ON A MULTIVARIABLE EXTENSION OF THE HERMITE AND RELATED POLYNOMIALS

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ABSTRACT. In this paper, some limit relations between multivariable Hermite polynomials (MHP) and some other multivariable polynomials are given, a class of multivariable polynomials is defined via generating function, which include (MHP) and multivariable Gegenbauer polynomials (MGP) and with the help of this generating function various recurrence relations are obtained to this class. Integral representations of MHP and MGP are also given. Furthermore, general families of multilinear and multilateral generating functions are obtained and their applications are presented.

1. INTRODUCTION

The classical Hermite polynomials $H_n(x)$ of degree n are defined by the Rodrigues formula

$$H_n(x) = (-1)^n e^{x^2} D_x^n \left(e^{-x^2}\right) \; , \; \left(D_x := \frac{d}{dx}\right) \tag{1.1}$$

or, equivalently, by

$$H_n(x) = (2x)^n \; {}_2F_0\left(-\tfrac{n}{2}, -\tfrac{n}{2} + \tfrac{1}{2}; -; -\tfrac{1}{x^2}\right) \tag{1.2}$$

where ${}_2F_0$ denotes the familiar hypergeometric function which corresponds to the special case $r-2 = s = 0$ of the generalized hypergeometric function ${}_rF_s$ with r numerator and s denominator parameters.

It is well-known that these polynomials are orthogonal over the interval $(-\infty, \infty)$ with respect to the weight function $\omega(x) = e^{-x^2}$. In fact, we have

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the following relation

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x)\, dx = 2^n n! \sqrt{\pi} \delta_{m,n}$$

$$(m, n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\})$$

where $\delta_{m,n}$ denotes the Kronecker delta.

The Hermite polynomials are limiting cases of the Jacobi polynomials, and we have the relationship [9]

$$H_n(x) = n! \lim_{\nu \to \infty} \left\{ \frac{\nu^{-\frac{n}{2}} (2\nu)_n}{\left(\nu+\frac{1}{2}\right)_n} P_n^{\left(\nu-\frac{1}{2}, \nu-\frac{1}{2}\right)} \left(x/\sqrt{\nu}\right) \right\}. \tag{1.3}$$

As a result of this, the following relation between Gegenbauer polynomials and Hermite polynomials holds:

$$H_n(x) = n! \lim_{\nu \to \infty} \left\{ \nu^{-n/2} C_n^{\nu} \left(x/\sqrt{\nu}\right) \right\}. \tag{1.4}$$

A systematic investigation of a multivariable extension of the Hermite polynomials $H_n(x)$ is defined by

$$H_{\mathbf{n}}(\mathbf{x}) = H_{n_1,\ldots,n_r}(x_1, \ldots, x_r) = H_{n_1}(x_1) \ldots H_{n_r}(x_r) \tag{1.5}$$

where $\mathbf{x} = (x_1, \ldots, x_r)$ and $|\mathbf{n}| = n_1 + \ldots + n_r$; $n_1, \ldots, n_r \in \mathbb{N}_0$ (see [5]). The multivariable Hermite polynomials $H_{\mathbf{n}}(\mathbf{x})$ (MHP) are orthogonal with respect to the weight function

$$\omega(x_1, \ldots, x_r) = \omega_1(x_1) \ldots \omega_r(x_r) = e^{-\left(x_1^2 + \ldots + x_r^2\right)}$$

over the domain

$$\Omega = \{(x_1, \ldots, x_r) : \ -\infty < x_i < \infty \ ; \ i = 1, 2, \ldots, r\}.$$

In fact, we have

$$\begin{aligned} & \int_{\Omega} \omega(x_1, \ldots, x_r) H_{\mathbf{n}}(\mathbf{x}) H_{\mathbf{m}}(\mathbf{x})\, d\mathbf{x} \qquad (1.6) \\ = & \int_{-\infty}^{\infty} H_{n_1}(x_1) H_{m_1}(x_1) e^{-x_1^2} dx_1 \times \ldots \times \int_{-\infty}^{\infty} H_{n_r}(x_r) H_{m_r}(x_r) e^{-x_r^2} dx_r \\ = & \pi^{r/2} \prod_{i=1}^{r} 2^{n_i} n_i! \delta_{m_i, n_i} \\ & (m_i, n_i \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \ ; \ i = 1, 2, \ldots, r) \end{aligned}$$

where $dx = dx_1 \ldots dx_r$.

We organize the paper as follows:

In section 2, some limit relations between MHP and other special functions such as multivariable extended Jacobi polynomials (MEJP) and multivariable Lagrange polynomials (MLP) are given. In section 3, some generating functions are obtained for MHP and multivariable Gegenbauer polynomials (MGP). Furthermore, general class of polynomials is defined via generating function and with the help of this generating function several recurrence formulas for MHP and MGP are given. In section 4, some integral representations are obtained for MHP and MGP and in section 5, multilinear and multilateral generating functions are derived for MHP and MGP. In section 6, some applications of the results obtained in section 5 are presented.

2. Some Limit Relations for Multivariable Hermite Polynomials

We first deal with the classical Hermite polynomials in order to get various limit relations involving other well-known polynomials. Fujiwara [6] studied the polynomial $F_n^{(\alpha,\beta)}(x;a,b,c)$ called extended Jacobi polynomial (EJP) and defined by the Rodrigues formula

$$\begin{aligned} F_n^{(\alpha,\beta)}(x;a,b,c) &= \frac{(-c)^n}{n!}(x-a)^{-\alpha}(b-x)^{-\beta} \\ &\times D_x^n\left\{(x-a)^{n+\alpha}(b-x)^{n+\beta}\right\} \quad , \ (c>0)\,. \end{aligned} \tag{2.1}$$

The following relation between the polynomials EJPs and Jacobi polynomials holds [11]:

$$F_n^{(\alpha,\beta)}(x;a,b,c) = \{c(a-b)\}^n P_n^{(\alpha,\beta)}\left(\frac{2(x-a)}{a-b}+1\right). \tag{2.2}$$

