On the construction of radially Moore digraphs

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Abstract

A digraph with maximum out-degree d and radius k has at most $1+d+\cdots+d^k$ vertices, as the Moore bound states. Regular digraphs attaining such a bound and whose diameter is at most $k+1$ are called *radially Moore digraphs*. Knor [4] proved that these extremal digraphs do exist for any value of $d \geq 1$ and $k \geq 1$. In this paper, we introduce a digraph operator, based on the line digraph, which allow us to construct new radially Moore digraphs and recover the known ones. Besides, we show that for $k = 2$ a radially Moore digraph with as many central vertices as the degree do exist.

Keywords: Center, eccentricity, radius, diameter, line digraph, Moore digraph, radially Moore digraph.

1 Introduction

The modelization of interconnection networks by graphs motivated the study of the optimization problem known as the *degree/diameter problem* (see [5] for a survey of it). In the directed case, given the values of the maximum out-degree d and the diameter k, there is a natural upper bound $n_{d,k}$ for the largest order of a digraph with these two parameters,

$$n_{d,k} = 1 + d + \cdots + d^k,$$

referred to as the *Moore bound*. Digraphs attaining such a bound are called *Moore digraphs*. In particular, all vertices of a Moore digraph have the same degree (d) and the same eccentricity (k). It is well known that Moore digraphs do only exist in the trivial cases, $d = 1$ or $k = 1$, which correspond to the directed cycle of order $k+1$ and the complete digraph of order $d+1$, respectively (see [6, 2]). This has lead to the study of digraphs

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'close' to the Moore ones. One way to do it is by allowing the existence of vertices with eccentricity just one more than the value they should have. In this context, regular digraphs of degree d, radius k, diameter at most $k+1$ and order equal to $n_{d,k}$ are known as *radially Moore digraphs*. Knor [4] proved that these extremal digraphs do exist for any value of $d \geq 1$ and $k \geq 1$. His construction, defined as a digraph on an alphabet, has just one central vertex, if $d > 1$ and $k > 1$. In this paper, we introduce a digraph operator, based on the line digraph, which allow us to obtain new radially Moore digraphs and recover the known ones (see Sections 2 and 3). Besides, we show that for $k = 2$ a radially Moore digraph with d central vertices do exist (see Section 4).

Terminology and notation

A *digraph* $G = (V, E)$ consists of a finite nonempty set V of objects called *vertices* and a (multi)set E of ordered pairs of vertices called *arcs* (loops and multiple arcs are allowed). The *order* of G is the cardinality of its set of vertices V, denoted by $|V|$. If (u, v) is an arc, it is said that u is *adjacent to* v and also that v is *adjacent from* u. We also represent an arc (u, v) as uv. The (multi)set of vertices that are adjacent from [to] a given vertex v is denoted by $N^+(v)$ [resp. $N^-(v)$] and its cardinality is the *out-degree* of v, $d^+(v)$ [resp. *in-degree* of v, $d^-(v)$]. If $d^+(v) = d^-(v) = d$, for all $v \in V$, then G is said to be *regular* of degree d. A *path* of length h from a vertex u to a vertex v ($u \to v$ path) is a sequence of $h+1$ distinct vertices, $u = u_0, u_1, \ldots, u_{h-1}, u_h = v$, such that each pair $u_{i-1}u_i$ is an arc of G. The length of a shortest $u \to v$ path is the *distance* from u to v, denoted by $\mathrm{dist}(u, v)$. The *out-eccentricity* of a vertex v, $e^+(v)$, is the maximum distance from v to any vertex in G. Analogously, the *in-eccentricity* of v, $e^-(v)$, is the maximum distance from any vertex in G to v. A vertex u is said to be an *out-eccentric* [resp. *in-eccentric*] vertex of v if $d(v, u) = e^+(v)$ [resp. $d(u, v) = e^-(v)$]. The *eccentricity* of a vertex v, $e(v)$, is the maximum between its out-eccentricity and in-eccentricity. The *radius* of G, $\mathrm{rad}(G)$, is the minimum value of all its vertex eccentricities. The *center* of G, $C(G)$, is the set of vertices of G with minimum eccentricity. The *out-center* and *in-center* of G, $C^+(G)$ and $C^-(G)$, are defined in a similar way. The *diameter* of G, $\mathrm{diam}(G)$, is the maximum value of all its vertex (out-)eccentricities. Reader is referred to Chartrand and Lesniak [3] for additional graph concepts.

