otherwise f is an SEDF on G and the result follows by Theorem D. Consider two cases.

Case 1. There is a non-pendant edge $e = uv \in Z$.
If e is not a bridge, then $G - e$ is connected and $f|_{G-e}$ is an MEDF of $G - e$. By the inductive hypothesis we have

$$f(E(G)) = f|_{G-e}(E(G-e)) \geq n - (m-1) > n - m. \qquad (a)$$

If e is a bridge and G_1 and G_2 are the connected components of $G - e$, then obviously $f|_{G_1}$ and $f|_{G_2}$ are MEDFs of G_1 and G_2, respectively. By the inductive hypothesis we have

$$\begin{aligned} f(E(G)) &= f|_{G_1}(E(G_1)) + f|_{G_2}(E(G_2)) \\ &\geq |V(G_1)| + |V(G_2)| - (|E(G_1)| + |E(G_2)|) \qquad (b) \\ &> n - m. \end{aligned}$$

Case 2. The only edges e for which $f(e) = 0$ are pendant edges. Let $e \in Z$. Then the function f, restricted to $G - e$, is obviously an MEDF for $G - e$ and by inductive hypothesis we have

$$f(E(G)) = f(E(G - e)) \geq (n - 1) - (m - 1) = n - m.$$

This completes the proof. □

Now we characterize all simple connected graphs G for which $\gamma'_m(G) = n - m$. The following two lemmas explore the structure of γ'_m-functions of such graphs.

Lemma 3. Let T be a tree of order $n \geq 2$ with $\gamma'_m(T) = 1$. If f is a $\gamma'_m(T)$-function, then

1. $f(e) = 1$ for each non-pendant edge $e \in E(T)$;
2. $f(v) = 1$ for each vertex v of degree greater than 1;
3. $f[e] = 1$ for each edge $e \in E(T)$.

Proof. The proof is by induction on n. The statements are obviously true for $n = 2$. Assume $n \geq 3$ and the statements are true for all trees T of order less than n. Let T be a tree of order n and f be a $\gamma'_m(T)$-function. We may assume $Z = \{e \in E(T) \mid f(e) = 0\} \neq \emptyset$ for otherwise f is an SEDF on T and the result follows by Theorem C. By (b) all edges in Z are pendant edges. Let $e = uv \in Z$ in which $\deg(u) = 1$. Then the function f, restricted to $T - u$, is obviously an MEDF for $T - u$ and we have

$$f(E(T - u)) = f(E(T)) = 1.$$

By Theorem 2, $\gamma'_m(T - u) = 1$ and so the function f, restricted to $T - u$, is a $\gamma'_m(T - u)$-function. If $\deg(v) \geq 3$, then the results follows by inductive hypothesis. Let $\deg(v) = 2$ and $vw \in E(T)$. Since $f[uv] \geq 1$, we have $f(vw) = 1$. It follows that $f[uv] = 1$ and $f(v) = 1$. Now the result follows by inductive hypothesis and the proof is complete. □

Lemma 4. Let G be a simple connected graph of order $n \geq 2$ and size m with $\gamma'_m(G) = n - m$. Let f be a γ'_m-function for G. Then

1. $f(e) = 1$ for each non-pendant edge $e \in E(G)$;
2. $f(v) = 1$ for each vertex v of degree greater than 1;
3. $f[e] = 1$ for each edge $e \in E(G)$.

Proof. Let k be the number of cycles of G. The proof is by induction on k. The statements are true for $k = 0$ by Lemma 3. Assume the statements are true for all simple connected graphs G for which the number of cycles is less than k, where $k \geq 1$. Let G be a simple connected graph with k cycles. Let f be a $\gamma'_m(G)$-function.

(1) Let, to the contrary, $e = uv$ be a non-pendant edge such that $f(e) = 0$ or -1. By (a) and (b), we have $f(e) = -1$. Consider two cases.

Case 1. e is not a bridge. Obviously, $f|_{G-e}$ is an MEDF for $G - e$. Moreover, $G - e$ has at most $k - 1$ cycles. By Theorem 2 we have

$$n-(m-1) \leq f|_{G-e}(E(G-e)) = f(E(G))+1 = n-m+1 = n-(m-1).$$

Therefore, f, restricted to $G - e$, is a γ'_m-function. Now let e' be an edge at u in $G - e$. Then by the inductive hypothesis we have $f|_{G-e}[e'] = 1$. Hence, $f[e'] = 0$ in G, a contradiction.

Case 2. e is a bridge. Let G_1 and G_2 be the connected components of $G - e$ with $u \in V(G_1)$. Obviously, $f|_{G_1}$ and $f|_{G_2}$ are MEDFs of G_1 and G_2, respectively. We have

$$n - m = f(E(G)) = -1 + f|_{G_1}(E(G_1)) + f|_{G_2}(E(G_2)) \geq n - m.$$

This implies that $f|_{G_1}$ and $f|_{G_2}$ are γ'_m-functions of G_1 and G_2, respectively. Without loss of generality we may assume the number of cycles of G_1 is less than k. Let $e' \in E(G_1)$ be an edge at u. By the inductive hypothesis we have $f|_{G_1}[e'] = 1$ which implies $f[e'] = 0$ in G, a contradiction.

