Solutions of fractional systems of difference equations

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Abstract

This paper is devoted to study the form of the solutions and the periodicity of the following rational systems of rational difference equations

$$x_{n+1} = \frac{x_{n-5}}{1 - x_{n-5}y_{n-2}}, \; y_{n+1} = \frac{y_{n-5}}{\pm 1 \pm y_{n-5}x_{n-2}},$$

with initial conditions are real numbers.

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1 Introduction

Recently, there has been great interest in studying difference equation systems. One of the reasons for this is the necessity for some techniques that can be used in investigating equations arising in mathematical models describing real life situations in population biology, economics, probability theory, genetics, psychology, etc. There are many papers related to the difference equations systems for example, Cinar [2] studied the solutions of the system of difference equations

$$x_{n+1} = \frac{m}{y_n}, \; y_{n+1} = \frac{py_n}{x_{n-1}y_{n-1}}.$$

The behavior of positive solutions of the system

$$x_{n+1} = \frac{x_{n-1}}{1 + x_{n-1}y_n}, \; y_{n+1} = \frac{y_{n-1}}{1 + y_{n-1}x_n}.$$

has been studied by Kurbanli et al. [10].

In [14] Yalçınkaya investigated the sufficient condition for the global asymptotic stability of the following system of difference equations

$$z_{n+1} = \frac{t_n z_{n-1} + a}{t_n + z_{n-1}},\ t_{n+1} = \frac{z_n t_{n-1} + a}{z_n + t_{n-1}}.$$

Similar difference equations and nonlinear systems of rational difference equations were investigated see [1]-[14].

We consider in this paper, the solution of the systems of difference equations

$$x_{n+1} = \frac{x_{n-5}}{1 - x_{n-5} y_{n-2}},\ y_{n+1} = \frac{y_{n-5}}{\pm 1 \pm y_{n-5} x_{n-2}},$$

with real numbers initial conditions.

2 On the system: $x_{n+1} = \frac{x_{n-5}}{1 - y_{n-2} x_{n-5}},\ y_{n+1} = \frac{y_{n-5}}{1 + x_{n-2} y_{n-5}}$

In this section, we investigate the solution of the system of two difference equations

$$x_{n+1} = \frac{x_{n-5}}{1 - y_{n-2} x_{n-5}}, \quad y_{n+1} = \frac{y_{n-5}}{1 + x_{n-2} y_{n-5}}, \tag{1}$$

where $n \in \mathbb{N}_0$ and the initial conditions are arbitrary real numbers with $x_{-5}y_{-2}$, $x_{-4}y_{-1}$, $x_{-3}y_0 \neq 1$ and $x_{-2}y_{-5}$, $x_{-1}y_{-4}$, $x_0 y_{-3} \neq -1$.

The following theorem is devoted to the form of the solution of system (1).

Theorem 1. Suppose that $\{x_n, y_n\}$ are solutions of system (1). Then for $n = 0, 1, 2, \ldots,$

$$\begin{aligned}
x_{6n-5} &= \frac{f}{(1-pf)^n},\quad x_{6n-4} = \frac{e}{(1-eh)^n},\quad x_{6n-3} = \frac{d}{(1-dg)^n},\\
x_{6n-2} &= c(1+sc)^n,\quad x_{6n-1} = b(1+br)^n,\quad x_{6n} = a(1+aq)^n,\\
y_{6n-5} &= \frac{s}{(1+sc)^n},\quad y_{6n-4} = \frac{r}{(1+br)^n},\quad y_{6n-3} = \frac{q}{(1+aq)^n},\\
y_{6n-2} &= p(1-pf)^n,\quad y_{6n-1} = h(1-eh)^n,\quad y_{6n} = g(1-dg)^n.
\end{aligned}$$

Proof: For $n = 0$ the result holds. Now suppose that $n > 1$ and that our assumption holds for $n - 1$. That is,

$$\begin{aligned}
x_{6n-11} &= \frac{f}{(1-pf)^{n-1}},\quad x_{6n-10} = \frac{e}{(1-eh)^{n-1}},\quad x_{6n-9} = \frac{d}{(1-dg)^{n-1}},\\
x_{6n-8} &= c(1+sc)^{n-1},\quad x_{6n-7} = b(1+br)^{n-1},\quad x_{6n-6} = a(1+aq)^{n-1},\\
y_{6n-11} &= \frac{s}{(1+sc)^{n-1}},\quad y_{6n-10} = \frac{r}{(1+br)^{n-1}},\quad y_{6n-9} = \frac{q}{(1+aq)^{n-1}},\\
y_{6n-8} &= p(1-pf)^{n-1},\quad y_{6n-7} = h(1-eh)^{n-1},\quad y_{6n-6} = g(1-dg)^{n-1}.
\end{aligned}$$