2 Central line digraph operator

We recall that in the line digraph $L(G)$ of a digraph G, each vertex represents an arc of G and a vertex uv is adjacent to a vertex wz if and only if $v = w$. Thus, when G is a regular digraph of degree d, so it is $L(G)$, and the order of $L(G)$ is d times the order of G. Next lemma shows the relationship between vertex eccentricities in G and $L(G)$. It can be proved using the arguments given by Aigner in [1, Theorem 5].

Lemma 1. *Let $G = (V, E)$ be a digraph and let $L(G)$ be its line digraph. Let uv be an arc of G. Then,*

$$e^+(v) \leq e^+(uv) \leq 1 + e^+(v) \quad \text{and} \quad e^-(u) \leq e^-(uv) \leq 1 + e^-(u).$$

Moreover,

$e^+(uv) = e^+(v)$ if and only if u is the unique out-eccentric vertex of v and $d^+(u) = 1$,

$e^-(uv) = e^-(u)$ if and only if v is the unique in-eccentric vertex of u and $d^-(v) = 1$.

Although the line digraph of a radially Moore digraph G of degree d and radius k have 'good' eccentricity properties, for becoming a radially Moore digraph, the order of $L(G)$ is $n_{d,k+1} - 1$. So, we need to add conveniently an extra vertex to $L(G)$ in order to get a radially Moore digraph.

Let G be a regular digraph of degree $d \geq 1$, radius $k \geq 1$ and order $n_{d,k}$. Let λ be a central vertex of G and let $N^+(\lambda) = \{v_1, \ldots, v_d\}$. Then, $\text{dist}(v_i, \lambda) = k$, $i = 1, \ldots, k$. As a consequence, for each vertex $v_i \in N^+(\lambda)$ there is a unique and distinct vertex $w_i \in N^-(\lambda)$ such that $\text{dist}(v_i, w_i) = k - 1$ (see Figure 1).

We define the λ-*central line digraph* of G, denoted by $L_\lambda(G)$, as the line digraph of G with an extra vertex λ' and where we replace each arc of the form $(w_i\lambda, \lambda v_i)$ by the arcs $(w_i\lambda, \lambda')$ and $(\lambda', v_i\lambda)$, $1 \leq i \leq d$ (Figure 2 shows the differences between the line digraph and the λ-central line digraph). In particular, we can apply this new operator to a radially Moore digraph. Next proposition gives some properties of $L_\lambda(G)$.

Proposition 1. *Let G be a regular digraph of degree $d \geq 2$, radius k and order $n_{d,k}$. Let λ be a central vertex of G. Then, $L_\lambda(G)$ is a regular digraph of degree d, order $n_{d,k+1}$ and radius $k+1$. Moreover, λ' is a central vertex of $L_\lambda(G)$.*

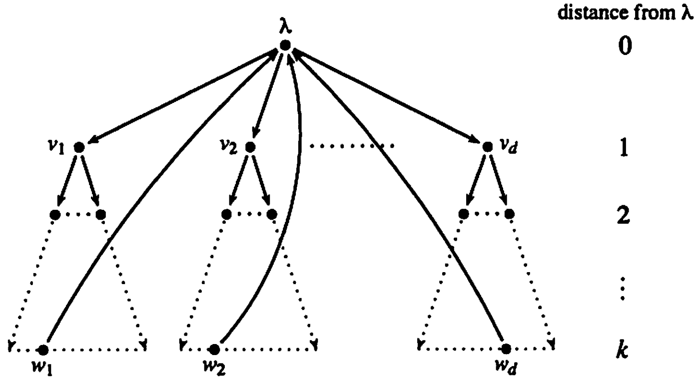


Figure 1: Structure of a regular digraph of degree d, radius k and order $n_{d,k}$ pendant from a central vertex λ.