It is well-known that the following equality between Jacobi and classical Lagrange polynomials $g_n^{(\alpha,\beta)}(x,y)$ (see [4]):

$$P_n^{(-\alpha-n,-\beta-n)}\left(\frac{x+y}{x-y}\right) = (y-x)^{-n} g_n^{(\alpha,\beta)}(x,y) \tag{2.3}$$

holds. We now recall the Chan-Chyan-Srivastava (CCS) polynomials $g_n^{(\alpha_1,\dots,\alpha_r)}(x_1,\dots,x_r)$ (see [4]), which is a multivariable extension of the classical Lagrange polynomials, generated by

$$\prod_{j=1}^{r}\left\{(1-x_jt)^{-\alpha_j}\right\} = \sum_{n=0}^{\infty} g_n^{(\alpha_1,\dots,\alpha_r)}(x_1,\dots,x_r)\,t^n\ , \tag{2.4}$$

where $|t| < \min\left\{|x_1|^{-1},\dots,|x_r|^{-1}\right\}$. In [1], Altın et.al show that the following relation between Jacobi polynomials and CCS polynomials holds:

$$\begin{aligned}
&\mathrm{g}_n^{(\alpha_1,\dots,\alpha_r)}(x_1,\dots,x_r)\\
&= \sum_{n_1+\dots+n_{r-1}=n} \prod_{i=1}^{r-2} \binom{-\alpha_i}{n_i} (-1)^{n_i} x_i^{n_i} (x_{r-1}-x_r)^{n_{r-1}}\\
&\quad \times P_{n_{r-1}}^{(-\alpha_r-n_{r-1},-\alpha_{r-1}-n_{r-1})}\left(\frac{x_r+x_{r-1}}{x_r-x_{r-1}}\right). \qquad (2.5)
\end{aligned}$$

As a result of these properties, we can give the next results.

Theorem 2.1. *For the classical Hermite polynomials $H_n(x)$, we have*
(i) $H_n(x) = n!\{c(a-b)\}^{-n}$

$$\times \lim_{\nu\to\infty}\left\{\frac{\nu^{-\frac{n}{2}}(2\nu)_n}{\left(\nu+\frac{1}{2}\right)_n} F_n^{(\nu-\frac{1}{2},\nu-\frac{1}{2})}\left(a+\tfrac{a(x-\sqrt{\nu})}{2\sqrt{\nu}}-\tfrac{b(x-\sqrt{\nu})}{2\sqrt{\nu}};a,b,c\right)\right\},$$

(ii) $H_n(x) = n!(-2x)^{-n}\lim_{\nu\to\infty}\left\{\frac{\nu^{-\frac{n}{2}}(2\nu)_n}{\left(\nu+\frac{1}{2}\right)_n}\mathrm{g}_n^{(-\nu-n+\frac{1}{2},-\nu-n+\frac{1}{2})}\left(x,\frac{x^2-x\sqrt{\nu}}{x+\sqrt{\nu}}\right)\right\}$,

(iii) $$\lim_{\nu\to\infty}\left\{\mathrm{g}_n^{(\alpha_1,\dots,\alpha_{r-2},-\nu-n_{r-1}+\frac{1}{2},-\nu-n_{r-1}+\frac{1}{2})}\left(x_1,\dots,x_{r-2},\frac{x_r^2-x_r\sqrt{\nu}}{x_r+\sqrt{\nu}},x_r\right)\right.$$
$$\left.\times\, \nu^{-\frac{n_{r-1}}{2}}\frac{(2\nu)_{n_{r-1}}}{\left(\nu+\frac{1}{2}\right)_{n_{r-1}}}\right\}$$

$$= \sum_{n_1+\dots+n_{r-1}=n}\left\{\prod_{i=1}^{r-2}\binom{-\alpha_i}{n_i}(-1)^{n_i}x_i^{n_i}\right\}(-2x_r)^{n_{r-1}}\frac{H_{n_{r-1}}(x_r)}{n_{r-1}!}.$$

Proof. (i) From (1.3) and (2.2), we have

$$\begin{aligned}
H_n(x) &= n!\lim_{\nu\to\infty}\left\{\frac{\nu^{-\frac{n}{2}}(2\nu)_n}{\left(\nu+\frac{1}{2}\right)_n}P_n^{(\nu-\frac{1}{2},\nu-\frac{1}{2})}(x/\sqrt{\nu})\right\}\\
&= \lim_{\nu\to\infty}\left\{\frac{\nu^{-\frac{n}{2}}(2\nu)_n}{\left(\nu+\frac{1}{2}\right)_n}F_n^{(\nu-\frac{1}{2},\nu-\frac{1}{2})}\left(a+\tfrac{a(x-\sqrt{\nu})}{2\sqrt{\nu}}-\tfrac{b(x-\sqrt{\nu})}{2\sqrt{\nu}};a,b,c\right)\right\}\\
&\quad\times n!\{c(a-b)\}^{-n}.
\end{aligned}$$

(ii) It is enough to use (1.3) and (2.3).
(iii) It follows from (1.3) and (2.5) immediately. □

Now, we recall the multivariable extended Jacobi polynomials defined by [1]

$$F_{\mathbf{n}}^{(\alpha_1,\dots,\alpha_r;\beta_1,\dots,\beta_r)}(\mathbf{x}) := F_{n_1}^{(\alpha_1,\beta_1)}(x_1;a_1,b_1,c_1)\dots F_{n_r}^{(\alpha_r,\beta_r)}(x_r;a_r,b_r,c_r)$$

where $\mathbf{x}=(x_1,\dots,x_r)$ and $|\mathbf{n}| = n_1+\dots+n_r;\ n_1,\dots,n_r\in\mathbb{N}_0$.

Therefore, using the polynomials $F_{\mathbf{n}}^{(\alpha_1,\dots,\alpha_r;\beta_1,\dots,\beta_r)}(\mathbf{x})$ and $H_{\mathbf{n}}(\mathbf{x})$, we get the next theorem.

Theorem 2.2. *The multivariable Hermite polynomials* $H_{\mathbf{n}}(\mathbf{x})$ *satisfy the following relation:*

$$H_{\mathbf{n}}(\mathbf{x}) = \left(\prod_{i=1}^{r} n_i!\left\{c_i(a_i-b_i)\right\}^{-n_i}\right)$$
$$\times \lim_{(\nu_1,\ldots,\nu_r)\to\infty}\left\{F_{\mathbf{n}}^{\left(\nu_1-\frac{1}{2},\ldots,\nu_r-\frac{1}{2};\nu_1-\frac{1}{2},\ldots,\nu_r-\frac{1}{2}\right)}\left(\mathbf{a}+\frac{\mathbf{a}(\mathbf{x}-\sqrt{\nu})}{2\sqrt{\nu}}-\frac{\mathbf{b}(\mathbf{x}-\sqrt{\nu})}{2\sqrt{\nu}}\right)\right.$$
$$\left.\times\prod_{i=1}^{r}\frac{\nu_i^{-\frac{n_i}{2}}(2\nu_i)_{n_i}}{\left(\nu_i+\frac{1}{2}\right)_{n_i}}\right\}$$

where

$$\mathbf{a}+\frac{\mathbf{a}(\mathbf{x}-\sqrt{\nu})}{2\sqrt{\nu}}-\frac{\mathbf{b}(\mathbf{x}-\sqrt{\nu})}{2\sqrt{\nu}}$$
$$:=\left(a_1+\frac{a_1(x_1-\sqrt{\nu_1})}{2\sqrt{\nu_1}}-\frac{b_1(x_1-\sqrt{\nu_1})}{2\sqrt{\nu_1}},\ldots,a_r+\frac{a_r(x_r-\sqrt{\nu_r})}{2\sqrt{\nu_r}}-\frac{b_r(x_r-\sqrt{\nu_r})}{2\sqrt{\nu_r}}\right).$$

