Sums of products of generalized Fibonacci and Lucas numbers

Hacène Belbachir[1] and Farid Bencherif[2]

USTHB, Department of Mathematics,
P.B. 32 El Alia, 16111, Algiers, Algeria.
hbelbachir@usthb.dz or hacenebelbachir@gmail.com
fbencherif@usthb.dz or fbencherif@gmail.com

Abstract

We establish several formulae for sums and alternating sums of products of generalized Fibonacci and Lucas numbers. In particular, we extend some results of Z. Čerin, and of Z. Čerin and G. M. Gianella.

Keywords. Fibonacci numbers; Lucas numbers; Pell numbers; Alternating sums; Integer sequences.
AMS Classification. 11B39, 11Y55.

1 Introduction and the main result

Let p and q be two integers such that $pq \neq 0$ and $\Delta := p^2 - 4q \neq 0$. We define sequences of generalized Fibonacci and Lucas numbers $(U_n) = (U_n^{(p,q)})$ and $(V_n) = (V_n^{(p,q)})$, for all n, by induction

$$\begin{cases} U_0 = 0, \; U_1 = 1, \; U_n = pU_{n-1} - qU_{n-2}, \\ V_0 = 2, \; V_1 = p, \; V_n = pV_{n-1} - qV_{n-2}. \end{cases}$$

Sequences of Fibonacci (F_n), Lucas (L_n), Pell (P_n), Pell-Lucas (Q_n), Jacobsthal (J_n), Jacobsthal-Lucas (j_n), listed as A000045, A00032, A000129, A002203, A001045, A014551 respectively in Sloane [16], are special cases of sequences (U_n) and (V_n). In fact $(F_n, L_n) = (U_n^{(1,-1)}, V_n^{(1,-1)})$ for $n \geq 0$, $(P_n, Q_n) = (U_n^{(2,-1)}, V_n^{(2,-1)})$ for $n \geq 0$, and $(J_n, j_n) = (U_n^{(1,-2)}, V_n^{(1,-2)})$ for $n \geq 0$.

[1] The research is supported by LAID3 Laboratory of USTHB University.
[2] The research is supported by LATN Laboratory of USTHB University.

For two integers r and s and for all sequences $X = (X_m)_{m\in\mathbb{Z}}$ and $Y = (Y_m)_{m\in\mathbb{Z}}$, set

$$S_n^{(r,s)}(X,Y) := \sum_{i=0}^{n} X_{r+2i}Y_{s+2i} \quad \text{and} \quad A_n^{(r,s)}(X,Y) := \sum_{i=0}^{n} (-1)^i X_{r+2i}Y_{s+2i}.$$

We also set $S_n^{(r,s)}(X) := S_n^{(r,s)}(X,X)$ and $A_n^{(r,s)}(X) := A_n^{(r,s)}(X,X)$.

Sums involving Fibonacci, Lucas, Pell and Pell-Lucas numbers and some generalizations have been studied by several authors. For example, for trigonometric sums, see Melham [12] and Belbachir & Bencherif [1], for reciprocal and power sums, see Melham [13, 14], and for the sum of squares see Long [11]. The sums $S_n^{(r,s)}(X)$ and $A_n^{(r,s)}(X)$, for $r-s \in \{0,1\}$ and $X \in \{F, L, P, Q, J, j\}$, were studied by Čerin, and Čerin & Gianella: $S_n^{(r,s)}(F)$ [2], $A_n^{(r,s)}(F)$ [3] and $A_n^{(r,s)}(L)$ [5], $S_n^{(r,s)}(J)$ and $A_n^{(r,s)}(J)$ [8], $S_n^{(r,s)}(j)$ and $A_n^{(r,s)}(j)$, [9], $S_n^{(r,s)}(P)$ [4], $S_n^{(r,s)}(Q)$ and $A_n^{(r,s)}(Q)$ [6], $A_n^{(r,s)}(P)$ and again $S_n^{(r,s)}(P)$ [7].

In this article, we give simplified expressions for the sums $S_n^{(r,s)}(X,Y)$ and $A_n^{(r,s)}(X,Y)$, where X and Y can be either U or V, when $q = \pm 1$. The results obtained can be applied to $(U,V) \in \{(F,L),(P,Q),(J,j)\}$ for $(p,q) \in \{(3,-1),(3,1),(4,-1),(4,1),(5,-1),(5,1)\}$ listed in [16] as ($A006190$, $A006497$), ($A001906$, $A005248$), ($A001076$, $A014448$), ($A001353$, $A003500$), ($A052918$, $A087130$), ($A004254$, $A003501$). We also give a unified improvement of recent papers of Čerin [2, 3, 5] and Čerin & Gianella [4, 6, 7] cited above. However, the formulae obtained in the case $q = \pm 1$ do not allow to find the formulae obtained by Čerin [8], and [9] (case $q = 2$). Indeed, it is only under the condition $q = \pm 1$ that we can obtain the simple identities leading to our results. This is the main reason for which we suppose $q = \pm 1$ in all what follows, except for Lemma 5.