Proof. From its own definition, $L_\lambda(G)$ is a regular digraph of degree d and order $n_{d,k+1}$. As a consequence, the radius of $L_\lambda(G)$ is $\geq k+1$. Since $N^+(\lambda') = N^+_{L(G)}(w_i\lambda)$ for all $w_i \in N^-(\lambda)$ (see Figure 2), and using Lemma 1,

$$e^+(\lambda') = \max_{1\leq i\leq d}\{e^+_{L(G)}(w_i\lambda)\} = 1 + e^+(\lambda) = 1 + k.$$

(We remark that in the first equality we use that $\text{dist}(\lambda', w_i\lambda) = k+1$). Analogously, since $N^-(\lambda') = N^-_{L(G)}(\lambda v_i)$ for all $v_i \in N^+(\lambda)$,

$$e^-(\lambda') = \max_{1\leq i\leq d}\{e^-_{L(G)}(\lambda v_i)\} = 1 + e^-(\lambda) = 1 + k.$$

Hence, $\text{rad}(L_\lambda(G)) = k+1$ and λ' is a central vertex of $L_\lambda(G)$. □

We cannot guarantee that the diameter of $L_\lambda(G)$ is just one more than the diameter of G, since some arcs of $L(G)$ have been removed. This happens, for instance, with the digraph G shown in Figure 3, where $\text{rad}(L_\lambda(G)) = 3$ and $\text{diam}(L_\lambda(G)) = 5$. So, the λ-central line digraph of a radially Moore digraph is not, in general, another radially Moore digraph.

Given a regular digraph G of degree $d \geq 2$, radius k and order $n_{d,k}$, and given a central vertex λ of G, Proposition 1 allow us recursively define the *n-iterated λ-central line digraph of* G, denoted by $L^n_\lambda(G)$, as follows $L^n_\lambda(G) = L_{\lambda^{(n-1)}}(L^{n-1}_\lambda(G))$, $n \geq 2$, where $L^1_\lambda(G) = L_\lambda(G)$ and $\lambda^{(n-1)}$ denotes the extra vertex added to $L(L^{n-1}_\lambda(G))$. Notice that $L^n_\lambda(G)$ is a regular digraph of degree d, radius $k+n$ and order $n_{d,k+n}$, for $n \geq 1$.

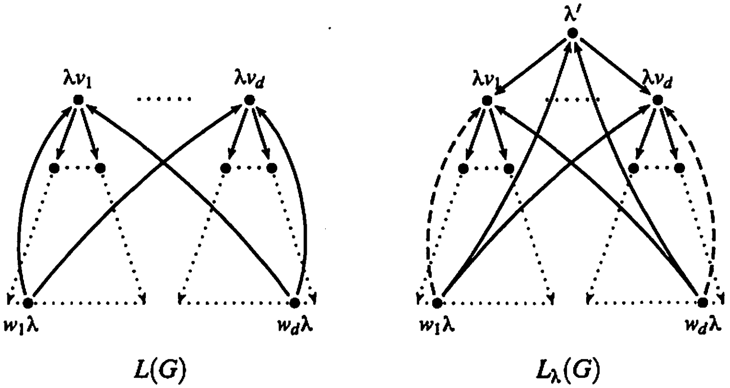


Figure 2: The line digraph $L(G)$ and the λ-central line digraph $L_\lambda(G)$. Dashed lines indicate removed arcs from $L(G)$.

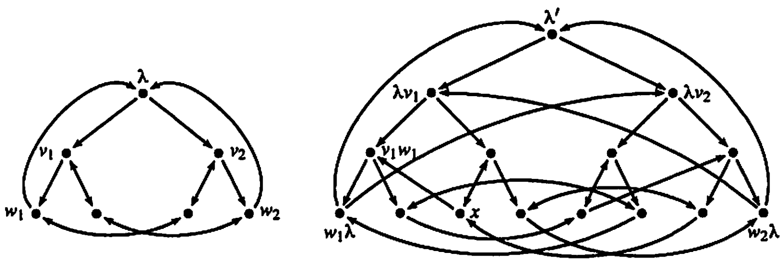


Figure 3: A radially Moore digraph G of degree 2 and radius 2, with $\lambda \in C(G)$, and its λ-central line digraph $L_\lambda(G)$. Let us observe that $\mathrm{dist}(v_1w_1, x) = 5$.

Under certain conditions on G, $\{L_\lambda^n(G)\}_{n\geq 1}$ become a family of radially Moore digraphs, as the following theorem states.

Theorem 1. *Let G be a radially Moore digraph with degree $d \geq 2$ and radius $k \geq 1$. Let us assume that λ is a central vertex of G for which all the following conditions hold:*

(a) For any vertex x in G, there exists at least one vertex $v \in N^+(\lambda)$ such that $\mathrm{dist}(v, x) = k$.