Proof. By (1.5) and Theorem 2.1 (i), we can write that

$$H_{\mathbf{n}}(\mathbf{x}) = H_{n_1}(x_1)\ldots H_{n_r}(x_r)$$
$$= \prod_{i=1}^{r} n_i!\left\{c_i(a_i-b_i)\right\}^{-n_i}$$
$$\times\lim_{\nu_i\to\infty}\left\{F_{n_i}^{\left(\nu_i-\frac{1}{2},\nu_i-\frac{1}{2}\right)}\left(a_i+\frac{a_i(x_i-\sqrt{\nu_i})}{2\sqrt{\nu_i}}-\frac{b_i(x_i-\sqrt{\nu_i})}{2\sqrt{\nu_i}};a_i,b_i,c_i\right)\right.$$
$$\left.\times\frac{\nu_i^{-\frac{n_i}{2}}(2\nu_i)_{n_i}}{\left(\nu_i+\frac{1}{2}\right)_{n_i}}\right\}$$
$$= \left(\prod_{i=1}^{r} n_i!\left\{c_i(a_i-b_i)\right\}^{-n_i}\right)$$
$$\times\lim_{(\nu_1,\ldots,\nu_r)\to\infty}\left\{F_{\mathbf{n}}^{\left(\nu_1-\frac{1}{2},\ldots,\nu_r-\frac{1}{2};\nu_1-\frac{1}{2},\ldots,\nu_r-\frac{1}{2}\right)}\left(\mathbf{a}+\frac{\mathbf{a}(\mathbf{x}-\sqrt{\nu})}{2\sqrt{\nu}}-\frac{\mathbf{b}(\mathbf{x}-\sqrt{\nu})}{2\sqrt{\nu}}\right)\right.$$
$$\left.\times\prod_{i=1}^{r}\frac{\nu_i^{-\frac{n_i}{2}}(2\nu_i)_{n_i}}{\left(\nu_i+\frac{1}{2}\right)_{n_i}}\right\}.$$

□

3. Generating Functions and Recurrence Relations

The multivariable Gegenbauer polynomials $C_{\mathbf{n}}^{(\nu_1,\ldots,\nu_r)}(\mathbf{x})$ of degree $|\mathbf{n}|$ are defined as follows (see [5])

$$C_{\mathbf{n}}^{(\nu_1,\ldots,\nu_r)}(\mathbf{x}) = C_{n_1,\ldots,n_r}^{(\nu_1,\ldots,\nu_r)}(x_1,\ldots,x_r) = C_{n_1}^{\nu_1}(x_1)\ldots C_{n_r}^{\nu_r}(x_r). \qquad (3.1)$$

In this section, we obtain some generating functions and recurrence relations for MHP $H_{\mathbf{n}}(\mathbf{x})$ and MGP $C_{\mathbf{n}}^{(\nu_1,\ldots,\nu_r)}(\mathbf{x})$. Firstly, recall that the classical Hermite polynomials are generated by (see [9])

$$\sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} = \exp\left(2xt - t^2\right) \tag{3.2}$$

and

$$\sum_{n=0}^{\infty} H_{m+n}(x) \frac{t^n}{n!} = \exp\left(2xt - t^2\right) H_m(x - t) \tag{3.3}$$

for $m \in \mathbb{N}_0$.

Furthermore, for the classical Gegenbauer polynomials, we have (see [10])

$$\sum_{n=0}^{\infty} C_n^{\nu}(x) t^n = \left(1 - 2xt + t^2\right)^{-\nu} \tag{3.4}$$

and

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{(\mu)_n} C_n^{\nu}(x) t^n = F_1\left[\lambda, \nu, \nu; \mu; \left(x + \sqrt{x^2 - 1}\right) t, \left(x - \sqrt{x^2 - 1}\right) t\right] \tag{3.5}$$

where $\lambda, \mu \in \mathbb{R}$ and F_1 is the first kind of Appell's double hypergeometric functions

$$F_1\left[\alpha, \beta, \beta'; \gamma; x, y\right] = \sum_{r,s=0}^{\infty} \frac{(\alpha)_{r+s} (\beta)_r \left(\beta'\right)_s}{(\gamma)_{r+s}} \frac{x^r}{r!} \frac{y^s}{s!}, \quad \max\{|x|, |y|\} < 1.$$

Using the above expressions we can give the following results.

Theorem 3.1. *For the polynomials MHP* $H_{\mathrm{n}}(x_1, ..., x_r)$, *we have*

(i) $$\sum_{n_1, ..., n_r = 0}^{\infty} H_{\mathrm{n}}(x_1, ..., x_r) \frac{t_1^{n_1}}{n_1!} ... \frac{t_r^{n_r}}{n_r!}$$

$$= \prod_{i=1}^{r} \exp\left(2x_i t_i - t_i^2\right), \tag{3.6}$$

(ii) $$\sum_{n=0}^{\infty} Y_{n+m_1+...+m_r}(x_1, ..., x_r) t^n$$

$$= \left\{\prod_{i=1}^{r} \exp\left(2x_i t - t^2\right)\right\} H_{\mathrm{m}}(\mathrm{x} - t) \tag{3.7}$$

where

$$\begin{aligned} & Y_{n+m_1+\ldots+m_r}(x_1,\ldots,x_r) \\ = & \sum_{n_1=0}^{n}\sum_{n_2=0}^{n-n_1}\cdots\sum_{n_{r-1}=0}^{n-n_1-\ldots-n_{r-2}} \xi_n(n_1,\ldots,n_{r-1}) \\ & \times H_{n+m_1-(n_1+\ldots+n_{r-1}),n_1+m_2,\ldots,n_{r-1}+m_r}(x_1,\ldots,x_r), \end{aligned}$$

$$H_{\mathbf{m}}(\mathbf{x}-t) = H_{m_1,\ldots,m_r}(x_1-t,\ldots,x_r-t)$$

and also

$$\xi_n(n_1,\ldots,n_{r-1}) = \frac{1}{(n-(n_1+\ldots+n_{r-1}))!n_1!\ldots n_{r-1}!}.$$

Theorem 3.2. *The polynomials MGP $C_{\mathbf{n}}^{(\nu_1,\ldots,\nu_r)}(\mathbf{x})$ hold as follows:*