(2) Let $e = uv$ be a non-bridge edge of G. By Part 1 we have $f(e) = 1$. Let G' be obtained from $G - e$ by adding new pendant edges uw_1 and vw_2 at u and v, respectively. Define $g : E(G') \longrightarrow \{-1, 0, 1\}$ by:

$$g(uw_1) = g(vw_2) = 1 \text{ and } g(a) = f(a) \text{ if } a \in E(G-e).$$

Then g is an MEDF of G' and $g(E(G')) = f(E(G)) + 1 = (n - m) + 1 = |V(G')| - |E(G')|$. Thus g is a γ'_m-function of G'. Since G' has at most $k - 1$ cycles the result follows by the inductive hypothesis on G'.

(3) Part 3 follows from Parts 1 and 2. □

Define $\mathcal{F}_0$ to be the collection of all simple connected graphs of order $n \geq 2$ in which $\ell(v) \geq (\deg(v) - 1)/2$ for every vertex v.

Theorem 5. Let G be a simple connected graph of order $n \geq 2$ and size m. Then $\gamma'_m(G) = n - m$ if and only if $G \in \mathcal{F}_0$.

Proof. If $\gamma'_m(G) = n - m$, then by Lemma 4 it is straightforward to see that $G \in \mathcal{F}_0$. Conversely, let $G \in \mathcal{F}_0$. By Theorem 2 we have $\gamma'_m(G) \geq n - m$. By induction on k, the number of cycles of G, we find an MEDF of G, say g, such that $g(E(G)) = n - m$. Let $k = 0$. Assume G_1 is obtained from G by deleting exactly one pendant edge from each vertex of even degree. Obviously G_1 has no vertex of even degree and $\ell(v) \geq \lfloor(\deg(v) - 1)/2\rfloor$ for every vertex $v \in V(G_1)$. Then by Theorem C, $\gamma'_s(G_1) = |V(G_1)| - |E(G_1)|$. Let f be a $\gamma'_s(G_1)$-function. Then $f(E(G_1)) = |V(G_1)| - |E(G_1)|$. Define $g : E(G) \longrightarrow \{-1, 0, 1\}$ by

$$g(e) = f(e) \text{ if } e \in E(G_1) \text{ and } g(e) = 0 \text{ if } e \in E(G) - E(G_1).$$

Obviously g is an MEDF on G and

$$g(E(G)) = f(E(G_1)) = |V(G_1)| - |E(G_1)| = n - m.$$

Now assume there is an MEDF with the required property for every simple connected graph G for which the number of cycles is less than k, where $k \geq 1$. Let G be a simple connected graph with k cycles. Let $e = uv$ be a non-bridge edge in G. By assumption there are pendant edges, say uu', vv' at u and v, respectively. Let $G' = G \setminus \{uv, uu', vv'\}$. Obviously, $G' \in \mathcal{F}_0$. So by the inductive hypothesis there is an MEDF f of G' with $f(E(G')) = |V(G')| - |E(G')| = n - m + 1$. If $\deg_{G'}(u) \geq 2$, then by Lemma 4 we have $f(u) = 1$. Let $\deg_{G'}(u) = 1$ and $uu_1 \in E(G)$. Since $G \in \mathcal{F}_0$ it follows that $\ell_{G'}(u_1) \geq (\deg(u_1) + 1)/2$. Hence f assigns 1 to at least a pendant edge at u_1 or assigns 0 to at least two pendant edge incident to u_1. First let f assigns 0 to at least two pendant edge incident to u_1. Without loss of generality we may assume $f(u_1w) = f(u_1u) = 0$ where u_1w, u_1u are pendant edges at u_1. Then the mapping $h : E(G) \longrightarrow \{-1, 0, +1\}$ defined by

$$h(u_1u) = 1, h(u_1w) = -1 \text{ and } h(e) = f(e) \text{ if } e \in E(G') \setminus \{u_1u, u_1w\},$$

is an MEDF of G' for which $h(E(G')) = f(E(G'))$ and $h(u_1u) = 1$. Thus, we may assume f assigns 1 to at least a pendant edge at u_1. Without loss of generality we may assume $f(uu_1) = 1$ and hence, $f(u) = 1$. Similarly, we may assume $f(v) = 1$. Define $g : E(G) \longrightarrow \{-1, 0, 1\}$ by:

$$g(uu') = g(vv') = -1,\ g(uv) = 1 \text{ and } g(a) = f(a) \text{ if } a \in E(G').$$

Then g is an MEDF of G and $g(E(G)) = n - m$. This completes the proof. □

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