(b) For any vertex x in G, there exists at least one vertex $w \in N^-(\lambda)$ such that $\mathrm{dist}(x, w) = k$.

(c) $\mathrm{diam}(G - \lambda) \leq k + 1$, *where $G - \lambda$ denotes the deletion of vertex λ of G.*

Then, $L_\lambda^n(G)$ is a radially Moore digraph of degree d and radius $k+n$, for every $n \geq 1$.

Proof. First, we prove that $L_\lambda(G)$ inherits property (a) from G. Let us consider a vertex $x = x_1x_2 \in L_\lambda(G)$. By assumption (a), there exists at least a vertex $v \in N^+(\lambda)$ such that $\text{dist}(v, x_1) = k$. Then, $\text{dist}(\lambda v, x_1x_2) = k+1$ and $\lambda v \in N^+(\lambda')$. We can use the same argument to prove that $L_\lambda(G)$ inherits property (b) from G. Next, we show that $L_\lambda(G)$ inherits property (c). Let x_1x_2, y_1y_2 be two vertices of $L_\lambda(G) - \lambda'$. Notice that these two vertices can also be considered in $L_\lambda(G)$. We distinguish three cases:

(1) $x_2 \neq \lambda$ and $y_1 \neq \lambda$. Since $\text{diam}(G - \lambda) \leq k+1$, there is a path from vertex x_2 to vertex y_1 in G of length at most $k+1$. Since it does not contain the vertex λ, there exists a path from x_1x_2 to y_1y_2 in $L_\lambda(G) - \lambda'$ of length at most $k+2$.

(2) $x_2 = \lambda$. Then, $x_1 = w$ for a vertex $w \in N^-(\lambda)$. Since $e^+(\lambda') = k+1$, there exists a unique vertex $v \in N^+(\lambda)$ such that $\text{dist}(\lambda v, y_1y_2) \leq k$ in $L_\lambda(G)$. This distance bound also holds in $L_\lambda(G) - \lambda'$. So, if $w\lambda$ is adjacent to λv in $L_\lambda(G)$ then $\text{dist}(x_1x_2, y_1y_2) \leq k+1$ in $L_\lambda(G) - \lambda'$. Otherwise, $w\lambda$ is adjacent to $\lambda v'$ in $L_\lambda(G)$, for all $v' \in N^+(\lambda), v' \neq v$, by definition. Besides, at least one of these vertices $v' \in N^+(\lambda), v' \neq v$, satisfies that $\text{dist}(\lambda v', y_1y_2) = k+1$ in $L_\lambda(G) - \lambda'$, since $L_\lambda(G)$ inherits property (a) from G, $\text{dist}(\lambda v, y_1y_2) \leq k$ and $\text{dist}(\lambda v', \lambda') = k+1$. As a consequence, $\text{dist}(x_1x_2, y_1y_2) = k+2$ in $L_\lambda(G) - \lambda'$.

(3) $y_1 = \lambda$. This case can be proved analogously to (2).

Hence, $\text{diam}(L_\lambda(G) - \lambda') \leq k+2$ and, since $e(\lambda') = k+1$, then the diameter of $L_\lambda(G)$ is $\leq k+2$. Therefore, $L_\lambda(G)$ becomes a radially Moore digraph of degree d and radius $k+1$. Moreover, since $L_\lambda(G)$ inherits properties (a), (b) and (c) from G, then $L_\lambda^n(G)$ is a radially Moore digraph of degree d and radius $k+n$, for every $n \geq 1$. □

Corollary 1. *Let λ be a central vertex of a regular digraph G of radius $k = 1$, degree $d \geq 2$ and order $1+d$. Let us assume that* $\text{diam}(G - \lambda) \leq 2$. *Then, $L_\lambda^n(G)$ is a radially Moore digraph of degree d and radius $k+n$, for every $n \geq 1$.*

Proof. Since $e(\lambda) = 1$, any other vertex x of G is adjacent from and to λ. Hence, $\text{diam}(G) \leq 2$ and G is a radially Moore digraph. Moreover, condition (a) [resp. (b)], given in Theorem 1, means that every vertex x of G should be adjacent from [resp. to] a vertex $v \in N^+(\lambda)$. Thus, if $\text{diam}(G-\lambda) \leq 2$ then (a) and (b) are satisfied. Hence, G fulfills assumptions of Theorem 1 and result holds. □

Clearly, Corollary 1 can be applied to the complete digraph K_{d+1} of order $d+1$, which is in fact the unique digraph without loops nor multiple arcs satisfying the assumptions of such corollary.