(i) $\sum_{n_1,\ldots,n_r=0}^{\infty} C_{\mathbf{n}}^{(\nu_1,\ldots,\nu_r)}(x_1,\ldots,x_r)\, t_1^{n_1}\ldots t_r^{n_r}$

$$=\prod_{i=1}^{r}\left(1-2x_it_i+t_i^2\right)^{-\nu_i}, \tag{3.8}$$

(ii) $\sum_{n=0}^{\infty} u_n^{(\nu_1,\ldots,\nu_r)}(x_1,\ldots,x_r)\, t^n$

$$=\prod_{i=1}^{r} F_1\left[\lambda_i,\nu_i,\nu_i;\mu_i;\left(x_i+\sqrt{x_i^2-1}\right)t,\left(x_i-\sqrt{x_i^2-1}\right)t\right] \tag{3.9}$$

where

$$\begin{aligned} u_n^{(\nu_1,\ldots,\nu_r)}(x_1,\ldots,x_r) = & \sum_{n_1=0}^{n}\sum_{n_2=0}^{n-n_1}\cdots\sum_{n_{r-1}=0}^{n-n_1-\ldots-n_{r-2}} \delta_n(n_1,\ldots,n_{r-1}) \\ & \times C_{n-(n_1+\ldots+n_{r-1}),n_1,\ldots,n_{r-1}}^{(\nu_1,\ldots,\nu_r)}(x_1,\ldots,x_r) \end{aligned}$$

and also

$$\delta_n(n_1,\ldots,n_{r-1}) = \frac{(\lambda_1)_{n-(n_1+\ldots+n_{r-1})}(\lambda_2)_{n_1}\cdots(\lambda_r)_{n_{r-1}}}{(\mu_1)_{n-(n_1+\ldots+n_{r-1})}(\mu_2)_{n_1}\cdots(\mu_r)_{n_{r-1}}}.$$

In order to obtain some recurrence relations we need the following lemma.

Lemma 3.3. *Let a generating function for $f_{n_1,\ldots,n_r}(x_1,\ldots,x_r)$ be*

$$\Psi(2x_1t_1-t_1^2,\ldots,2x_rt_r-t_r^2) = \sum_{n_1,\ldots,n_r=0}^{\infty} f_{n_1,\ldots,n_r}(x_1,\ldots,x_r)t_1^{n_1}\ldots t_r^{n_r} \tag{3.10}$$

where $f_{n_1,...,n_r}(x_1,...,x_r)$ *is a polynomial of degree* n_i *with respect to* x_i*(of total degree* $n = n_1 + ... + n_r$*), provided that*

$$\Psi(u_1,...,u_r) = \Psi_1(u_1)...\Psi_r(u_r) \; ; \; u_i = 2x_it_i - t_i^2 \; ; \; i = 1,2,...,r$$
$$\Psi_i(u_i) = \sum_{n_i=0}^{\infty} \gamma_{n_i} u_i^{n_i} \; , \; \gamma_0 \neq 0.$$

Then we have

$$x_i \frac{\partial}{\partial x_i} f_{n_1,...,n_r}(x_1,...,x_r)$$
$$= \frac{\partial}{\partial x_i} f_{n_1,...,n_{i-1},n_i-1,n_{i+1},...,n_r}(x_1,...,x_r) + n_i f_{n_1,...,n_r}(x_1,...,x_r), \; n_i \geq 1. \tag{3.11}$$

Proof. Differentiating (3.10) with respect to x_i and t_i and making necessary arrangements, we obtain the desired relation. □

As a result of Lemma 3.3, considering (3.6), we can write that

$$f_{n_1,...,n_r}(x_1,...,x_r) = \frac{H_{\mathbf{n}}(x_1,...,x_r)}{n_1!...n_r!},$$
$$\gamma_{n_i} = \frac{1}{n_i!} \; , \; i = 1,2,...,r.$$

With the help of the Lemma 3.3 and also considering (3.11), one can easily obtain the next result.

Theorem 3.4. *For the polynomials* $H_{\mathbf{n}}(x_1,...,x_r)$, *we have*

$$x_i \frac{\partial}{\partial x_i} H_{n_1,...,n_r}(x_1,...,x_r) - n_i \frac{\partial}{\partial x_i} H_{n_1,...,n_{i-1},n_i-1,n_{i+1},...,n_r}(x_1,...,x_r)$$
$$= n_i H_{n_1,...,n_r}(x_1,...,x_r) \; , \; n_i \geq 1.$$

As a consequence of Lemma 3.3, considering (3.8), we have

$$f_{n_1,...,n_r}(x_1,...,x_r) = C_{\mathbf{n}}^{(\nu_1,...,\nu_r)}(x_1,...,x_r),$$
$$\gamma_{n_i} = \frac{(\nu_i)_{n_i}}{n_i!} \; , \; i = 1,2,...,r.$$

With the help of the Lemma 3.3 and also considering (3.11), we can easily find the next result.

Theorem 3.5. *The polynomials* $C_{\mathbf{n}}^{(\nu_1,...,\nu_r)}(x_1,...,x_r)$ *satisfy the following recurrence relation:*

$$x_i \frac{\partial}{\partial x_i} C_{n_1,...,n_r}^{(\nu_1,...,\nu_r)}(x_1,...,x_r) - \frac{\partial}{\partial x_i} C_{n_1,...,n_{i-1},n_i-1,n_{i+1},...,n_r}^{(\nu_1,...,\nu_r)}(x_1,...,x_r)$$
$$= n_i C_{n_1,...,n_r}^{(\nu_1,...,\nu_r)}(x_1,...,x_r) \; , \; n_i \geq 1.$$

As a result of Theorem 3.5, for $\nu_i = 1/2$, $(i = 1, 2, ..., r)$, the multivariable Legendre polynomials $P_{n_1,...,n_r}(x_1, ..., x_r)$ defined by

$$P_{\mathbf{n}}(x_1, ..., x_r) = P_{n_1,...,n_r}(x_1, ..., x_r) = P_{n_1}(x_1) ... P_{n_r}(x_r)$$

hold:

Corollary 3.6. *For the multivariable Legendre polynomials $P_{\mathbf{n}}(x_1, ..., x_r)$, we have*

$$x_i \frac{\partial}{\partial x_i} P_{n_1,...,n_r}(x_1, ..., x_r) - \frac{\partial}{\partial x_i} P_{n_1,...,n_{i-1},n_i-1,n_{i+1},...,n_r}(x_1, ..., x_r)$$
$$= n_i P_{n_1,...,n_r}(x_1, ..., x_r) \ , \quad n_i \geq 1.$$

where

$$P_{\mathbf{n}}(\mathbf{x}) = P_{n_1,...,n_r}(x_1, ..., x_r) = P_{n_1}(x_1) ... P_{n_r}(x_r) .$$

Similar to Lemma 3.3, we also get the following result.