To state our main result, consider the following sequences (a_n), (b_n) and (c_n) defined, for $n \in \mathbb{Z}$, by

$$a_n = \frac{U_{2n}}{U_2}, \quad b_n = \frac{V_{4n+2}}{V_2}, \quad c_n = \frac{U_{4n+4}}{U_4}.$$

Note, that these sequences are well defined when $pq \neq 0$ and $q = \pm 1$. However, $U_2 = p \neq 0$, $V_2 = p^2 - 2q \neq 0$, and $U_4 = pV_2 \neq 0$. They satisfy the following recurrence relations

$$\begin{cases} a_{-1} = -1, & a_0 = 0, \\ a_n = V_2 a_{n-1} - a_{n-2}, \end{cases} \quad \begin{cases} b_{-1} = 1, & b_0 = 1, \\ b_n = V_4 b_{n-1} - b_{n-2}, \end{cases} \quad \begin{cases} c_{-1} = 0, & c_0 = 1, \\ c_n = V_4 c_{n-1} - c_{n-2}. \end{cases}$$

The sequences of integers (a_n), (b_n) and (c_n), when $q = -1$ and $p = 1$ (resp. 2) are listed in Sloane [16] as A001906, A049685 and A004187 (respectively A001109, A077420 and A029547).

The following Theorem is the main result of this paper. For an integer $n \geq 0$, let $\varepsilon_n = (1 + (-1)^n)/2$.

Theorem 1 *For all integers r, s, and for all positive integer n, we have*

$$\begin{aligned}
\Delta S_n^{(r,s)}(U) &= p^{-1}[U_{4n+r+s+2} - U_{r+s-2}] - (n+1)\,q^r V_{s-r}, & (1)\\
&= a_{n+1} V_{2n+r+s} - (n+1)\,q^r V_{s-r}. & (2)\\
S_n^{(r,s)}(V) &= p^{-1}[U_{4n+r+s+2} - U_{r+s-2}] + (n+1)\,q^r V_{s-r}, & (3)\\
&= a_{n+1} V_{2n+r+s} + (n+1)\,q^r V_{s-r}. & (4)\\
\Delta S_n^{(r,s)}(U,V) &= p^{-1}[V_{4n+r+s+2} - V_{r+s-2}] - (n+1)\,\Delta q^r U_{s-r}, & (5)\\
&= a_{n+1}\Delta U_{2n+r+s} - (n+1)\,\Delta q^r U_{s-r}. & (6)\\
\Delta S_n^{(r,s)}(V,U) &= p^{-1}[V_{4n+r+s+2} - V_{r+s-2}] + (n+1)\,\Delta q^r U_{s-r}, & (7)\\
&= a_{n+1}\Delta U_{2n+r+s} + (n+1)\,\Delta q^r U_{s-r}. & (8)\\
\Delta A_n^{(r,s)}(U) &= V_2^{-1}[V_{r+s-2} + (-1)^n V_{4n+r+s+2}] - \varepsilon_n q^r V_{s-r}, & (9)\\
&= \begin{cases} b_m V_{4m+r+s} - q^r V_{s-r}, & n = 2m;\\ -p\Delta c_m U_{4m+r+s+2}, & n = 2m+1. \end{cases} & (10)\\
A_n^{(r,s)}(V) &= V_2^{-1}[V_{r+s-2} + (-1)^n V_{4n+r+s+2}] + \varepsilon_n q^r V_{s-r}, & (11)\\
&= \begin{cases} b_m V_{4m+r+s} + q^r V_{s-r}, & n = 2m;\\ -p\Delta c_m U_{4m+r+s+2}, & n = 2m+1. \end{cases} & (12)\\
A_n^{(r,s)}(U,V) &= V_2^{-1}[U_{r+s-2} + (-1)^n U_{4n+r+s+2}] - \varepsilon_n q^r U_{s-r}, & (13)\\
&= \begin{cases} b_m U_{4m+r+s} - q^r U_{s-r}, & n = 2m;\\ -p c_m V_{4m+r+s+2}, & n = 2m+1. \end{cases} & (14)\\
A_n^{(r,s)}(V,U) &= V_2^{-1}[U_{r+s-2} + (-1)^n U_{4n+r+s+2}] + \varepsilon_n q^r U_{s-r}, & (15)\\
&= \begin{cases} b_m U_{4m+r+s} + q^r U_{s-r}, & n = 2m;\\ -p c_m V_{4m+r+s+2}, & n = 2m+1. \end{cases} & (16)
\end{aligned}$$