Example 1. *Let λ be any vertex of the complete digraph K_{d+1}. Then, $L_\lambda^n(K_{d+1})$ is a radially Moore digraph of degree d and radius $n+1$, for every $n \geq 1$ and $d \geq 2$.*

Next example also satisfies conditions of Corollary 1 and, consequently, it can be used to construct another family of radially Moore digraphs, which contain loops.

Example 2. *Let G be the digraph with vertex set $V = \{\lambda, 0, 1, \cdots, d-1\}$ and arc set*

$$\begin{aligned} E &= \{(\lambda, i) \mid 0 \leq i \leq d-1\} \cup \{(i, \lambda) \mid 0 \leq i \leq d-1\} \\ &\quad \cup \{(i, i+j \mod d) \mid 0 \leq i \leq d-1 \text{ and } 0 \leq j \leq d-2\}. \end{aligned}$$

Then, $L_\lambda^n(G)$ is a radially Moore digraph of degree d and radius $n+1$, for every $n \geq 1$ and $d \geq 3$.

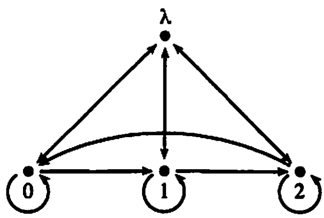


Figure 4: The digraph given in Example 2 for the case of $d = 3$.

Next corollary gives a simpler but stronger sufficient condition, which only involves the centrality of the neighborhood of the chosen central vertex of the digraph.

Corollary 2. *Let G be a radially Moore digraph with degree $d \geq 2$ and radius $k \geq 1$. Let us assume that λ is a central vertex of G for which all the following conditions hold:*

(a)' $N^+(\lambda) \subseteq C^+(G)$,

(b)' $N^-(\lambda) \subseteq C^-(G)$,

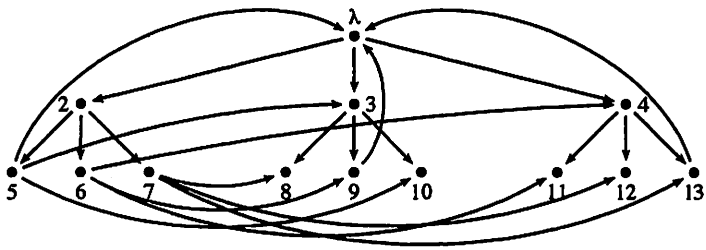
(c) $\text{diam}(G - \lambda) \leq k+1$.

Then, $L_\lambda^n(G)$ is a radially Moore digraph of degree d and radius $k+n$, for every $n \geq 1$.

Proof. First, we prove that if $N^+(\lambda) \subseteq C^+(G)$ then condition (a) of Theorem 1 holds. Let x be a vertex of G. By assumption $\text{dist}(v,x) \leq k$ for all $v \in N^+(\lambda)$. Moreover, since $e^+(\lambda) = k$ then, there exists a unique vertex $v' \in N^+(\lambda)$ such that $\text{dist}(v',x) \leq k-1$. Hence, $\text{dist}(v,x) = k$ for all $v \in N^+(\lambda), v \neq v'$. Analogously, it can be proved that $N^-(\lambda) \subseteq C^-(G)$ implies condition (b) of Theorem 1. Hence, G satisfies all conditions of Theorem 1 and the result follows. □

Let us consider a vertex λv, where $\lambda v \in N^+(\lambda')$. If condition (a)' in Corollary 2 holds, then $e^+(\lambda v) = 1 + e^+(v) = 1 + k$. Hence, $N^+(\lambda') \subseteq C^+(L_\lambda(G))$ and, consequently, $L_\lambda(G)$ inherits condition (a)' (analogously, $L_\lambda(G)$ inherits condition (b)'). Furthermore, in the case of degree $d = 2$ condition (a) [resp. (b)] is equivalent to condition (a)' [resp. (b)']. So, taking as a basis a radially Moore digraphs G satisfying all assumptions of Corollary 2, we get a family of radially Moore digraphs $\{L_\lambda^n(G)\}_{n\geq 1}$ with at least $d+1$ vertices in the out-center of $L_\lambda^n(G)$ and $d+1$ vertices in the in-center of $L_\lambda^n(G)$. The complete digraph K_{d+1} fulfill these properties. The following digraph is another example for the case of radius $k = 2$ and degree $d = 3$.