Lemma 3.7. *Let a generating function for $f_{n_1,...,n_r}(x_1, ..., x_r)$ be*

$$e^{-t_1^2 - ... - t_r^2} \Phi(x_1 t_1, ..., x_r t_r) = \sum_{n_1,...,n_r=0}^{\infty} f_{n_1,...,n_r}(x_1, ..., x_r) t_1^{n_1} ... t_r^{n_r} \quad (3.12)$$

where $f_{n_1,...,n_r}(x_1, ..., x_r)$ is a polynomial of degree n_i with respect to x_i(of total degree $n = n_1 + ... + n_r$), provided that

$$\Phi(u_1, ..., u_r) = \Phi_1(u_1) ... \Phi_r(u_r) \ ; \ u_i = x_i t_i \ ; \ i = 1, 2, ..., r$$
$$\Phi_i(u_i) = \sum_{n_i=0}^{\infty} \varphi_{n_i} u_i^{n_i} \ , \ \varphi_0 \neq 0.$$

Then we have

$$n_i f_{n_1,...,n_r}(x_1, ..., x_r) + 2 f_{n_1,...,n_{i-1},n_i-2,n_{i+1},...,n_r}(x_1, ..., x_r)$$
$$= x_i \frac{\partial}{\partial x_i} f_{n_1,...,n_r}(x_1, ..., x_r) \ , \ n_i \geq 2. \quad (3.13)$$

As a result of Lemma 3.7, if we choice

$$f_{n_1,...,n_r}(x_1, ..., x_r) = \frac{H_{n_1,...,n_r}(x_1, ..., x_r)}{n_1! ... n_r!}$$
$$\varphi_{n_i} = \frac{2^{n_i}}{n_i!} \ , \ i = 1, 2, ..., r$$

and also consider (3.13), one can easily obtain the next result.

Theorem 3.8. *For the polynomials MHP $H_{n_1,...,n_r}(x_1, ..., x_r)$, we have the following recurrence relation:*

$$x_i \frac{\partial}{\partial x_i} H_{n_1,...,n_r}(x_1, ..., x_r) - n_i H_{n_1,...,n_r}(x_1, ..., x_r)$$
$$= 2 n_i (n_i - 1) H_{n_1,...,n_{i-1},n_i-2,n_{i+1},...,n_r}(x_1, ..., x_r) \ , \ n_i \geq 1.$$

By Theorem 3.4 and Theorem 3.8, we can give as follows.

Corollary 3.9. *The polynomials MHP* $H_{n_1,\ldots,n_r}(x_1,\ldots,x_r)$ *hold:*

$$\frac{\partial}{\partial x_i}H_{n_1,\ldots,n_r}(x_1,\ldots,x_r)=2n_iH_{n_1,\ldots,n_{i-1},n_i-1,n_{i+1},\ldots,n_r}(x_1,\ldots,x_r)$$

and

$$\begin{aligned}&2x_iH_{n_1,\ldots,n_{i-1},n_i-1,n_{i+1},\ldots,n_r}(x_1,\ldots,x_r)\\ =\;& H_{n_1,\ldots,n_r}(x_1,\ldots,x_r)+\frac{\partial}{\partial x_i}H_{n_1,\ldots,n_{i-1},n_i-1,n_{i+1},\ldots,n_r}(x_1,\ldots,x_r).\end{aligned}$$

4. Integral Representations

In this section, we give various integral representations for families of polynomials generated by (3.6) and (3.8). It is well-known that the classical Hermite polynomials have the following integral representation [9] :

$$H_n(x)=2^{n+1}\exp\left(x^2\right)\int_x^{\infty}\exp\left(-t^2\right)t^{n+1}P_n(x/t)\,dt \tag{4.1}$$

where $P_n(x)$ is Legendre polynomials of degree n.

Now, we get the following result.

Theorem 4.1. *For the polynomials* $H_{\mathbf{n}}(x_1,\ldots,x_r)$, *we have*

$$\begin{aligned}H_{\mathbf{n}}(x_1,\ldots,x_r) &= \int_{x_1}^{\infty}\int_{x_2}^{\infty}\ldots\int_{x_r}^{\infty}\exp\left(-t_1^2-\ldots-t_r^2\right)t_1^{n_1+1}\ldots t_r^{n_r+1}P_{\mathbf{n}}(\mathbf{x}/\mathbf{t})\,\mathbf{dt}\\ &\quad\times 2^{n_1+\ldots+n_r+r}\exp\left(x_1^2+\ldots+x_r^2\right)\end{aligned}$$

where

$$\begin{aligned}P_{\mathbf{n}}(\mathbf{x}/\mathbf{t}) &= P_{n_1,\ldots,n_r}(x_1/t_1,\ldots,x_r/t_r)=P_{n_1}(x_1/t_1)\ldots P_{n_r}(x_r/t_r),\\ \mathbf{dt} &= dt_1\ldots dt_r.\end{aligned}$$

Proof. It is straightforward from (1.5) and (4.1). □

Theorem 4.2. *The polynomials MGP* $C_{\mathbf{n}}^{(\nu_1,\ldots,\nu_r)}(x_1,\ldots,x_r)$ *have the following integral representation:*

$$\begin{aligned}C_{\mathbf{n}}^{(\nu_1,\ldots,\nu_r)}(x_1,\ldots,x_r) &= \frac{1}{\Gamma(\nu_1)\ldots\Gamma(\nu_r)}\int_0^{\infty}\int_0^{\infty}\ldots\int_0^{\infty}\xi_1^{\nu_1-1}\ldots\xi_r^{\nu_r-1}e^{-(\xi_1+\ldots+\xi_r)}\\ &\quad\times S(n_1,\ldots,n_r)\mathbf{d\xi}\end{aligned}$$

where

$$S(n_1,...,n_r)=\prod_{j=1}^{r}\sum_{m_j=0}^{\left[\frac{n_j}{2}\right]}\frac{2^{n_j-2m_j}(-1)^{m_j}\left(x_j\xi_j\right)^{n_j-2m_j}\left(\xi_j\right)^{m_j}}{(n_j-2m_j)!m_j!}$$

and $\mathbf{d}\boldsymbol{\xi}=d\xi_1...d\xi_r$ *and* $\mathrm{Re}(\nu_i)>0\quad(i=1,...,r)$.