Now it follows from Eq.(1) that

$$
\begin{aligned}
x_{6n-5} &= \frac{x_{6n-11}}{1-y_{6n-8}x_{6n-11}} = \frac{\frac{f}{(1-pf)^{n-1}}}{\left(1-p(1-pf)^{n-1}\frac{f}{(1-pf)^{n-1}}\right)} \\
&= \frac{\frac{f}{(1-pf)^{n-1}}}{(1-pf)} = \frac{f}{(1-pf)^{n}}, \\
y_{6n-5} &= \frac{y_{6n-11}}{1+x_{6n-8}y_{6n-11}} = \frac{\frac{s}{(1+sc)^{n-1}}}{\left(1+c(1+sc)^{n-1}\frac{s}{(1+sc)^{n-1}}\right)} \\
&= \frac{\frac{s}{(1+sc)^{n-1}}}{(1+cs)} = \frac{s}{(1+sc)^{n}},
\end{aligned}
$$

Also, we see from Eq.(1) that

$$
\begin{aligned}
x_{6n-2} &= \frac{x_{6n-8}}{1-y_{6n-5}x_{6n-8}} = \frac{c(1+sc)^{n-1}}{\left(1-\frac{s}{(1+sc)^{n}}c(1+sc)^{n-1}\right)} \\
&= \frac{c(1+sc)^{n-1}}{\left(1-\frac{sc}{(1+sc)}\right)}\left(\frac{(1+sc)}{(1+sc)}\right) = \frac{c(1+sc)^{n}}{1+sc-sc} = c(1+sc)^{n}, \\
y_{6n-2} &= \frac{y_{6n-8}}{1+x_{6n-5}y_{6n-8}} = \frac{p(1-pf)^{n-1}}{\left(1+p(1-pf)^{n-1}\frac{f}{(1-pf)^{n}}\right)} \\
&= \frac{p(1-pf)^{n-1}}{\left(1+\frac{pf}{(1-pf)}\right)}\left(\frac{1-pf}{1-pf}\right) = \frac{p(1-pf)^{n}}{(1-pf+pf)} = p(1-pf)^{n}.
\end{aligned}
$$

By the same way, we can prove the other relations. The proof is complete.

Lemma 1. Let $\{x_n, y_n\}$ be a positive solution of system (1), then $\{y_n\}$ is bounded and converges to zero.

Proof: It follows from Eq.(1) that

$$y_{n+1} = \frac{y_{n-5}}{1+x_{n-2}y_{n-5}} \leq y_{n-5}.$$

Then the subsequences $\{y_{6n-5}\}_{n=0}^{\infty}$, $\{y_{6n-4}\}_{n=0}^{\infty}$, $\{y_{6n-3}\}_{n=0}^{\infty}$, $\{y_{6n-2}\}_{n=0}^{\infty}$, $\{y_{6n-1}\}_{n=0}^{\infty}$, $\{y_{6n}\}_{n=0}^{\infty}$ are decreasing and so are bounded from above by $M = \max\{y_{-5}, y_{-4}, y_{-3}, y_{-2}, y_{-1}, y_0\}$.

Lemma 2. The solutions of system (1) has unboundedness solutions except in the following case.

Theorem 2. System (1) has a periodic solution of period twelve iff $pf = eh = dg = 2$, $br = aq = sc = -2$ and it will be taken the form $\{x_n\} = \{f, e, d, c, b, a, -f, -e, -d, -c, -b, -a, f, e, ...\}$,
$\{y_n\} = \{s, r, q, p, h, g, -s, -r, -q, -p, -h, -g, s, r, ...\}$.
Proof: First suppose that there exists a prime period twelve solution

$$\begin{aligned} \{x_n\} &= \{f, e, d, c, b, a, -f, -e, -d, -c, -b, -a, f, e, ...\}, \\ \{y_n\} &= \{s, r, q, p, h, g, -s, -r, -q, -p, -h, -g, s, r, ...\}, \end{aligned}$$

of system (1), we see from the form of the solution of system (1) that

$$\begin{aligned} \pm f &= \frac{f}{(1-pf)^n}, \quad \pm e = \frac{e}{(1-eh)^n}, \quad \pm d = \frac{d}{(1-dg)^n}, \\ \pm c &= c(1+sc)^n, \quad \pm b = b(1+br)^n, \quad \pm a = a(1+aq)^n, \end{aligned}$$

and

$$\begin{aligned} \pm s &= \frac{s}{(1+sc)^n}, \quad \pm r = \frac{r}{(1+br)^n}, \quad \pm q = \frac{q}{(1+aq)^n}, \\ \pm p &= p(1-pf)^n, \quad \pm h = h(1-eh)^n, \quad \pm g = g(1-dg)^n. \end{aligned}$$