Example 3. *Let G be the digraph shown in the picture*



with the extra arcs: $(8,5), (8,4), (8,6), (9,7), (9,11), (10,13), (10,2), (10,12), (11,5), (11,6), (11,10), (12,2), (12,7), (12,9), (13,3), (13,8)$. *It can be checked that G is a radially Moore digraph such that $C^+(G) = \{\lambda, 2, 3, 4\}$, $C^-(G) = \{\lambda, 5, 9, 13\}$ and $\text{diam}(G - \lambda) = 3$. As a consequence, $L_\lambda^n(G)$ is radially Moore digraph of degree 3 and radius $2+n$, for every $n \geq 1$.*

It remains, as an open problem, to find general constructions of radially Moore digraphs under assumptions of Corollary 2.

Question 1. *Find new families of radially Moore digraphs with degree $d \geq 3$ and radius $k \geq 2$ satisfying conditions given in Corollary 2.*

3 The family of radially Moore digraphs $\{L_\lambda^n(K_{d+1})\}_{n\geq 1}$

At present, the unique known family of radially Moore digraphs was given by Knor in [4]. We denote these digraphs as knor$[d,k]$. They are defined as follows: the set V of vertices consists of all strings of length at most k over an alphabet of d symbols, $\{1,2,\dots,d\}$; that is,

$$V = \{\lambda\} \cup \{e_1e_2\dots e_{k'} \mid 1 \leq k' \leq k \text{ and } 1 \leq e_i \leq d,\ i = 1,\dots,k'\},$$

where the empty word is denoted by λ.

For each string $a = e_1e_1\dots e_1e_2\dots e_{k'}, e_2 \neq e_1$, Knor defined $\overline{a} = e_2\dots e_{k'}$. Notice that $\overline{a} = \lambda$ if and only if $a = e_1\dots e_1$. If $l(a)$ denotes de length of a string $a \in V$, the set E of arcs is defined by,

$$\begin{aligned} E = &\{(a,ae) \mid l(a) \leq k-1 \text{ and } 1 \leq e \leq d\}\\ &\cup\{(a,\overline{a}e) \mid l(a) = k,\ \overline{a} \neq \lambda \text{ and } 1 \leq e \leq d\}\\ &\cup\{(a,e) \mid l(a) = k,\ \overline{a} = \lambda,\ \text{i.e. } a = e_1\dots e_1,\ \text{and } 1 \leq e \leq d, e \neq e_1\}\\ &\cup\{(a,\lambda) \mid l(a) = k \text{ and } \overline{a} = \lambda\}. \end{aligned}$$

It is known that knor$[d,k]$ is regular of degree d, radius k, diameter $k+1$ and order $n_{d,k}$ (see [4]). Next proposition states the relation between Knor digraphs and the λ-central line digraph operator.

Proposition 2. *The digraph* knor$[d,k]$ *is isomorphic to* L_λ(knor$[d,k-1]$), *for every* $d \geq 2$ *and* $k \geq 2$.

Proof. Let us denote by V the set of vertices of knor$[d,k]$ and by V_L the set of vertices of L_λ(knor$[d,k-1]$). Then,

$$\begin{aligned} V_L = &\{\lambda'\} \cup \{(\lambda,e) \mid 1 \leq e \leq d\} \cup \{(a,ae) \mid 1 \leq l(a) \leq k-2 \text{ and } 1 \leq e \leq d\}\\ &\cup\{(a,\overline{a}e) \mid l(a) = k-1,\ \overline{a} \neq \lambda \text{ and } 1 \leq e \leq d\}\\ &\cup\{(a,e) \mid l(a) = k-1,\ \overline{a} = \lambda,\ \text{i.e. } a = e_1\dots e_1,\ \text{and } 1 \leq e \leq d, e\neq e_1\}\\ &\cup\{(a,\lambda) \mid l(a) = k-1 \text{ and } \overline{a} = \lambda\}. \end{aligned}$$