Corollary 2 *For all integers r, s, t, and for all positive integer n, we have*

$$\begin{aligned}
2p\Delta S_n^{(r+t,s-t)}(U) &= \lambda_1 V_{4n+s} + \lambda_2 U_{4n+s} - \lambda_3 V_s & (17)\\
&\quad -\lambda_4 U_s - 2(n+1)pq^{r+t} V_{s-r-2t}.\\
2pS_n^{(r+t,s-t)}(V) &= \lambda_1 V_{4n+s} + \lambda_2 U_{4n+s} - \lambda_3 V_s & (18)\\
&\quad -\lambda_4 U_s + 2(n+1)pq^{r+t} V_{s-r-2t}.\\
2V_2\Delta A_n^{(r+t,s-t)}(U) &= \lambda_2(-1)^n V_{4n+s} + \lambda_1(-1)^n \Delta U_{4n+s} & (19)\\
&\quad +\lambda_4 V_s + \lambda_3\Delta U_s - 2\varepsilon_n q^{r+t} V_2 V_{s-r-2t}.\\
2V_2 A_n^{(r+t,s-t)}(V) &= \lambda_2(-1)^n V_{4n+s} + \lambda_1(-1)^n \Delta U_{4n+s} & (20)\\
&\quad +\lambda_4 V_s + \lambda_3\Delta U_s + 2\varepsilon_n q^{r+t} V_2 V_{s-r-2t}.
\end{aligned}$$

with $\lambda_1 = U_{r+2}$, $\lambda_2 = V_{r+2}$, $\lambda_3 = U_{r-2}$, $\lambda_4 = V_{r-2}$.

Corollary 3 *For all integers r, s, t, and for all positive integer n, we have*

$$\Delta\left(S_n^{(s,s+t)}(U)-q^{s-r}S_n^{(r,r+t)}(U)\right)$$
$$=S_n^{(s,s+t)}(V)-q^{s-r}S_n^{(r,r+t)}(V), \tag{21}$$
$$=a_{n+1}\Delta U_{s-r}U_{2n+r+s+t}. \tag{22}$$
$$\Delta S_n^{(s,s+t)}(U)+q^{s-r}S_n^{(r,r+t)}(V)$$
$$=S_n^{(s,s+t)}(V)+\Delta q^{s-r}S_n^{(r,r+t)}(U), \tag{23}$$
$$=a_{n+1}V_{s-r}V_{2n+r+s+t}. \tag{24}$$
$$S_n^{(s,s+t)}(U,V)-q^{s-r}S_n^{(r,r+t)}(U,V)$$
$$=S_n^{(s,s+t)}(V,U)-q^{s-r}S_n^{(r,r+t)}(V,U), \tag{25}$$
$$=a_{n+1}U_{s-r}V_{2n+r+s+t}. \tag{26}$$
$$\Delta\left(A_n^{(s,s+t)}(U)-q^{s-r}A_n^{(r,r+t)}(U)\right)$$
$$=A_n^{(s,s+t)}(V)-q^{s-r}A_n^{(r,r+t)}(V), \tag{27}$$
$$=\begin{cases}\Delta b_m U_{s-r}U_{4m+r+s+t}, & n=2m;\\ -p\Delta c_m U_{s-r}V_{4m+r+s+t+2}, & n=2m+1.\end{cases} \tag{28}$$
$$\Delta A_n^{(s,s+t)}(U)+q^{s-r}A_n^{(r,r+t)}(V)$$
$$=A_n^{(s,s+t)}(V)+\Delta q^{s-r}A_n^{(r,r+t)}(U), \tag{29}$$
$$=\begin{cases}b_m V_{s-r}V_{4m+r+s+t}, & n=2m;\\ -p\Delta c_m V_{s-r}U_{4m+r+s+t+2}, & n=2m+1.\end{cases} \tag{30}$$
$$A_n^{(s,s+t)}(U,V)-q^{s-r}A_n^{(r,r+t)}(U,V)$$
$$=A_n^{(s,s+t)}(V,U)-q^{s-r}A_n^{(r,r+t)}(V,U), \tag{31}$$
$$=\begin{cases}b_m U_{s-r}V_{4m+r+s+t}, & n=2m;\\ -p\Delta c_m U_{s-r}U_{4m+r+s+t+2}, & n=2m+1.\end{cases} \tag{32}$$

Corollary 4 *For all integers r, s, t, j, positive integers n, m, and for* $\lambda=-1,0$ *or* 1*, we have*