Then we get

$$1 - pf = 1 - eh = 1 - dg = 1 + sc = 1 + br = 1 + aq = -1.$$

Thus

$$pf = eh = dg = 2, \quad br = aq = sc = -2.$$

Second assume that $pf = eh = dg = 2$, $br = aq = sc = -2$. Then we see from the form of the solution of system (1) that

$$\begin{aligned} x_{6n-5} &= (-1)^n f, \quad x_{6n-4} = (-1)^n e, \quad x_{6n-3} = (-1)^n d, \\ x_{6n-2} &= (-1)^n c, \quad x_{6n-1} = (-1)^n b, \quad x_{6n} = (-1)^n a, \end{aligned}$$

and

$$\begin{aligned} y_{6n-5} &= (-1)^n s, \quad y_{6n-4} = (-1)^n r, \quad y_{6n-3} = (-1)^n q, \\ y_{6n-2} &= (-1)^n p, \quad y_{6n-1} = (-1)^n h, \quad y_{6n} = (-1)^n g. \end{aligned}$$

Thus we have a periodic solution of period twelve and the proof is complete.
Example 1. We consider interesting numerical example for the difference system (1) with the initial conditions $x_{-5} = .19$, $x_{-4} = .3$, $x_{-3} = -.2$, $x_{-2} = .31$, $x_{-1} = -.41$, $x_0 = .21$, $y_{-5} = -.16$, $y_{-4} = .23$, $y_{-3} = .12$, $y_{-2} = .14$, $y_{-1} = .18$, *and* $y_0 = .3$. (See Fig. 1).

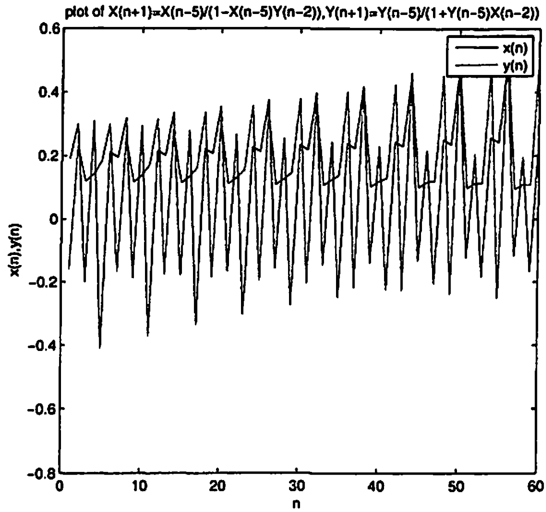

Figure 1.

3 On the system: $x_{n+1} = \frac{x_{n-5}}{1-y_{n-2}x_{n-5}}$, $y_{n+1} = \frac{y_{n-5}}{1-x_{n-2}y_{n-5}}$

In this section, we obtain the form of the solution of the system of two difference equations

$$x_{n+1} = \frac{x_{n-5}}{1 - y_{n-2}x_{n-5}}, \quad y_{n+1} = \frac{y_{n-5}}{1 - x_{n-2}y_{n-5}}, \tag{2}$$

where $n \in \mathbb{N}_0$ and the initial conditions are arbitrary real numbers.

Theorem 3. Assume that $\{x_n, y_n\}$ are solutions of system (2). Then for $n = 0, 1, 2, \ldots,$

$$\begin{aligned}
x_{6n-5} &= f\prod_{i=0}^{n-1} \frac{(1-(2i)pf)}{(1-(2i+1)pf)}, \quad x_{6n-4} = e\prod_{i=0}^{n-1} \frac{(1-(2i)eh)}{(1-(2i+1)eh)},\\
x_{6n-3} &= d\prod_{i=0}^{n-1} \frac{(1-(2i)dg)}{(1-(2i+1)dg)}, \quad x_{6n-2} = c\prod_{i=0}^{n-1} \frac{(1-(2i+1)sc)}{(1-(2i+2)sc)},\\
x_{6n-1} &= b\prod_{i=0}^{n-1} \frac{(1-(2i+1)br)}{(1-(2i+2)br)}, \quad x_{6n} = a\prod_{i=0}^{n-1} \frac{(1-(2i+1)aq)}{(1-(2i+2)aq)},\\
y_{6n-5} &= s\prod_{i=0}^{n-1} \frac{(1-(2i)sc)}{(1-(2i+1)sc)}, \quad y_{6n-4} = r\prod_{i=0}^{n-1} \frac{(1-(2i)br)}{(1-(2i+1)br)},\\
y_{6n-3} &= q\prod_{i=0}^{n-1} \frac{(1-(2i)aq)}{(1-(2i+1)aq)}, \quad y_{6n-2} = p\prod_{i=0}^{n-1} \frac{(1-(2i+1)pf)}{(1-(2i+2)pf)},\\
y_{6n-1} &= h\prod_{i=0}^{n-1} \frac{(1-(2i+1)eh)}{(1-(2i+2)eh)}, \quad y_{6n} = g\prod_{i=0}^{n-1} \frac{(1-(2i+1)dg)}{(1-(2i+2)dg)},
\end{aligned}$$