Notice that each word $a' \in V$, with $l(a') \geq 2$, can be written as $a' = ae$, where $l(a) = l(a') - 1$ and $1 \leq e \leq d$. We prove that the bijection $\Psi : V \rightarrow V_L$, defined as follows,

$$\begin{aligned}\Psi(\lambda) &= \lambda', \\ \Psi(e) &= (\lambda, e), \text{ where } 1 \le e \le d,\end{aligned}$$

$$\Psi(ae) = \begin{cases} (a, ae), & \text{where } 1 \le l(a) \le k-2 \text{ and } 1 \le e \le d, \\ (a, \overline{a}e), & \text{where } l(a) = k-1, \overline{a} \ne \lambda \text{ and } 1 \le e \le d, \\ (a, e), & \text{where } l(a) = k-1, a = e_1 \ldots e_1 \text{ and } 1 \le e \le d, e \ne e_1, \\ (a, \lambda), & \text{where } l(a) = k-1, a = e \ldots e \text{ and } 1 \le e \le d, \end{cases}$$

is an isomorphism between knor$[d, k]$ and $L_\lambda(\text{knor}[d, k-1])$.

Let $a' \in V$. Clearly, the application Ψ preserve the adjacency relationship if $l(a') \le 1$. Let us suppose that $2 \le l(a') \le k-1$. Then $a' = ae$ is adjacent to aee', $1 \le e' \le d$. Besides, $\Psi(ae) = (a, ae)$ and $\Psi(aee')$ may take several values, depending on the length of ae:

If $l(ae) \le k-2$ then $\Psi(aee') = (ae, aee')$.

Otherwise $l(ae) = k-1$ and

$$\Psi(aee') = \begin{cases} (ae, \overline{ae}e'), & \text{if} \quad \overline{ae} \ne \lambda, \\ (ae, e'), & \text{if} \quad a = e \ldots e \text{ and } e \ne e', \\ (ae, \lambda), & \text{if} \quad a = e \ldots e \text{ and } e = e'. \end{cases}$$

Thus, in any case, $\Psi(aee')$ is adjacent from $\Psi(ae)$ by definition of $L_\lambda(\text{knor}[d, k-1])$.

Let us suppose that $l(a') = k$. Then,

$$a' = ae \text{ is adjacent to } \begin{cases} \overline{ae}e', & \text{if} \quad \overline{ae} \ne \lambda \text{ and } 1 \le e' \le d, \\ e', & \text{if} \quad a = e \ldots e \text{ and } e' \ne e, \\ \lambda, & \text{if} \quad a = e \ldots e \text{ and } e' = e. \end{cases}$$

If $\overline{a} \ne \lambda$ then $\Psi(ae) = (a, \overline{a}e)$ and, since $\overline{ae} \ne \lambda$, ae is adjacent to $\overline{ae}e'$. Besides,

$$\Psi(\overline{ae}e') = \begin{cases} (\overline{ae}, \overline{ae}e'), & \text{if} \quad 1 \le l(\overline{ae}) \le k-2, \\ (\overline{ae}, \overline{\overline{ae}}e'), & \text{if} \quad l(\overline{ae}) = k-1. \end{cases}$$

Let us observe that $\overline{a}e = \overline{ae}$, if $\overline{a} \ne \lambda$. Hence, $\Psi(ae)$ is adjacent to $\Psi(\overline{ae}e')$.

If $ae = e_1 \ldots e_1 e, e_1 \ne e$, then ae is adjacent to $\overline{ae}e' = ee', 1 \le e' \le d$. On the other hand, $\Psi(ae) = (a, e)$, which is adjacent to the vertex $\Psi(ee') = (e, ee')$.

If $ae = e \dots e$ then ae is adjacent to λ and $e', e' \neq e$. Besides, $\Psi(ae) = (a, \lambda)$, $\Psi(\lambda) = \lambda'$ and $\Psi(e') = (\lambda, e')$. Hence, $\Psi(ae)$ is adjacent to $\Psi(\lambda)$ and $\Psi(e')$ by definition of $L_\lambda(\text{knor}[d, k-1])$.

□

For the case $k = 1$, knor$[d, 1]$ is the complete digraph K_{d+1}. Applying Proposition 2, we get the following result:

Corollary 3. *The digraph* knor$[d, k]$ *is isomorphic to* $L_\lambda^{k-1}(K_{d+1})$, *for every* $d \geq 2$ *and* $k \geq 2$.