$$\begin{aligned} S_n^{(s,s+t)}(U) &= a_{n+1}U_{s-r+2\lambda}U_{2n+r+s+t-2\lambda} \\ &\quad +\lambda q^{s-r}U_{r-\lambda-1}U_{r+t-\lambda-1}+q^{s-r}S_{n-\lambda}^{(r,r+t)}(U). \end{aligned} \tag{33}$$
$$\begin{aligned} S_n^{(s,s+t)}(V) &= a_{n+1}V_{s-r+2\lambda}V_{2n+r+s+t-2\lambda} \\ &\quad -\lambda q^{s-r}\Delta U_{r-\lambda-1}U_{r+t-\lambda-1}-q^{s-r}\Delta S_{n-\lambda}^{(r,r+t)}(U). \end{aligned} \tag{34}$$
$$\Delta A_{2m}^{(s,s+t)}(U) = \Delta b_m U_{2m+s+j}U_{2m+s-j+t}+q^{s+j}b_m V_{t-2j}-q^s V_t, \tag{35}$$
$$= b_m V_{2m+s+j}V_{2m+s-j+t}-q^{s+j}b_m V_{t-2j}-q^s V_t. \tag{36}$$
$$A_{2m+1}^{(s,s+t)}(U) = -pc_m(U_{2m+j+2}V_{2m+2s+t-j}-q^j U_{2s+t-2j-2}). \tag{37}$$

$$A_{2m}^{(s,s+t)}(V) = \Delta b_m U_{2m+s+j} U_{2m+s-j+t} + q^{s+j} b_m V_{t-2j} + q^s V_t, \quad (38)$$
$$= b_m V_{2m+s+j} V_{2m+s-j+t} - q^{s+j} b_m V_{t-2j} + q^s V_t. \quad (39)$$
$$A_{2m+1}^{(s,s+t)}(V) = -p\Delta c_m (U_{2m+j+2} V_{2m+2s+t-j} + q^j U_{2s+t-2j-2}). \quad (40)$$

Before proving our results, let us note that our formulae were tested on a computer. However, we obtained them without the use of computers. The proofs are direct and do not require mathematical induction.

2 Proof of the main result

We shall use the following Lemmas. Many of the identities of Lemmas 5 and 6 are known (see, for instance, [10]). Although the proof of these results is simple, we have outlined it in each lemma for the convenience of the reader.

Lemma 5 *For all integers n, m, q and h, we have*

1. $U_{-n} = -q^{-n} U_n$,
2. $V_{-n} = q^{-n} V_n$,
3. $\Delta U_n U_m = V_{n+m} - q^m V_{n-m}$,
4. $V_n V_m = V_{n+m} + q^m V_{n-m}$,
5. $U_n V_m = U_{n+m} + q^m U_{n-m}$,
6. $V_n U_m = U_{n+m} - q^m U_{n-m}$,
7. $U_n U_{m+h} - U_{n+h} U_m = q^m U_h U_{n-m}$,
8. $V_n V_{m+h} - V_{n+h} V_m = -q^m \Delta U_h U_{n-m}$,
9. $V_n V_{m+h} - \Delta U_{n+h} U_m = q^m V_h V_{n-m}$,
10. $U_n V_{m+h} - U_{n+h} V_m = -q^m U_h V_{n-m}$,
11. $V_n V_m + \Delta U_n U_m = 2V_{n+m}$,
12. $V_n V_m - \Delta U_n U_m = 2q^m V_{n-m}$,
13. $U_n V_m + V_n U_m = 2U_{n+m}$,
14. $U_n V_m - V_n U_m = 2q^m U_{n-m}$.

Proof. We use Binet's forms $\frac{\alpha^n - \beta^n}{\alpha - \beta}$ and $\alpha^n + \beta^n$ of U_n and V_n, where α and β are the roots of $x^2 - px + q = 0$. We notice that: $\alpha + \beta = p$, $\alpha\beta = q$, $(\alpha - \beta)^2 = \Delta$. Let us prove the relation 9: $V_n V_{m+h} - \Delta U_{n+h} U_m = (\alpha^n + \beta^n)(\alpha^{m+h} + \beta^{m+h}) - (\alpha^{n+h} - \beta^{n+h})(\alpha^m - \beta^m) = (\alpha\beta)^m (\alpha^h + \beta^h)(\alpha^{n-m} + \beta^{n-m}) = q^m V_h V_{n-m}$. ∎

Lemma 6 *For all integer r, and for all positive integer n, we have*

1. $\Delta U_2 \sum_{i=0}^{n} U_{r+4i} = V_{4n+r+2} - V_{r-2} = \Delta U_{2n+r} U_{2n+2} = p\Delta a_{n+1} U_{2n+r}$,
2. $U_2 \sum_{i=0}^{n} V_{r+4i} = U_{4n+r+2} - U_{r-2} = V_{2n+r} U_{2n+2} = p a_{n+1} V_{2n+r}$,
3. $V_2 \sum_{i=0}^{n} (-1)^i U_{r+4i} = (-1)^n U_{4n+r+2} + U_{r-2}$

$$= \begin{cases} U_{2n+r}V_{2n+2} = V_2 b_m U_{4m+r}, & n = 2m; \\ -V_{2n+r}U_{2n+2} = -V_2 p c_m V_{4m+r+2}, & n = 2m+1. \end{cases}$$