where $x_{-5}=f,\ x_{-4}=e,\ x_{-3}=d,\ x_{-2}=c,\ x_{-1}=b,\ x_0=a,\ y_{-5}=s,\ y_{-4}=r,\ y_{-3}=q,\ y_{-2}=p,\ y_{-1}=h,\ y_0=g.$

Proof: For $n=0$ the result holds. Now suppose that $n>1$ and that our assumption holds for $n-1$. That is,

$$\begin{aligned}
x_{6n-11} &= f\prod_{i=0}^{n-2}\frac{(1-(2i)pf)}{(1-(2i+1)pf)},\quad x_{6n-10}=e\prod_{i=0}^{n-2}\frac{(1-(2i)eh)}{(1-(2i+1)eh)},\\
x_{6n-9} &= d\prod_{i=0}^{n-2}\frac{(1-(2i)dg)}{(1-(2i+1)dg)},\quad x_{6n-8}=c\prod_{i=0}^{n-2}\frac{(1-(2i+1)sc)}{(1-(2i+2)sc)},\\
x_{6n-7} &= b\prod_{i=0}^{n-2}\frac{(1-(2i+1)br)}{(1-(2i+2)br)},\quad x_{6n-6}=a\prod_{i=0}^{n-2}\frac{(1-(2i+1)aq)}{(1-(2i+2)aq)},\\
y_{6n-11} &= s\prod_{i=0}^{n-2}\frac{(1-(2i)sc)}{(1-(2i+1)sc)},\quad y_{6n-10}=r\prod_{i=0}^{n-2}\frac{(1-(2i)br)}{(1-(2i+1)br)},\\
y_{6n-9} &= q\prod_{i=0}^{n-2}\frac{(1-(2i)aq)}{(1-(2i+1)aq)},\quad y_{6n-8}=p\prod_{i=0}^{n-2}\frac{(1-(2i+1)pf)}{(1-(2i+2)pf)},\\
y_{6n-7} &= h\prod_{i=0}^{n-2}\frac{(1-(2i+1)eh)}{(1-(2i+2)eh)},\quad y_{6n-6}=g\prod_{i=0}^{n-2}\frac{(1-(2i+1)dg)}{(1-(2i+2)dg)}.
\end{aligned}$$

It follows from Eq.(2) that

$$\begin{aligned}
x_{6n-5} &= \frac{x_{6n-11}}{1-y_{6n-8}x_{6n-11}}\\
&= \frac{f\prod_{i=0}^{n-2}\frac{(1-(2i)pf)}{(1-(2i+1)pf)}}{\left(1-p\prod_{i=0}^{n-2}\frac{(1-(2i+1)pf)}{(1-(2i+2)pf)}f\prod_{i=0}^{n-2}\frac{(1-(2i)pf)}{(1-(2i+1)pf)}\right)}\\
&= \frac{f\prod_{i=0}^{n-2}\frac{(1-(2i)pf)}{(1-(2i+1)pf)}}{\left(1-pf\prod_{i=0}^{n-2}\frac{(1-(2i)pf)}{(1-(2i+2)pf)}\right)}=\frac{f\prod_{i=0}^{n-2}\frac{(1-(2i)pf)}{(1-(2i+1)pf)}}{\left(1-\frac{pf}{(1-(2n-2)pf)}\right)}\\
&= f\prod_{i=0}^{n-2}\frac{(1-(2i)pf)}{(1-(2i+1)pf)}\frac{(1-(2n-2)pf)}{(1-(2n-2)pf-pf)}=f\prod_{i=0}^{n-1}\frac{(1-(2i)pf)}{(1-(2i+1)pf)},
\end{aligned}$$

and

$$y_{6n-5}=\frac{y_{6n-11}}{1-x_{6n-8}y_{6n-11}}=\frac{\prod_{i=0}^{n-2}\frac{(1-(2i)sc)}{(1-(2i+1)sc)}}{\left(1-c\prod_{i=0}^{n-2}\frac{(1-(2i+1)sc)}{(1-(2i+2)sc)}s\prod_{i=0}^{n-2}\frac{(1-(2i)sc)}{(1-(2i+1)sc)}\right)}$$

$$= \frac{\prod_{i=0}^{n-2}\frac{(1-(2i)sc)}{(1-(2i+1)sc)}}{\left(1-\frac{sc}{(1-(2n-2)sc)}\right)} = \prod_{i=0}^{n-2}\frac{(1-(2i)sc)}{(1-(2i+1)sc)}\frac{(1-(2n-2)sc)}{(1-(2n-2)sc-sc)}.$$

