L(2,1)-labeling of flower snark and related graphs *

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Abstract

An $L(2,1)$-labeling of a graph G is an assignment of nonnegative integers to the vertices of G such that adjacent vertices get numbers at least two apart, and vertices at distance two get distinct numbers. The $L(2,1)$-labeling number of G, $\lambda(G)$, is the minimum range of labels over all such labelings. In this paper, we determine the λ-numbers of flower snark and its related graphs for all $n \geq 3$.
Keywords: L(2,1)-labeling; L(2,1)-labeling number; Flower snark

1 Introduction

We consider only finite undirected graphs without loops or multiple edges. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$.

An $L(2,1)$-*labeling* of a graph G is a function f from the vertex set $V(G)$ to the set of all nonnegative integers such that $|f(u) - f(v)| \geq 2$ if

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$d(u,v) = 1$ and $|f(u) - f(v)| \geq 1$ if $d(u,v) = 2$, where $d(u,v)$ denotes the distance between vertex u and vertex v in G. The *span* of $L(2,1)$-labeling is the maximum difference between two labels. The minimum span of labels required for such an $L(2,1)$-labeling of G is called the $L(2,1)$-*labeling number* and denoted by $\lambda(G)$.

Griggs and Yeh [3] introduced the $L(2,1)$-labeling and conjectured that $\lambda(G) \leq \Delta^2$ for any graph G with maximum degree $\Delta \geq 2$. There are considerable articles studying the L(2,1)-labelings. Most of the papers are considering bounds or exact values of λ-numbers on particular classes of graphs and the conjecture has been proved to be true for some particular classes graphs (see references [1, 2, 5, 6, 9–16]). In the present paper, we will study the exact values of λ-numbers on a flower snark and its related graphs.

Let G_n be a simple nontrivial connected cubic graph with vertex set $V(G_n) = \{a_i, b_i, c_i, d_i : 0 \leq i \leq n-1\}$ and edge set $E(G_n) = \{a_i a_{i+1}, b_i b_{i+1}, c_i c_{i+1}, a_i d_i, b_i d_i, c_i d_i \ : \ 0 \leq i \leq n-1\}$, where the vertex labels are read modulo n. Let H_n be a graph obtained from G_n by replacing the edges $b_{n-1}b_0$ and $a_{n-1}a_0$ with $b_{n-1}a_0$ and $a_{n-1}b_0$ respectively. For odd $n \geq 5$, H_n is called a *flower snark*. G_n and H_n ($n = 3$ or even $n \geq 4$) are related graphs of flower snarks.

Figure 1.1 shows the flower snark H_5 and its related graph G_5.

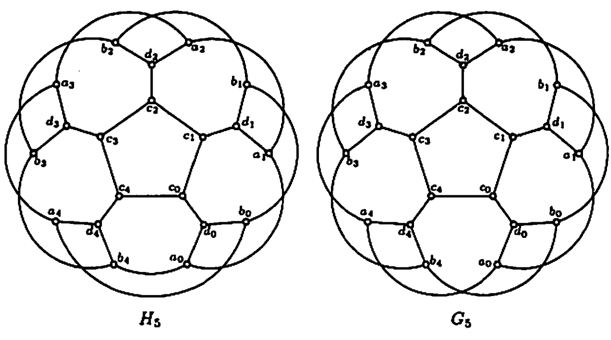

Figure 1.1. The flower snark H_5 and its related graph G_5

The *flower snark*, defined by Isaacs [4], is one of the most famous cubic graphs that have been studied extensively in recent years. Mohammad et al [7] determined the circular chromatic index of flower snarks. Mominul et al [8] showed the prime cordial labeling of flower snarks and its related graphs. Tong et al [17] determined the $(d,1)$-total numbers of flower snarks and related graphs. Zheng et al [19] studied the crossing number of the flower snark and its related graphs. Xi et al [18] proved that flower snarks and their related graphs are super vertex-magic. In this paper, we determine the exact values of $\lambda(H_n)$ and $\lambda(G_n)$ for all $n \geq 3$.