4. $V_2 \sum_{i=0}^{n} (-1)^i V_{r+4i} = (-1)^n V_{4n+r+2} + V_{r-2}$

$$= \begin{cases} V_{2n+r}V_{2n+2} = V_2 b_m V_{4m+r}, & n = 2m; \\ -\Delta U_{2n+r}U_{2n+2} = -V_2 p \Delta c_m U_{4m+r+2}, & n = 2m+1. \end{cases}$$

Proof. Let us prove the first relation. From the 3. of Lemma 5, we have: $\Delta U_2 \sum_{i=0}^{n} U_{r+4i} = \sum_{i=0}^{n}(V_{r+4i+2} - V_{r+4i-2}) = \sum_{i=1}^{n+1} V_{r+4i-2} - \sum_{i=0}^{n} V_{r+4i-2} = V_{4n+r+2} - V_{r-2} = \Delta U_{2n+r}U_{2n+2} = p\Delta a_{n+1}U_{2n+r}$. ■

Lemma 7 *For all integers r, s, and for all positive integer n, we have*

1. $S_n^{(r,s)}(V) + \Delta S_n^{(r,s)}(U) = 2p^{-1}[U_{4n+r+s+2} - U_{r+s-2}]$
$= 2a_{n+1}V_{2n+r+s}$.

2. $S_n^{(r,s)}(V) - \Delta S_n^{(r,s)}(U) = 2(n+1)q^r V_{s-r}$.

3. $\Delta(S_n^{(r,s)}(V,U) + S_n^{(r,s)}(U,V)) = 2p^{-1}[V_{4n+r+s+2} - V_{r+s-2}]$
$= 2a_{n+1}\Delta U_{2n+r+s}$.

4. $\Delta(S_n^{(r,s)}(V,U) - S_n^{(r,s)}(U,V)) = 2(n+1)\Delta q^r U_{s-r}$.

5. $A_n^{(r,s)}(V) + \Delta A_n^{(r,s)}(U) = 2V_2^{-1}[(-1)^n V_{4n+r+s+2} + V_{r+s-2}]$

$$= \begin{cases} 2b_m V_{4m+r+s}, & n = 2m; \\ -2p\Delta c_m U_{4m+r+s+2}, & n = 2m+1. \end{cases}$$

6. $A_n^{(r,s)}(V) - \Delta A_n^{(r,s)}(U) = 2\varepsilon_n q^r V_{s-r}$.

7. $A_n^{(r,s)}(V,U) + A_n^{(r,s)}(U,V) = 2V_2^{-1}[(-1)^n U_{4n+r+s+2} + U_{r+s-2}]$

$$= \begin{cases} 2b_m U_{4m+r+s}, & n = 2m; \\ -2pc_m V_{4m+r+s+2}, & n = 2m+1. \end{cases}$$

8. $A_n^{(r,s)}(V,U) - A_n^{(r,s)}(U,V) = 2\varepsilon_n q^r U_{s-r}$.

Proof. Let us prove the first and the fifth ones. Using relations of Lemma 5 and Lemma 6, we have

1. $S_n^{(r,s)}(V) + \Delta S_n^{(r,s)}(U) = \sum_{i=0}^{n}(V_{r+2i}V_{s+2i} + \Delta U_{r+2i}U_{s+2i})$
$= 2\sum_{i=0}^{n} V_{r+s+4i}$,
$= 2p^{-1}[U_{4n+r+s+2} - U_{r+s-2}] = 2a_{n+1}V_{2n+r+s}$.

5. $A_n^{(r,s)}(V) + \Delta A_n^{(r,s)}(U) = \sum_{i=0}^{n}(-1)^i(V_{r+2i}V_{s+2i} + \Delta U_{r+2i}U_{s+2i})$
$= 2\sum_{i=0}^{n}(-1)^i V_{r+s+4i}$,
$= 2V_2^{-1}[(-1)^n V_{4n+r+s+2} + V_{r+s-2}]$

$$= \begin{cases} 2b_m V_{4m+r+s}, & n = 2m; \\ -2p\Delta c_m U_{4m+r+s+2}, & n = 2m+1. \end{cases}$$ ■

Proof of Theorem 1. From 1 and 2 of Lemma 7, we have

$$\begin{aligned} \Delta S_n^{(r,s)}(U) &= ((S_n^{(r,s)}(V) + \Delta S_n^{(r,s)}(U) - (S_n^{(r,s)}(V) - \Delta S_n^{(r,s)}(U)))/2, \\ &= p^{-1}[U_{4n+r+s+2} - U_{r+s-2}] - (n+1)q^r V_{s-r} \\ &= a_{n+1}V_{2n+r+s} - (n+1)q^r V_{s-r}. \end{aligned}$$

$$S_n^{(r,s)}(V) = ((S_n^{(r,s)}(V) + \Delta S_n^{(r,s)}(U) + (S_n^{(r,s)}(V) - \Delta S_n^{(r,s)}(U)))/2, \\ = p^{-1}[U_{4n+r+s+2} - U_{r+s-2}] + (n+1)\, q^r V_{s-r} \\ = a_{n+1} V_{2n+r+s} + (n+1)\, q^r V_{s-r}.$$