Then we see that

$$y_{6n-5} = s\prod_{i=0}^{n-1}\frac{(1-(2i)sc)}{(1-(2i+1)sc)}.$$

Similarly we can prove the other relations. This completes the proof.

Lemma 3. If x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0, y_{-5}, y_{-4}, y_{-3}, y_{-2}, y_{-1} and y_0 arbitrary real numbers and let $\{x_n, y_n\}$ are solutions of system (4) then the following statements are true:

(i) If $x_{-5} = 0$, $y_{-2} \neq 0$, then we have $x_{6n-5} = 0$ *and* $y_{6n-2} = y_{-2}$.
(ii) If $x_{-4} = 0$, $y_{-1} \neq 0$, then we have $x_{6n-4} = 0$ *and* $y_{6n-1} = y_{-1}$.
(iii) If $x_{-3} = 0$, $y_0 \neq 0$, then we have $x_{6n-3} = 0$ *and* $y_{6n} = y_0$.
(iv) If $x_{-2} = 0$, $y_{-5} \neq 0$, then we have $x_{6n-2} = 0$ *and* $y_{6n-5} = y_{-5}$.
(v) If $x_{-1} = 0$, $y_{-4} \neq 0$, then we have $x_{6n-1} = 0$ *and* $y_{6n-4} = y_{-4}$.
(vi) If $x_0 = 0$, $y_{-3} \neq 0$, then we have $x_{6n} = 0$ *and* $y_{6n-3} = y_{-3}$.
(vii) If $y_{-5} = 0$, $x_{-2} \neq 0$, then we have $y_{6n-5} = 0$ *and* $x_{6n-2} = x_{-2}$.
(viii) If $y_{-4} = 0$, $x_{-1} \neq 0$, then we have $y_{6n-4} = 0$ *and* $x_{6n-1} = x_{-1}$.
(ix) If $y_{-3} = 0$, $x_0 \neq 0$, then we have $y_{6n-3} = 0$ *and* $x_{6n} = x_0$.
(x) If $y_{-2} = 0$, $x_{-5} \neq 0$, then we have $y_{6n-2} = 0$ *and* $x_{6n-5} = x_{-5}$.
(xi) If $y_{-1} = 0$, $x_{-4} \neq 0$, then we have $y_{6n-1} = 0$ *and* $x_{6n-4} = x_{-4}$.
(xii) If $y_0 = 0$, $x_{-3} \neq 0$, then we have $y_{6n} = 0$ *and* $x_{6n-3} = x_{-3}$.

Proof: The proof follows from the form of the solutions of system (2).

Example 2. For system (2), assume the initial conditions $x_{-5} = .19$, $x_{-4} = .3$, $x_{-3} = -.2$, $x_{-2} = .31$, $x_{-1} = -.41$, $x_0 = .21$, $y_{-5} = -.16$, $y_{-4} = .23$, $y_{-3} = .12$, $y_{-2} = .1$