Proof. If we use the identity for $\mathrm{Re}(\nu)>0$

$$a^{-\nu}=\frac{1}{\Gamma(\nu)}\int_0^\infty e^{-at}t^{\nu-1}dt \tag{4.2}$$

in the right hand side of the generating function (3.8), we have

$$\begin{aligned}
&\sum_{n_1,...,n_r=0}^{\infty} C_{\mathbf{n}}^{(\nu_1,...,\nu_r)}(x_1,...,x_r)\ t_1^{n_1}...t_r^{n_r}\\
=&\prod_{i=1}^{r}\left(1-2x_it_i+t_i^2\right)^{-\nu_i}\\
=&\frac{1}{\Gamma(\nu_1)...\Gamma(\nu_r)}\int_0^\infty\int_0^\infty...\int_0^\infty e^{-(\xi_1+...+\xi_r)}\xi_1^{\nu_1-1}...\xi_r^{\nu_r-1}\prod_{j=1}^{r}e^{2t_jx_j\xi_j}e^{-t_j^2\xi_j}\mathbf{d}\boldsymbol{\xi}\\
=&\frac{1}{\Gamma(\nu_1)...\Gamma(\nu_r)}\int_0^\infty\int_0^\infty...\int_0^\infty\left\{e^{-(\xi_1+...+\xi_r)}\xi_1^{\nu_1-1}...\xi_r^{\nu_r-1}\right.\\
&\left.\times\prod_{j=1}^{r}\sum_{n_j=0}^{\infty}\sum_{m_j=0}^{\left[\frac{n_j}{2}\right]}\frac{2^{n_j-2m_j}(-1)^{m_j}\left(x_j\xi_j\right)^{n_j-2m_j}\left(\xi_j\right)^{m_j}}{(n_j-2m_j)!m_j!}t_j^{n_j}\right\}\mathbf{d}\boldsymbol{\xi}
\end{aligned}$$

where $\mathbf{d}\boldsymbol{\xi}=d\xi_1...d\xi_r$. By identification of $t_1^{n_1}...t_r^{n_r}$, we obtain the desired. □

As a result of Theorem 4.2, if we get $\nu_i=1/2$, $(i=1,2,...,r)$, we give the next integral representation for the multivariable Legendre polynomials $P_{n_1,...,n_r}(x_1,...,x_r)$:

Corollary 4.3. *For the multivariable Legendre polynomials, we get*

$$\begin{aligned}
P_{n_1,...,n_r}(x_1,...,x_r) &= \frac{1}{\pi^{r/2}}\int_0^\infty\int_0^\infty...\int_0^\infty\xi_1^{-1/2}...\xi_r^{-1/2}e^{-(\xi_1+...+\xi_r)}\\
&\quad\times S(n_1,...,n_r)d\xi_1...d\xi_r
\end{aligned}$$

where

$$S(n_1,\ldots,n_r)=\prod_{j=1}^{r}\sum_{m_j=0}^{\left[\frac{n_j}{2}\right]}\frac{2^{n_j-2m_j}(-1)^{m_j}\left(x_j\xi_j\right)^{n_j-2m_j}\left(\xi_j\right)^{m_j}}{(n_j-2m_j)!m_j!}.$$

5. Multilinear and Multilateral Generating Functions

In recent years by making use of the familiar group-theoretic (Lie algebraic) method a certain mixed trilateral finite-series relationships have been proved for orthogonal polynomials (see, for instance, [10]). In this section, we derive several families of multilinear and multilateral generating functions for the MHP and MGP without using Lie algebraic techniques but, with the help of the similar method as considered in [3],[7],[8].

We begin by stating the following theorem.

Theorem 5.1. *Corresponding to an identically non-vanishing function* $\Omega_\mu(y_1,\ldots,y_s)$ *of* s *complex variables* $y_1,\ldots,y_s$ $(s\in\mathbb{N})$ *and of complex order* μ*, let*

$$\Lambda_{\mu,\psi}(y_1,\ldots,y_s;z):=\sum_{k=0}^{\infty}a_k\Omega_{\mu+\psi k}(y_1,\ldots,y_s)z^k \tag{5.1}$$

$$(a_k\neq 0,\ \ \mu,\psi\in\mathbb{C})$$

and

$$\Theta_{n,p,\mu,\psi}\left(x_1,\ldots,x_r;y_1,\ldots,y_s;\tau\right)\ :=\sum_{k=0}^{[n/p]}a_k\ Y_{n-pk+m_1+\ldots+m_r}\left(x_1,\ldots,x_r\right)\times\Omega_{\mu+\psi k}(y_1,\ldots,y_s)\tau^k \tag{5.2}$$

where $n,p\in\mathbb{N}$*. Then we have*

$$\sum_{n=0}^{\infty}\Theta_{n,p,\mu,\psi}\left(x_1,\ldots,x_r;y_1,\ldots,y_s;\frac{\eta}{t^p}\right)t^n = \left\{\prod_{i=1}^{r}\exp\left(2x_it-t^2\right)\right\}H_{\mathbf{m}}\left(\mathbf{x}-t\right)\Lambda_{\mu,\psi}(y_1,\ldots,y_s;\eta) \tag{5.3}$$

provided that each member of (5.3) exists.

Proof. For convenience, let S denote the first member of the assertion (5.3). Then, upon substituting for the polynomials

$$\Theta_{n,p,\mu,\psi}\left(x_1,\ldots,x_r;y_1,\ldots,y_s;\frac{\eta}{t^p}\right)$$

from the definition (5.2) into the left-hand side of (5.3), we obtain

$$S=\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]}a_k\ Y_{n-pk+m_1+\ldots+m_r}\left(x_1,\ldots,x_r\right)\Omega_{\mu+\psi k}(y_1,\ldots,y_s)\eta^k t^{n-pk}.$$

Replacing n by $n+pk$ and then using (3.6), we may write

$$\begin{aligned} S &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_k \; Y_{n+m_1+\ldots+m_r}\left(x_1,\ldots,x_r\right)\Omega_{\mu+\psi k}(y_1,\ldots,y_s)\eta^k t^n \\ &= \sum_{n=0}^{\infty} Y_{n+m_1+\ldots+m_r}\left(x_1,\ldots,x_r\right)t^n \sum_{k=0}^{\infty} a_k\Omega_{\mu+\psi k}(y_1,\ldots,y_s)\eta^k \\ &= \left\{\prod_{i=1}^{r}\exp\left(2x_i t - t^2\right)\right\} H_{\mathbf{m}}\left(\mathbf{x}-t\right)\Lambda_{\mu,\psi}(y_1,\ldots,y_s;\eta) \end{aligned}$$

which completes the proof. □

In a similar manner, by appealing to the formulas (3.5) and (3.7), we are led fairly easily to the following theorems, respectively.