Thus, we get (1), (2), (3) and (4). ■

Proof of Corollary 2. From the 11. and the 13. of Lemma 5, we get

$U_{4n+r+s+2} = (U_{4n+s}V_{r+2} + V_{4n+s}U_{r+2})/2,$
$U_{r+s-2} = (U_s V_{r-2} + V_s U_{r-2})/2,$
$V_{4n+r+s+2} = (V_{4n+s}V_{r+2} + \Delta U_{4n+s}U_{r+2})/2,$
$V_{r+s-2} = (V_s V_{r-2} + \Delta U_s U_{r-2})/2.$

It suffices to substitute these relations in (1), (3), (9) and (11) for $(r+t, s-t)$ instead of (r,s). ■

Proof of Corallary 3. Relations (2) and (4) give respectively

$\Delta S_n^{(s,s+t)}(U) = a_{n+1}V_{2n+2s+t} - (n+1)\, q^s V_t$
and $q^{s-r}\Delta S_n^{(r,r+t)}(U) = a_{n+1}q^{s-r}V_{2n+2r+t} - (n+1)\, q^s V_t,$
$S_n^{(s,s+t)}(V) = a_{n+1}V_{2n+2s+t} + (n+1)\, q^s V_t$
and $q^{s-r}\Delta S_n^{(r,r+t)}(V) = a_{n+1}q^{s-r}V_{2n+2r+t} + (n+1)\, q^s V_t$, then

$$\Delta(S_n^{(s,s+t)}(U) - q^{s-r}S_n^{(r,r+t)}(U)) = S_n^{(s,s+t)}(V) - q^{s-r}S_n^{(r,r+t)}(V), \\ = a_{n+1}(V_{2n+2s+t} - q^{s-r}V_{2n+2r+t}) \\ = a_{n+1}\Delta U_{s-r}U_{2n+r+s+t}.$$

Thus, we get (21) and (22). ■

Proof of Corallary 4. Let us prove the first one. For $\lambda = -1, 0, 1$, we easily check that $S_n^{(r,r+t)}(U) = S_{n-\lambda}^{(r,r+t)}(U) + \lambda U_{r+2n+1-\lambda}U_{r+t+2n+1-\lambda}$. From the 7. of Lemma 5, and noticing that $U_{-2\lambda} = -\lambda U_2$, we have $U_{s-r}U_{2n+r+s+t} = U_{s-r+2\lambda}U_{2n+r+s+t-2\lambda} - \lambda q^{s-r}U_2U_{2n+2r+t-2\lambda}$, and thus, from (22), we obtain

$$S_n^{(s,s+t)}(U) = a_{n+1}U_{s-r}U_{2n+r+s+t} + q^{s-r}S_n^{(r,r+t)}(U) \\ = a_{n+1}U_{s-r+2\lambda}U_{2n+r+s+t-2\lambda} + \lambda q^{s-r}E + q^{s-r}S_{n-\lambda}^{(r,r+t)}(U),$$

with $E = U_N U_{M+H} - U_{N+H}U_M$, where $N = r+t+2n+1-\lambda$, $M = 2n+2$ and $H = r - \lambda - 1$.
Using relation 7 of Lemma 5, we get $E = q^M U_H U_{N-M} = U_{r-\lambda-1}U_{r+t-\lambda-1}$, which completes the proof of (33). The proof of (34) is similar. Relations (35) and (36) follows from (10), noticing that using the 3 and 4 of Lemma 5, we have $V_{4m+2s+t} = \Delta U_{2m+s+j}U_{2m+s-j+t} + q^{s+j}V_{t-2j}$ and $V_{4m+2s+t} = V_{2m+s+j}V_{2m+s-j+t} - q^{s+j}V_{t-2j}$. ■

3 Applications: extensions of Čerin and Čerin & Gianella results

In [4, 6, 7] Čerin and Gianella consider $P_n := 2U_n^{(2,-1)}$ and $Q_n := V_n^{(2,-1)}$.

3.1 Odd and even terms of Fibonacci sequence

In [2], Z. Čerin improved some results of Rajesh and Leversha [15] by proving several sums of odd and even terms of the Fibonacci sequence. Relation (22) when $(p,q) = (1,-1)$ gives for

1. $(r,s,t) = (-2j, 2k-1, 0)$ relation $S_n^{(-2j,-2j)}(F) + S_n^{(2k-1,2k-1)}(F) = F_{2(n+1)}F_{2(k+j)-1}F_{2(k+n-j)-1}$, then, for $n \in \{2j+1, 2j\}$ we get E) and F) in [2].
2. $(r,s,t) = (-1, 2k, 0)$ and $n = j$, relation K) in [2].
3. $(r,s,t) = (-2j, 2k-1, 1)$ relation $S_n^{(-2j,-2j+1)}(F) + S_n^{(2k-1,2k)}(F) = F_{2(n+1)}F_{2(k+j)-1}F_{2(k+n-j)}$, by replacing n successively by 0 with $j=-1$, $2j$ with $j \geq 0$ and $2j+1$ with $j \geq 1$, we get R) in [2].