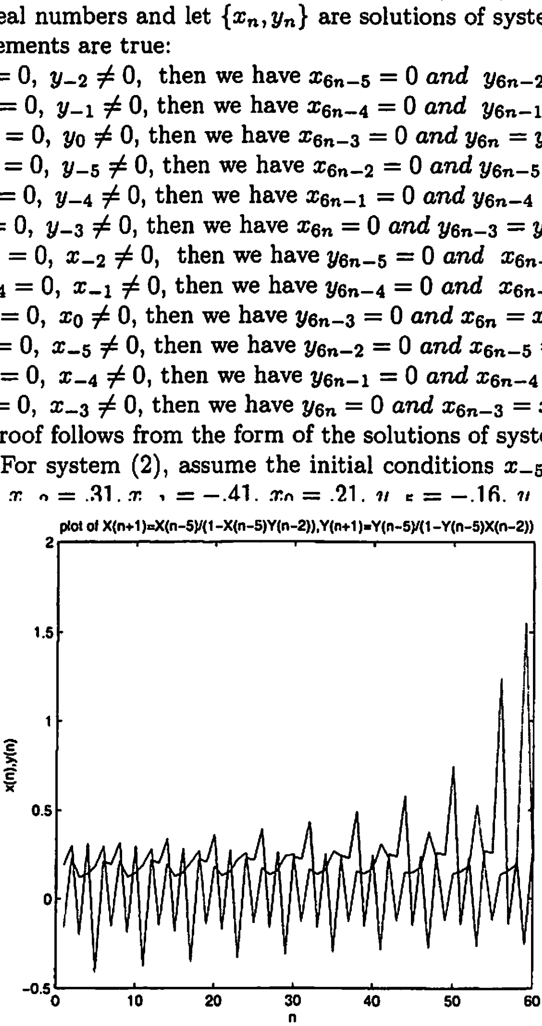

Figure 2.

Similarly we can prove the following systems.

4 On the system: $x_{n+1} = \frac{x_{n-5}}{1-y_{n-2}x_{n-5}}$, $y_{n+1} = \frac{y_{n-5}}{-1+x_{n-2}y_{n-5}}$

In this section, we get the solution of the system of the difference equations

$$x_{n+1} = \frac{x_{n-5}}{1 - y_{n-2}x_{n-5}}, \quad y_{n+1} = \frac{y_{n-5}}{-1 + x_{n-2}y_{n-5}}, \tag{3}$$

where $n \in \mathbb{N}_0$ and the initial conditions are arbitrary real numbers such that $x_{-5}y_{-2}$, $x_{-4}y_{-1}$, $x_{-3}y_0 \neq 1, \neq \frac{1}{2}$, and $x_{-2}y_{-5}$, $x_{-1}y_{-4}$, $x_0y_{-3} \neq \pm 1$.

Theorem 4. If $\{x_n, y_n\}$ are solutions of difference equation system (3). Then every solution of system (3) takes the following form for $n = 0, 1, 2, \ldots$,

$$\begin{aligned}
x_{12n-5} &= \frac{(-1)^n f(-1+2pf)^n}{(1-pf)^{2n}}, \quad x_{12n-4} = \frac{(-1)^n e(-1+2eh)^n}{(1-eh)^{2n}}, \\
x_{12n-3} &= \frac{(-1)^n d(-1+2dg)^n}{(1-dg)^{2n}}, \quad x_{12n-2} = c(1-sc)^n(1+sc)^n, \\
x_{12n-1} &= b(1-br)^n(1+br)^n, \ x_{12n} = a(1-aq)^n(1+aq)^n, \\
x_{12n+1} &= \frac{(-1)^n f(-1+2pf)^n}{(1-pf)^{2n+1}}, \ x_{12n+2} = \frac{(-1)^n e(-1+2eh)^n}{(1-eh)^{2n+1}}, \\
x_{12n+3} &= \frac{(-1)^n d(-1+2dg)^n}{(1-dg)^{2n+1}}, \quad x_{12n+4} = -c(-1+sc)^{n+1}(-1-sc)^n, \\
x_{12n+5} &= -b(-1+br)^{n+1}(-1-br)^n, \ x_{12n+6} = -a(-1+aq)^{n+1}(-1-aq)^n,
\end{aligned}$$

and

$$\begin{aligned}
y_{12n-5} &= \frac{s}{(-1+sc)^n(-1-sc)^n}, \quad y_{12n-4} = \frac{r}{(-1+br)^n(-1-br)^n}, \\
y_{12n-3} &= \frac{q}{(-1+aq)^n(-1-aq)^n}, \quad y_{12n-2} = \frac{(-1)^n p(1-pf)^{2n}}{(-1+2pf)^n}, \\
y_{12n-1} &= \frac{(-1)^n h(1-eh)^{2n}}{(-1+2eh)^n}, \ y_{12n} = \frac{(-1)^n g(1-dg)^{2n}}{(-1+2dg)^n}, \\
y_{12n+1} &= \frac{s}{(-1+sc)^{n+1}(-1-sc)^n}, \quad y_{12n+2} = \frac{r}{(-1+br)^{n+1}(-1-br)^n}, \\
y_{12n+3} &= \frac{q}{(-1+aq)^{n+1}(-1-aq)^n}, \quad y_{12n+4} = \frac{(-1)^n p(1-pf)^{2n+1}}{(-1+2pf)^{n+1}}, \\
y_{12n+5} &= \frac{(-1)^n h(1-eh)^{2n+1}}{(-1+2eh)^{n+1}}, \ y_{12n+6} = \frac{(-1)^n g(1-dg)^{2n+1}}{(-1+2dg)^{n+1}}.
\end{aligned}$$

where $x_{-5} = f$, $x_{-4} = e$, $x_{-3} = d$, $x_{-2} = c$, $x_{-1} = b$, $x_0 = a$, $y_{-5} = s$, $y_{-4} = r$, $y_{-3} = q$, $y_{-2} = p$, $y_{-1} = h$, $y_0 = g$.