Theorem 5.2. *Corresponding to an identically non-vanishing function $\Omega_\mu(y_1,\ldots,y_s)$ of complex variables $y_1,\ldots,y_s$ $(s\in\mathbb{N})$ and of complex order μ, let*

$$\Lambda_{\mu,v}(y_1,\ldots,y_s;z) := \sum_{k=0}^{\infty} a_k\Omega_{\mu+vk}(y_1,\ldots,y_s)z^k$$

where $a_k \neq 0$, $\mu, v\in\mathbb{C}$. Then, we have

$$\begin{aligned} &\sum_{n_1=0}^{\infty}\sum_{n_2,\ldots,n_r=0}^{\infty}\sum_{k=0}^{[n_1/p]} \frac{a_k H_{n_1-pk,n_2,\ldots,n_r}\left(\mathbf{x}\right)}{(n_1-pk)!n_2!\ldots n_r!} t_2^{n_2}\ldots t_r^{n_r} \\ &\times\Omega_{\mu+vk}(y_1,\ldots,y_s)\eta^k t_1^{n_1-pk} \\ =\; &\Lambda_{\mu,v}(y_1,\ldots,y_s;\eta)\prod_{i=1}^{r}\exp\left(2x_i t_i - t_i^2\right) \end{aligned} \tag{5.4}$$

provided that each member of (5.4) exists.

Theorem 5.3. *Corresponding to an identically non-vanishing function $\Omega_\mu(y_1,\ldots,y_s)$ of complex variables $y_1,\ldots,y_s$ $(s\in\mathbb{N})$ and of complex order μ, let*

$$\Lambda_{\mu,v}(y_1,\ldots,y_s;z) := \sum_{k=0}^{\infty} a_k\Omega_{\mu+vk}(y_1,\ldots,y_s)z^k$$

where $a_k \neq 0\,,\ \mu, v \in \mathbb{C}$. *Then, we have*

$$\begin{aligned} &\sum_{n_1,n_2,\ldots,n_r=0}^{\infty} \sum_{k=0}^{[n_1/p]} a_k C^{(\nu_1,\ldots,\nu_r)}_{n_1-pk,n_2,\ldots,n_r}(\mathbf{x})\, t_2^{n_2}\ldots t_r^{n_r} \\ &\times \Omega_{\mu+vk}(y_1,\ldots,y_s)\eta^k t_1^{n_1-pk} \\ = \;& \Lambda_{\mu,v}(y_1,\ldots,y_s;\eta) \prod_{i=1}^{r} \left(1-2x_i t_i + t_i^2\right)^{-\nu_i} \end{aligned} \tag{5.5}$$

provided that each member (5.5) of exists.

6. Further Consequences

We can give many applications of our theorems obtained in the previous sections with help of appropriate choices of the multivariable functions

$$\Omega_{\mu+\psi k}(y_1,\ldots,y_s)\ (k \in \mathbb{N}_0\,,\ s \in \mathbb{N})$$

in terms of simpler function of one and more variables. For example, if we set

$$s = r \ \text{ and } \ \Omega_{\mu+\psi k}(y_1,\ldots,y_r) = h^{(\gamma_1,\ldots,\gamma_r)}_{\mu+\psi k}(y_1,\ldots,y_r)$$

in Theorem 5.1, where the multivariable Lagrange-Hermite polynomials (see [2])

$$h_n^{(\alpha_1,\ldots,\alpha_r)}(x_1,\ldots,x_r)$$

are generated by

$$\prod_{j=1}^{r} \left\{ \left(1 - x_j t^j\right)^{-\alpha_j} \right\} = \sum_{n=0}^{\infty} h_n^{(\alpha_1,\ldots,\alpha_r)}(\mathbf{x})\, t^n \tag{6.1}$$

where $|t| < \min\left\{ |x_1|^{-1}, |x_2|^{-1/2}, \ldots, |x_r|^{-1/r} \right\}$. Then, we obtain the following result which provides a class of bilateral generating functions for the multivariable Lagrange-Hermite polynomials and multivariable Hermite polynomials(MHP).

Corollary 6.1. *If* $\Lambda_{\mu,\psi}(y_1,\ldots,y_r;z) := \sum_{k=0}^{\infty} a_k h^{(\gamma_1,\ldots,\gamma_r)}_{\mu+\psi k}(y_1,\ldots,y_r) z^k$ $(a_k \neq 0\,,\ \psi, \mu \in \mathbb{C})$ *and*

$$\begin{aligned} \Theta_{n,p,\mu,\psi}(x_1,\ldots,x_r;y_1,\ldots,y_r;\tau) \ : &= \sum_{k=0}^{[n/p]} a_k\ Y_{n-pk+m_1+\ldots+m_r}(x_1,\ldots,x_r) \\ &\times h^{(\gamma_1,\ldots,\gamma_r)}_{\mu+\psi k}(y_1,\ldots,y_r)\tau^k \end{aligned}$$

where $n, p \in \mathbb{N}$, then we have

$$\sum_{n=0}^{\infty} \Theta_{n,p,\mu,\psi}\left(x_1, ..., x_r; y_1, ..., y_r; \frac{\eta}{t^p}\right) t^n \tag{6.2}$$

$$= \left\{\prod_{i=1}^{r} \exp\left(2x_i t - t^2\right)\right\} H_{\mathbf{m}}\left(\mathbf{x}-t\right) \Lambda_{\mu,\psi}(y_1, ..., y_r; \eta)$$

provided that each member of (6.2) exists.

Remark 6.1. *Using the generating function (6.1) for the multivariable Lagrange-Hermite polynomials and taking $a_k = 1$, $\mu = 0$, $\psi = 1$, we have*

$$\sum_{n=0}^{\infty} \sum_{k=0}^{[n/p]} Y_{n-pk+m_1+...+m_r}\left(x_1, ..., x_r\right) h_k^{(\gamma_1, ..., \gamma_r)}(y_1, ..., y_r)\eta^k t^{n-pk}$$

$$= \left\{\prod_{i=1}^{r} \exp\left(2x_i t - t^2\right) \left\{\left(1 - y_i \eta^i\right)^{-\gamma_i}\right\}\right\} H_{\mathbf{m}}\left(\mathbf{x}-t\right)$$

where

$$\left(|\eta| < \min\left\{|y_1|^{-1}, |y_2|^{-1/2}, ..., |y_r|^{-1/r}\right\}\right).$$

Also, choosing $s = r$ and $\Omega_{\mu+\psi k}(y_1, ..., y_r) = Y_{\mu+\psi k}(y_1, ..., y_r)$, $\mu, \psi \in \mathbb{N}_0$, in Theorem 5.1 we obtain the following class of bilinear generating functions for MHP.