3.2 Alternating Sums of Fibonacci Products

In [3], Z. Čerin considers alternating sums of odd and even terms of the Fibonacci sequence and alternating sums of their products. The following two relations, deduced from (28) and (10) generalize Čerin's equalities

$q^{s-r+1}\Delta^{-1}b_m V_{4m+2r+t} + q^s\Delta^{-1}V_t + A_{2m}^{(s,s+t)}(U) = b_m U_{s-r}U_{4m+r+s+t}$,

$pq^{s-r}c_m U_{4m+2r+t+2} + A_{2m+1}^{(s,s+t)}(U) = -pc_m U_{s-r}V_{4m+r+s+t+2}$.

Indeed, for $(p,q) = (1,-1)$, by replacing (r,s,t) successively by $(-2m-1, 2k, 0)$, $(-2m-2, 2k-1, 0)$, $(-2m-3, 2k-1, 1)$ in the first relation, respectively by $(-2m-3, 2k, 0)$, $(-2m-2, 2k-1, 0)$, $(-2m-1, 2k-1, 1)$ in the second relation, we get E), L) and Q) respectively relations D), K) and P) in [3].

3.3 Sums of squares and products of Pell numbers

In [4], Z. Čerin and G. M. Gianella establish several formulae for sums of squares of even Pell numbers, sums of squares of odd Pell numbers and sums of products of even and odd Pell numbers.

Relation (33) gives the five theorems in [4]. Indeed, it suffices to take $(p,q) = (2,-1)$ with $(r,s,t,\lambda) \in \{(0,2k,0,0), (1,2k+1,0,1), (0,2k+1,0,0), (1,2k+1,1,1), (0,2k,1,0), (1,2k,1,0), (0,2k,1,-1), (1,2k,1,-1)\}$.

3.4 Some alternating sums of Lucas numbers

In [5], Z. Čerin considers alternating sums of squares of odd and even terms of the Lucas sequence and alternating sums of their products. Taking $j = 2k-s-1$ in (38) and $j = 2s-2k-t+1$ in (40), we get the following generalization of Čerin's equalities

$$\mu_n + A_n^{(s,s+t)}(V) = \begin{cases} \Delta b_m U_{2k+2m-1}U_{2m-2k+2s+t+1}, & n = 2m; \\ -p\Delta c_m U_{2m-2k+2s-t+3}V_{2k+2m+2t-1}, & n = 2m+1. \end{cases}$$

where $\mu_{2m} = -q(b_m V_{t-4k+2s+2} + q^{s+1} V_t)$, $\mu_{2m+1} = pq^{t+1}\Delta c_m U_{4k-2s+3t-4}$. Indeed, for $(p,q) = (1,-1)$ and $(s,t) = \{(2k,0);(2k-1,0);(2k-1,1)\}$, and noticing that $c_m - c_{m-1} = b_m$ and $c_m + c_{m-1} = a_{2m+1}$, one obtains relations $D)$, $E)$, $K)$, $L)$, $P)$ and $Q)$ in [5].

3.5 On sums of squares of Pell-Lucas numbers

In [6], Z. Čerin and G. M. Gianella prove several formulae for sums of squares of even Pell-Lucas numbers, sums of squares of odd Pell-Lucas numbers, and sums of even and odd Pell-Lucas numbers.
For $\lambda = 0$ in (34), $j = s - 2k$ in (39) and $j = 2k + t$ in (40), we get the following extensions of Theorems 1 to 7 of Čerin and Gianella:
$S_n^{(s,s+t)}(V) = -\Delta q^{s-r} S_n^{(r,r+t)}(U) + a_{n+1} V_{s-r} V_{2n+r+s+t}$,

$$A_n^{(s,s+t)}(V) = \begin{cases} b_m\left(V_{2m-2k+2s}V_{2k+2m+t} - V_{4k-2s+t}\right) + q^s V_t, & n = 2m; \\ -p\Delta c_m\left(U_{2k+2m+t+2}V_{2m-2k+2s} + q^t U_{2s-t-4k-2}\right), & n = 2m+1. \end{cases}$$

Indeed, for $(p,q) = (2,-1)$, the first relation, with $(s,t) = (2k,0)$ and $r = 0,2,1,-1$ gives respectively Theorem 1 and relations (2.3), (2.4) and (2.5), with $(s,t) = (2k+1,0)$ and $r \in \{2,3\}$ gives Theorems 2 and 3, and with $(s,t) = (2k,1)$ and $r = 0$ gives Theorem 4. The second relation, with $(s,t) \in \{(2k,0),(2k+1,0),(2k,1)\}$ and observing that $V_2 b_m - 2q = \Delta U_{2m+1}^2$ (from 3 of Lemma 5), gives respectively Theorems 5, 6 and 7.

