A note on skew spectrum of graphs

Yanna Wang, Bo Zhou*
Department of Mathematics, South China Normal University,
Guangzhou 510631, P. R. China

Abstract

We give some properties of skew spectrum of a graph, especially, we answer negatively a problem concerning the skew characteristic polynomial and matching polynomial in [M. Cavers et al., Skew-adjacency matrices of graphs, Linear Algebra Appl. 436 (2012) 4512–4529].

1 Introduction

We consider simple graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. An orientation of G is a sign-valued function σ on the set of ordered pairs $\{(i,j),(j,i)|ij \in E(G)\}$ that specifies an orientation to each edge ij of G: If $ij \in E(G)$, then we take $\sigma(i,j) = 1$ when $i \to j$ and $\sigma(i,j) = -1$ when $j \to i$. The resulting oriented graph is denoted by G^σ. Both σ and G^σ are called orientations of G.

The skew adjacency matrix $S^\sigma = S(G^\sigma)$ of G^σ is the $\{0,1,-1\}$-matrix with (i,j)-entry equal to $\sigma(i,j)$ if $ij \in E(G^\sigma)$ and 0 otherwise. If there is no confusion, we simply write $S = [s_{i,j}]$ for S^σ. Thus $s_{i,j} = 1$ if $ij \in E(G^\sigma)$, -1 if $ji \in E(G^\sigma)$, and 0 otherwise. The (skew) characteristic polynomial of $S = S^\sigma$ is

$$p_S(x) = \det(xI - S) = x^n + s_1x^{n-1} + \cdots + s_{n-1}x + s_n,$$

where $n = |V(G)|$. Let $\rho(D)$ be the spectral radius of a square matrix D, i.e., the largest modulus of the eigenvalues of D. The spectral radius of G is the spectral radius of its adjacency matrix. The maximum skew spectral radius of G is defined as $\rho_s(G) = \max_S \rho(S)$, where the maximum is taken over all of the skew adjacency matrices S of G.

*Corresponding author. E-mail address: zhoubo@scnu.edu.cn

An odd-cycle graph is a graph with no even cycles (cycles of even lengths). In particular, a tree is an odd-cycle graph.

Let G be a graph with n vertices. Let $m_k(G)$ be the number of matchings in G that cover k vertices. Obviously, $m_k(G) = 0$ if k is odd. The matching polynomial of G is defined as

$$m(G,x) = \sum_{k=0}^{n}(-1)^{\frac{k}{2}} m_k(G) x^{n-k},$$

where $m_0(G) = 1$.

Let G be a graph on n vertices. After showing that G is an odd-cycle graph if and only if $p_S(x) = (-i)^n m(G, ix)$ for all skew adjacency matrices S of G (see [1, Lemma 5.4]), Cavers et al. [1] posed the following question:

Problem 1. If $p_S(x) = (-i)^n m(G, ix)$ for some skew adjacency matrix S of G, must G be an odd-cycle graph?

After showing that if G is an odd-cycle graph, then $\rho_s(G) = \rho(S)$ for all skew adjacency matrices S of G ([1, Lemma 6.2]), Cavers et al. [1] posed the following question:

Problem 2. If G is a connected graph and $\rho(S)$ is the same for all skew adjacency matrices S of G, must G be an odd-cycle graph?

In this note we answer Problem 1 negatively by constructing a class of graphs, and when G is a connected bipartite graph we answer Problem 2 affirmatively.

2 Preliminaries

Let $\mathcal{U}_k$ be the set of all collections U of (vertex) disjoint edges and even cycles in G that cover k vertices ($\mathcal{U}_k^e$ was used for this set in [1]). A routing $\overrightarrow{U}$ of $U \in \mathcal{U}_k$ is obtained by replacing each edge in U by a digon and each (even) cycle in U by a dicycle. For an orientation σ of a graph G and a routing $\overrightarrow{U}$ of $U \in \mathcal{U}_k$, let $\sigma(\overrightarrow{U}) = \prod_{(i,j)\in E(\overrightarrow{U})} \sigma(i,j)$. We say that $\overrightarrow{U}$ is positively (resp. negatively) oriented relative to σ if $\sigma(\overrightarrow{U}) = 1$ (resp. $\sigma(\overrightarrow{U}) = -1$). For $U \in \mathcal{U}_k$, let $c^+(U)$ (resp. $c^-(U)$) be the number of cycles in U that are positively (resp. negatively) oriented relative to σ when U is given a routing $\overrightarrow{U}$. Then $c(U) = c^+(U) + c^-(U)$ is the (total) number of cycles of U.