Example 3. Figure (3) shows the behavior of the solution of the difference system (3) with the initial conditions $x_{-5} = .19$, $x_{-4} = .13$, $x_{-3} = .35$, $x_{-2} =$

$.21,\ x_{-1} = -.11,\ x_0 = .21,\ y_{-5} = -.16,\ y_{-4} = -.33,\ y_{-3} = .12,\ y_{-2} = -.44,$ $y_{-1} = .58,$ *and* $y_0 = .3$

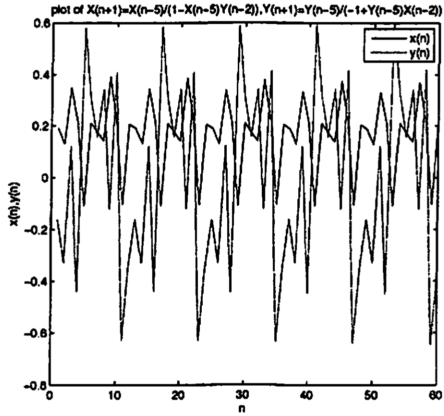

Figure 3.

5 On the system: $x_{n+1} = \frac{x_{n-5}}{1-y_{n-2}x_{n-5}},\ y_{n+1} = \frac{y_{n-5}}{-1-x_{n-2}y_{n-5}}$

In this section, we study the solution of the following system of the difference equations

$$x_{n+1} = \frac{x_{n-5}}{1 - y_{n-2}x_{n-5}}, \qquad y_{n+1} = \frac{y_{n-5}}{-1 - x_{n-2}y_{n-5}}, \tag{4}$$

where $n \in \mathbb{N}_0$ and the initial conditions are arbitrary real numbers with $x_{-5}y_{-2}$, $x_{-4}y_{-1}$, $x_{-3}y_0 \neq 1, \neq \frac{1}{2}$, and $x_{-2}y_{-5}$, $x_{-1}y_{-4}$, $x_0y_{-3} \neq \pm 1$.

Theorem 5. Suppose that $\{x_n, y_n\}$ are solutions of system (4). Then the solution for $n = 0, 1, 2, ...,$ is given by

$$\begin{aligned}
x_{12n-5} &= \frac{f}{(1-pf)^n(1+pf)^n}, \quad x_{12n-4} = \frac{e}{(1-eh)^n(1+eh)^n},\\
x_{12n-3} &= \frac{d}{(1-dg)^n(1+dg)^n}, \quad x_{12n-2} = \frac{c(1+sc)^{2n}}{(1+2sc)^n},\\
x_{12n-1} &= \frac{b(1+br)^{2n}}{(1+2br)^n}, \quad x_{12n} = \frac{a(1+aq)^{2n}}{(1+2aq)^n},\\
x_{12n+1} &= \frac{f}{(1-pf)^{n+1}(1+pf)^n}, \quad x_{12n+2} = \frac{e}{(1-eh)^{n+1}(1+eh)^n},\\
x_{12n+3} &= \frac{d}{(1-dg)^{n+1}(1+dg)^n}, \quad x_{12n+4} = \frac{c(1+sc)^{2n+1}}{(1+2sc)^{n+1}},\\
x_{12n+5} &= \frac{b(1+br)^{2n+1}}{(1+2br)^{n+1}}, \quad x_{12n+6} = \frac{a(1+aq)^{2n+1}}{(1+2aq)^{n+1}},
\end{aligned}$$

and

$$
\begin{aligned}
y_{12n-5} &= \frac{s(1+2sc)^n}{(1+sc)^{2n}}, \quad y_{12n-4} = \frac{r(1+2br)^n}{(1+br)^{2n}}, \\
y_{12n-3} &= \frac{q(1+2aq)^n}{(1+aq)^{2n}}, \quad y_{12n-2} = p(1-pf)^n(1+pf)^n, \\
y_{12n-1} &= h(1-eh)^n(1+eh)^n, \quad y_{12n} = g(1-dg)^n(1+dg)^n, \\
y_{12n+1} &= \frac{-s(1+2sc)^n}{(1+sc)^{2n+1}}, \quad y_{12n+2} = \frac{-r(1+2br)^n}{(1+br)^{2n+1}}, \\
y_{12n+3} &= \frac{-q(1+2aq)^n}{(1+aq)^{2n+1}}, \quad y_{12n+4} = -p(1+pf)^n(1-pf)^{n+1}, \\
y_{12n+5} &= -h(1+eh)^n(1-eh)^{n+1}, \; y_{12n+6} = -g(1+dg)^n(1-dg)^{n+1},
\end{aligned}
$$

where $x_{-5} = f,\ x_{-4} = e,\ x_{-3} = d,\ x_{-2} = c,\ x_{-1} = b,\ x_0 = a,\ y_{-5} = s,\ y_{-4} = r,\ y_{-3} = q,\ y_{-2} = p,\ y_{-1} = h,\ y_0 = g.$