Corollary 6.2. *If $\Lambda_{\mu,\psi}(y_1, ..., y_r; z) := \sum_{k=0}^{\infty} a_k Y_{\mu+\psi k}(y_1, ..., y_r) z^k$ where $a_k \neq 0$, $\mu, \psi \in \mathbb{C}$; and*

$$\Theta_{n,p,\mu,\psi}\left(x_1, ..., x_r; y_1, ..., y_r; \tau\right) \; := \sum_{k=0}^{[n/p]} a_k \; Y_{n-pk+m_1+...+m_r}\left(x_1, ..., x_r\right) \times Y_{\mu+\psi k}(y_1, ..., y_r)\tau^k$$

where $n, p \in \mathbb{N}$, then we have

$$\sum_{n=0}^{\infty} \Theta_{n,p,\mu,\psi}\left(x_1, ..., x_r; y_1, ..., y_r; \frac{\eta}{t^p}\right) t^n$$

$$= \left\{\prod_{i=1}^{r} \exp\left(2x_i t - t^2\right)\right\} H_{\mathbf{m}}\left(\mathbf{x}-t\right) \Lambda_{\mu,\psi}(y_1, ..., y_r; \eta) \tag{6.3}$$

provided that each member of (6.3) exists.

In particular, if we set

$$s = r \text{ and } \; \Omega_{\mu+\upsilon k}(y_1, ..., y_r) = u_{\mu+\upsilon k}^{(\nu_1, ..., \nu_r)}\left(y_1, ..., y_r\right)$$

in Theorem 5.2, where the polynomials $u_n^{(\nu_1,...,\nu_r)}(x_1,...,x_r)$ are generated by (3.9), then we obtain the following result which provides a class of bilateral generating functions for the multivariable Gegenbauer polynomials (MGP) and MHP.

Corollary 6.3. *If* $\Lambda_{\mu,v}(y_1,...,y_r;z) := \sum\limits_{k=0}^{\infty} a_k u_{\mu+vk}^{(\nu_1,...,\nu_r)}(y_1,...,y_r) z^k$,

$(a_k \neq 0, \mu, v \in \mathbb{N}_0)$. Then we have

$$\begin{aligned} & \sum_{n_1=0}^{\infty} \sum_{n_2,...,n_r=0}^{\infty} \sum_{k=0}^{[n_1/p]} \frac{a_k H_{n_1-pk,n_2,...,n_r}(\mathbf{x})}{(n_1-pk)!n_2!...n_r!} t_2^{n_2}...t_r^{n_r} \\ & \times u_{\mu+vk}^{(\nu_1,...,\nu_r)}(y_1,...,y_r)\eta^k t_1^{n_1-pk} \\ = \ & \Lambda_{\mu,v}(y_1,...,y_r;\eta)\prod_{i=1}^{r}\exp\left(2x_it_i-t_i^2\right) \end{aligned} \tag{6.4}$$

provided that each member of (6.4) exists.

Remark 6.2. *Using (3.9) and* $a_k = 1,\ \mu = 0,\ v = 1,$ *we have*

$$\begin{aligned} & \sum_{n_1=0}^{\infty} \sum_{n_2,...,n_r=0}^{\infty} \sum_{k=0}^{[n_1/p]} \frac{H_{n_1-pk,n_2,...,n_r}(\mathbf{x})}{(n_1-pk)!n_2!...n_r!} t_2^{n_2}...t_r^{n_r} u_k^{(\nu_1,...,\nu_r)}(y_1,...,y_r) \\ & \times \eta^k t_1^{n_1-pk} \\ = \ & \prod_{i=1}^{r} F_1\left[\lambda_i,\nu_i,\nu_i;\mu_i;\left(y_i+\sqrt{y_i^2-1}\right)\eta,\left(y_i-\sqrt{y_i^2-1}\right)\eta\right] \\ & \times \exp\left(2x_it_i-t_i^2\right) \end{aligned}$$

where

$$\begin{aligned} & u_k^{(\nu_1,...,\nu_r)}(y_1,...,y_r) \\ = \ & \sum_{k_1=0}^{k}\sum_{k_2=0}^{k-k_1}\cdots\sum_{k_{r-1}=0}^{k-k_1-...-k_{r-2}} C_{k-(k_1+...+k_{r-1}),k_1,...,k_{r-1}}^{(\nu_1,...,\nu_r)}(y_1,...,y_r) \\ & \times\ \delta_k(k_1,...,k_{r-1}) \end{aligned}$$

and also

$$\delta_k(k_1,...,k_{r-1}) = \frac{(\lambda_1)_{k-(k_1+...+k_{r-1})}(\lambda_2)_{k_1}\cdots(\lambda_r)_{k_{r-1}}}{(\mu_1)_{k-(k_1+...+k_{r-1})}(\mu_2)_{k_1}\cdots(\mu_r)_{k_{r-1}}}.$$

Furthermore, for every suitable choice of the coefficients a_k $(k \in \mathbb{N}_0)$, if the multivariable function $\Omega_{\mu+\psi k}(y_1,...,y_s)$ $(s \in \mathbb{N})$ is expressed as an appropriate product of several simpler functions, the assertions of Theorems 5.1, 5.2 and 5.3 can be applied in order to derive various families of multilinear and multilateral generating functions for MHP and MGP.

REFERENCES

[1] A. Altın, R. Aktaş and E. Erkuş-Duman, On a multivariable extension for the extended jacobi polynomials, *J. Math. Anal. Appl.* 353 (2009), 121-133.

[2] A. Altın and E. Erkuş, On a multivariable extension of the Lagrange-Hermite polynomials, *Integral Transforms Spec. Funct.* 17 (2006), 239-244.

[3] A. Altın and E. Erkuş and M. A. Özarslan, Families of linear generating functions for polynomials in two variables, *Integral Transforms Spec. Funct.* 17 (2006), no. 5, 315 320.

[4] W.-C. C. Chan, C.-J. Chyan and H. M. Srivastava, The Lagrange polynomials in several variables, *Integral Transforms Spec. Funct.* 12 (2001), 139-148.

[5] C.F. Dunkl and Y. Xu, Orthogonal polynomials of several variables, Cambridge univ. press, New York, 2001.

[6] I. Fujiwara, A unified presentation of classical orthogonal polynomials, *Math. Japon.* 11 (1966), 133-148.

[7] S.-D. Lin, I.-C. Chen, H.M. Srivastava, Certain classes of finite-series relationships and generating functions involving the generalized Bessel polynomials, Appl. Math. Comput. 137 (2003), 261 275.

[8] M. A. Özarslan, A. Altın, Some families of generating functions for the multiple orthogonal polynomials associated with modified Bessel K-functions, J. Math. Anal. Appl. 297 (2004), no. 1, 186 193.

[9] E. D. Rainville, Special Functions, The Macmillan Company, New York, 1960.

[10] H. M. Srivastava and H. L. Manocha, A Treatise on Generating Functions, Halsted Press (Ellis Horwood Limited, Chichester)/ John Wiley and Sons, New York, 1984.

[11] G. Szegö, Orthogonal polynomials, fourth ed., Amer. Math. Soc. Colloq. Publ.,vol. 23, 1975.

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