Lemma 1. [1, eq. (8)] *Let S be a skew adjacency matrix of G. Then $s_k = 0$ if k is odd and*

$$s_k = m_k(G) + \sum_{\substack{U\in\mathcal{U}_k \\ c(U)>0}} (-1)^{c^+(U)} 2^{c(U)} \text{ if } k \text{ is even.}$$

The following lemma is obtained from parts 2 and 3 of Lemma 6.2 in [1].

Lemma 2. *Let G be a connected bipartite graph,*

$$A = \begin{bmatrix} O & B \\ B^{\mathsf{T}} & O \end{bmatrix}$$

the adjacency matrix of G, and

$$S = \begin{bmatrix} O & B \\ -B^{\mathsf{T}} & O \end{bmatrix}$$

and

$$\tilde{S} = \begin{bmatrix} O & \widetilde{B} \\ -\widetilde{B}^{\mathsf{T}} & O \end{bmatrix}$$

two skew adjacency matrices of G. Then $\rho(A) = \rho_s(G)$, and $\rho(S) = \rho(\widetilde{S})$ if and only if $\widetilde{S} = DSD^{-1}$ for some $\{-1, 1\}$-diagonal matrix D.

Lemma 3. [1, Theorem 4.2] *The skew adjacency matrices of a graph G are all cospectral if and only if G is an odd-cycle graph.*

3 Results

First we give a negative answer to Problem 1.

Theorem 1. *For integer $m \geq 2$, let G be the graph consisting of two $2m$-vertex cycles C_1 and C_2 with exactly one common vertex. Let σ be an orientation of G such that the cycle C_1 (resp. C_2) is positively (resp. negatively) oriented relative to σ. Let $S = S(G^\sigma)$ and let $n = 4m-1$. Then $p_S(x) = (-i)^n m(G, ix)$.*

Proof. It is sufficient to show that $s_k = m_k(G)$ for even k. By Lemma 1, we only need to show that

$$\sum_{\substack{U \in \mathcal{U}_k \\ c(U)>0}} (-1)^{c^+(U)} 2^{c(U)} = 0 \text{ for even } k.$$

This is obvious when $k < 2m$. Suppose that k is even with $2m \leq k \leq 4m-2$. Let $C_1 = v_1 v_2 \dots v_{2m} v_1$ and $C_2 = v_1' v_2' \dots v_{2m}' v_1'$ with $v_1 = v_1'$.

Let $\mathcal{U}_k^1$ be the subset of $\mathcal{U}_k$ consisting of C_1 and $\frac{1}{2}(k-2m)$ disjoint edges in C_2 and $\mathcal{U}_k^2$ the subset of $\mathcal{U}_k$ consisting of C_2 and $\frac{1}{2}(k-2m)$ disjoint edges in C_1. Obviously, $\mathcal{U}_k^1 \cap \mathcal{U}_k^2 = \emptyset$. For any $U \in \mathcal{U}_k$ with $c(U) > 0$, $U \in \mathcal{U}_k^1$ or $U \in \mathcal{U}_k^2$. There is a bijection from $\mathcal{U}_k^1$ to $\mathcal{U}_k^2$ which maps $U \in \mathcal{U}_k^1$ consisting of