The previous characterization, together with Corollary 1, provide us an alternative proof that Knor digraphs, knor$[d, k]$, are a family of radially Moore digraphs of degree d and radius k.

4 Central vertices in radially Moore digraphs

Since Moore digraphs do not exist for degree $d \geq 2$ and radius $k \geq 2$, one can ask how 'close' are radially Moore digraphs from a theoretical Moore digraph, where each vertex would have eccentricity k. An attempt to answer this question is to look at the center of a radially Moore digraph: when the more central vertices it has, the more similar it would be to a theoretical Moore digraph. Knor [4] proved that, for the case of degree 2, the number of central vertices is bounded from above by one half of the order of the digraph.

The family of radially Moore digraphs $\{L_\lambda^n(K_{d+1})\}_{n \geq 1}$ have just one central vertex (see [4]). Besides, it is not complicated to find radially Moore digraphs of degree 2 and radius 2 containing two central vertices, as Knor pointed out in [4]. In fact, an exhaustive search made by computer, shows that there are exactly 18 radially Moore digraphs of degree 2 and radius 2. Three of them with two central vertices and the rest with a single central vertex.

Next result shows that a radially Moore digraph of radius 2 with as many central vertices as the degree can be constructed.

Proposition 3. *For any positive integer $d \geq 2$, there exists a radially Moore digraph G of degree d and radius $k = 2$ containing d central vertices.*

Proof. We construct such a digraph $G = (V, E)$ and show that G is a radially Moore digraph of radius 2 and degree d, containing exactly d central vertices (see Fig. 5 for a representation of G for the case of degree 2).

Let $V = \{0\} \cup \{i \mid 1 \le i \le d\} \cup \{ij \mid 1 \le i \le d,\ 1 \le j \le d\}$ and

$$\begin{aligned} E \ = \ & \{(0,i) \mid 1 \le i \le d\} \cup \{(i,ij) \mid 1 \le i \le d,\ 1 \le j \le d\} \\ & \cup \{(ii,0) \mid 1 \le i \le d\} \cup \{(ii,jj) \mid 1 \le i \le d,\ 1 \le j \le d,\ i \ne j\} \\ & \cup \{(ij,j) \mid 1 \le i \le d,\ 1 \le j \le d,\ i \ne j\} \\ & \cup \{(ij,jk) \mid 1 \le i \le d,\ 1 \le j \le d,\ 1 \le k \le d,\ i \ne j,\ j \ne k\}. \end{aligned}$$

Clearly the order of G is $n_{d,2} = 1 + d + d^2$ and from its own definition G is regular of degree d. The set of out-neighbors of a vertex $i \in V$ is $N^+(i) = \{i1, i2, \ldots, id\}$, $1 \le i \le d$. Since any vertex $v \notin N^+(i)$, $v \ne i$, is adjacent from a vertex $ij \in N^+(i)$ for suitable j, we have $e^+(i) = 2$. Moreover, since $N^-(i) = \{0\} \cup \{ji \mid 1 \le j \le d,\ j \ne i\}$ and any vertex $v \notin N^-(i)$, $v \ne i$, is adjacent to a vertex $ji \in N^-(i)$, we have $e^-(i) = 2$. Thus, $e(i) = 2$. As every vertex v of G, $v \ne 0$, is adjacent to a vertex with eccentricity 2 and from (possibly another) vertex with eccentricity 2, we have $e(v) \le 3$. As a consequence, $e(0) \le 3$ as well. So, $\mathrm{diam}(G) \le 3$. In fact $e^+(ij) = e^-(ij) = e^-(0) = 3$ and $e^+(0) = 2$. Hence, G is a radially Moore digraph of radius 2 with $C(G) = \{1, \ldots, d\}$. □

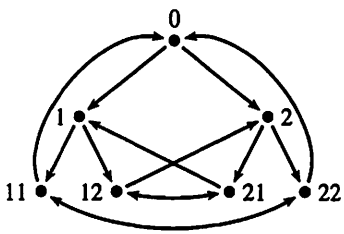


Figure 5: A radially Moore digraph G of degree 2 and radius 2. Notice that $C(G) = \{1, 2\}$.

It seems difficult to obtain families of radially Moore digraphs with radius $k > 2$ and with more than one central vertex.

Question 2. *Is it possible to construct a radially Moore digraph G with degree $d \ge 2$, radius $k > 2$ and with $|C(G)| \ge 2$?*

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