Example 4. Figure (3) shows the dynamics of the solution of system (4) with the initial conditions $x_{-5} = .19$, $x_{-4} = .13$, $x_{-3} = -.35$, $x_{-2} = .21$, $x_{-1} = -.11$, $x_0 = .09$, $y_{-5} = -.16$, $y_{-4} = -.33$, $y_{-3} = .12$, $y_{-2} = .04$, $y_{-1} = -.1$'

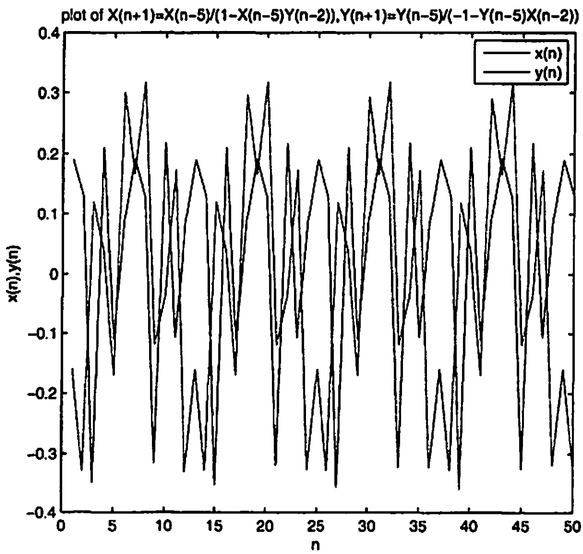

Figure 4.

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References

[1] N. Battaloglu, C. Cinar and I. Yalçınkaya, The dynamics of the difference equation, ARS Combinatoria, 97 (2010), 281-288.

[2] C. Cinar, I. Yalcinkaya and R. Karatas, On the positive solutions of the difference equation system $x_{n+1} = m/y_n$, $y_{n+1} = py_n/x_{n-1}y_{n-1}$, J. Inst. Math. Comp. Sci., 18 (2005), 135-136.

[3] E. M. Elabbasy, H. El-Metwally and E. M. Elsayed, Some properties and expressions of solutions for a class of nonlinear difference equation, Utilitas Mathematica, 87 (2012), 93-110.

[4] E. M. Elabbasy , H. El-Metwally and E. M. Elsayed, Global behavior of the solutions of difference equation, Adv. Differ. Equ., 2011, 2011: 28.

[5] E. M. Elsayed, Behavior of a rational recursive sequences, Studia Univ. " Babes-Bolyai ", Mathematica, LVI (1) (2011), 27–42.

[6] E. M. Elsayed, Solution and attractivity for a rational recursive sequence, Dis. Dyn. Nat. Soc., Volume 2011, Article ID 982309, 17 pages.

[7] E. M. Elsayed, Solutions of rational difference system of order two, Math. Comput. Mod., 55 (2012), 378–384.

[8] E. M. Elsayed, On the solution of some difference equations, Eur. J. Pure App. Math., 4 (3) (2011), 287-303.

[9] E. M. Elsayed and M. M. El-Dessoky, Dynamics and behavior of a higher order rational recursive sequence, Adv. Differ. Equ., 2012, 2012:69.

[10] A. S. Kurbanli, C. Cinar and I. Yalçınkaya, On the behavior of positive solutions of the system of rational difference equations, Math. Comp. Mod., 53 (2011), 1261-1267.

[11] N. Touafek and E. M. Elsayed, On the solutions of systems of rational difference equations, Math. Comput. Mod., 55 (2012), 1987–1997.

[12] N. Touafek and E. M. Elsayed, On the periodicity of some systems of nonlinear difference equations, Bull. Math. Soc. Sci. Math. Roumanie, Tome 55 (103) (2) (2012), 217–224.

[13] I. Yalcinkaya, On the global asymptotic stability of a second-order system of difference equations, Dis. Dyn. Nat. Soc., Vol. 2008, Article ID 860152 (2008), 12 pages.

[14] I. Yalçınkaya, On the global asymptotic behavior of a system of two nonlinear difference equations, ARS Combinatoria, 95 (2010), 151-159.