3.6 On sums of Pell numbers

In [7], Z. Čerin and G. M. Gianella improve their paper [4]. They prove twenty four formulae for sums of a finite number of consecutive terms of various integer sequences related to Pell numbers. Twelve of these formulae can be deduced from relations (17) and (19). Indeed for $(p,q) = (2,-1)$, these two relations become

$$\begin{aligned} S_n^{(r+t,s-t)}(P) &= \tfrac{1}{16}P_{r+2}Q_{4n+s} + \tfrac{1}{16}Q_{r+2}P_{4n+s} - \tfrac{1}{16}P_{r-2}Q_s \\ &\quad -\tfrac{1}{16}Q_{r-2}P_s - \tfrac{(-1)^{r+t}}{2}(n+1)Q_{s-r-2t}, \\ A_n^{(r+t,s-t)}(P) &= \tfrac{(-1)^n}{24}Q_{r+2}Q_{4n+s} + \tfrac{(-1)^n}{12}P_{r+2}P_{4n+s} + \tfrac{1}{24}Q_{r-2}Q_r \\ &\quad +\tfrac{1}{12}P_{r-2}P_s - \tfrac{(-1)^{r+t}}{2}\varepsilon_n Q_{s-r-2t}. \end{aligned}$$

By replacing (r,s,t) successively by $(0,4k,2k)$, $(2,4k,2k-1)$, $(1,2k,k-1)$, $(1,4k,2k-1)$, $(3,4k,2k-2)$, we obtain relations (2.8), (2.9), (2.14), (2.15), (2.16), (2.11), (2.12), (2.20), (2.21), (2.22) in [7]. For $p = 6$, $q = 1$, $r = k$, $s = k+1$, $j = 2k$, using Binet's formulae and noticing that $P_{2n} = 4U_n^{(6,1)}$ and $Q_{2n} = V_n^{(6,1)}$ in relations (17) and (19), we get (2.17) and (2.23).

Acknowledgement. *The authors are grateful to the referee and would like to thank him/her for comments and suggestions which improved the quality of this paper.*

References

[1] H. Belbachir and F. Bencherif, *Unimodality of sequences associated to Pell numbers,* to appear in Ars Combinatoria (2008).

[2] Z. Čerin, *Properties of odd and even terms of the Fibonacci sequence,* Demonstratio Mathematica, 39 (1) (2006), 55-60.

[3] Z. Čerin, *Alternating Sums of Fibonacci Products*, Atti del Seminario Matematico e Fisico dell'Università di Modena e Reggio Emilia, 53 (2005), 331–344.

[4] Z. Čerin and G. M. Gianella, *formulae for sums of squares and products of Pell numbers*, Acc. Sc. Torino - Atti Sci. Fis. 140 (2006), 113–122.

[5] Z. Čerin, *Some alternating sums of Lucas numbers,* Central European Journal of Mathematics, 3 (1), (2005), 1–13.

[6] Z. Čerin and G. M. Gianella, *On sums of squares of Pell-Lucas numbers,* INTEGERS: Electronic Journal of Combinatorial Number Theory, 6 (2006), #A15.

[7] Z. Čerin and G. M. Gianella, *On sums of Pell numbers*, Acc. Sc. Torino - Atti Sci. Fis. 140 (to appear).

[8] Z. Čerin, *Sums of squares and products of Jacobsthal numbers,* Journal of Integer Sequences, Vol. 10 (2007), Art. 07.2.5.

[9] Z. Čerin, *Formulae for sums of Jacobsthal-Lucas numbers*, International Mathematical Forum 2, 40 (2007), 1969–1984.

[10] T. Koshy, *Fibonacci and Lucas numbers with applications*, Wiley-Interscience, NY, (2001).

[11] C. T. Long, *Discovering Fibonacci identities*, The Fibonacci Quarterly, 24, (1986), 160–167.

[12] R. Melham, *Sums involving Fibonacci and Pell numbers.* Portugaliae Mathematica, Vol. 56, Fasc. 3, (1999).

[13] R. Melham, *On sums of powers of terms in a linear recurrence.* Portugaliae Mathematica, Vol. 56, Fasc. 4, (1999).

[14] R. Melham, *A generalization of a result of André-Jeannin concerning summation of reciprocals.* Portugaliae Mathematica, Vol. 57, Fasc. 1, (2000).

[15] V. Rajesh and G. Leversha, *Some properties of odd terms of the Fibonacci sequence*, Mathematical Gazette, 88 (2004), 85–86.

[16] N. J. A. Sloane, *The online Encyclopedia of Integer sequences*, Published electronically at http://www.research.att.com/~njas/ sequences, (2008).