C_1 and $\frac{1}{2}(k-2m)$ disjoint edges, say $v'_{i_1}v'_{i_1+1}, \ldots, v'_{i_s}v'_{i_s+1}$ in C_2 to $U' \in \mathcal{U}_k^2$ consisting of C_2 and $\frac{1}{2}(k-2m)$ disjoint edges $v_{i_1}v_{i_1+1}, \ldots, v_{i_s}v_{i_s+1}$ in C_1, where $s = \frac{1}{2}(k-2m)$ and $2 \leq i_1 < \cdots < i_s \leq 2m-1$. Thus $|\mathcal{U}_k^1| = |\mathcal{U}_k^2|$. Note that

$$\sum_{U \in \mathcal{U}_k^1} (-1)^{c^+(U)} 2^{c(U)} = (-1)^1 \cdot 2^1 \cdot |\mathcal{U}_k^1|$$

and

$$\sum_{U \in \mathcal{U}_k^2} (-1)^{c^+(U)} 2^{c(U)} = (-1)^0 \cdot 2^1 \cdot |\mathcal{U}_k^2|.$$

Thus

$$\sum_{\substack{U \in \mathcal{U}_k \\ c(U)>0}} (-1)^{c^+(U)} 2^{c(U)} = \sum_{U \in \mathcal{U}_k^1} (-1)^{c^+(U)} 2^{c(U)} + \sum_{U \in \mathcal{U}_k^2} (-1)^{c^+(U)} 2^{c(U)} = 0,$$

as desired. □

See Fig. 1 for an example with 7 vertices for Theorem 1 and its proof.

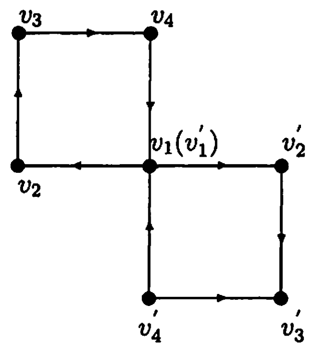

Fig. 1. A graph on 7 vertices with an orientation.

Now we give an observation on Problem 2, i.e., affirmative answer when G is a connected bipartite graph.

Theorem 2. *Let G be a connected bipartite graph. If $\rho(S)$ is the same for all skew adjacency matrices S of G, then G is a tree.*

Proof. Let

$$A = \begin{bmatrix} O & B \\ B^{\mathrm{T}} & O \end{bmatrix}$$

and

$$\overline{S} = \begin{bmatrix} O & B \\ -B^{\mathrm{T}} & O \end{bmatrix}$$

be an (ordinary) adjacency and a skew adjacency matrix of G. Let S be a skew adjacency matrix of G. Then $\rho(S) = \rho(\overline{S})$. By Lemma 2, there is a $\{-1, 1\}$-diagonal matrix D such that $S = D\overline{S}D^{-1}$, i.e., S is similar to $\overline{S}$, which implies that all skew adjacency matrices of G are cospectral. Thus by Lemma 3, G is an odd-cycle graph. Since G is connected and bipartite, G is a tree. □

Let G be a connected bipartite graph on n vertices. Let $K_{r,s}$ be the complete bipartite graph with r and s vertices in its two partite sets, respectively. Note that $\rho(G) < \rho(G + e)$ for an edge of the complement of G (following from the Perron-Frobenius theorem) and that $\rho(K_{r,s}) = \sqrt{rs}$. By Lemma 2, $\rho_s(G) = \rho(G) \leq \sqrt{\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil}$ with equality if and only if $G = K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$, cf. [1, Example 6.1].

Let G be a connected graph on n vertices. Let P_n be the path on n vertices. By [1, Lemma 6.4], $\rho_s(G) > \rho_s(G - e)$ for an edge e of G. Let T be a spanning tree of G. Then by Lemma 2 and a result of Collatz and Sinogowitz [2], $\rho_s(G) \geq \rho_s(T) = \rho(T) \geq \rho(P_n)$ with equality if and only if $G = P_n$, cf. [1, Example 6.3].

Acknowledgment. This work was supported by the Guangdong Provincial Natural Science Foundation of China (no. S2011010005539). The authors thank the referee for constructive comments.

References

[1] M. Cavers, S.M. Cioabă, S. Fallat, D.A. Gregory, W.H. Haemers, S.J. Kirkland, J.J. McDonald, M. Tsatsomeros, Skew-adjacency matrices of graphs, Linear Algebra Appl. 436 (2012) 4512–4529.

[2] L. Collatz, U. Sinogowitz, Spektren endlicher graphen, Abh. Math. Semin. Univ. Hamburg 21 (1957) 63–77.