2 Basic lemmas

Let M be a subgraph of $G \in \{H_n, G_n\}$ with the vertex set $V(M) = \{a_i, b_i, c_i, d_i, 0 \leq i \leq 2\}$ and edge set $E(M) = \{a_i d_i, b_i d_i, c_i d_i, 0 \leq i \leq 2\} \cup \{a_i a_{i+1}, b_i b_{i+1}, c_i c_{i+1}, 0 \leq i \leq 1\}$.

Lemma 2.1. $\lambda(M) \geq 6$.
Proof. Suppose on the contrary that f be a $L(2,1)$-labeling of M in {0,1,2,3,4,5}. According to the duality of L(2,1)-labeling, the cases $f(d_1) = 4, 3, 5$ fellow from $f(d_1) = 1, 2, 0$. We need only consider three cases as follows:

Case 1. $f(d_1) = 1$. Then $\{f(a_1), f(b_1), f(c_1)\} = \{3,4,5\}$. By the symmetry of a, b and c, say $(f(a_1), f(b_1), f(c_1)) = (3,4,5)$. Then $\{f(a_0), f(a_2)\} = \{0,5\}$ and $\{f(b_0), f(b_2)\} = \{0,2\}$. By the symmetry of i ($i = 0$ and i=2 are symmetrical), say $(f(a_0), f(a_2)) = (0,5)$. Then $(f(b_0), f(b_2)) = (2,0)$. It follows that $f(d_0) \in \{4,5\}$ (see figure 2.1(1)), a contradiction to $|f(d_0) - f(b_1)| \geq 1$ or $|f(d_0) - f(c_1)| \geq 1$.

Case 2. $f(d_1) = 2$. Then $\{f(a_1), f(b_1), f(c_1)\} = \{0,4,5\}$. By symmetry, say $(f(a_1), f(b_1), f(c_1)) = (0,4,5)$. Then $\{f(b_0), f(b_2)\} = \{0,1\}$. Say $(f(b_0), f(b_2)) = (0,1)$. Then $f(d_2) = 3$. It follows that $f(a_2) = 5$, $f(a_0) \in \{3,4\}$. If $f(a_0) = 3$, then $f(d_0) = 5$ (see figure 2.1(2)) , a contradiction to $|f(d_0) - f(c_1)| \geq 1$. If $f(a_0) = 4$, then $f(d_0) = 2$ and $f(c_0) = 5$ (see figure 2.1(2)), a contradiction to $|f(c_0) - f(c_1)| \geq 2$.

Case 3. $f(d_1) = 0$.
Case 3.1. $\{2,3\} \subset \{f(a_1), f(b_1), f(c_1)\}$. By symmetry, say $(f(a_1), f(b_1)) = (2,3)$. Then $\{f(a_0), f(a_2)\} = \{4,5\}$. Say $(f(a_0), f(a_2)) = (4,5)$. Then $f(b_2) = 1$. It follows $f(d_2)$=3 (see figure 2.1(3)), a contradiction to $|f(d_2) - f(b_1)| \geq 1$.

Case 3.2. $\{4,5\} \subset \{f(a_1), f(b_1), f(c_1)\}$. By symmetry, say $(f(a_1), f(b_1)) = (4,5)$. Then $\{f(a_0), f(a_2)\} = \{1,2\}$. Say $(f(a_0), f(a_2)) = (1,2)$. Then $f(d_0) = 3$. It follows $f(b_0) = 0$ (see figure 2.1(4)), a contradiction to $|f(b_0) - f(d_1)| \geq 1$. □

Corollary 2.2. $\lambda(H_n) \geq \lambda(M) \geq 6$, $\lambda(G_n) \geq \lambda(M) \geq 6$.

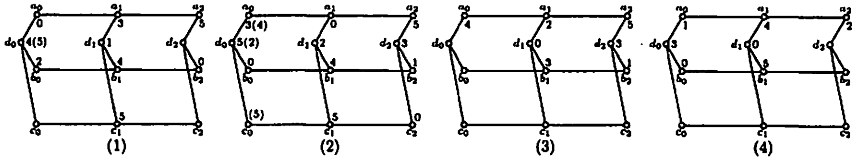

Figure 2.1. Cases of L(2,1)-labeling on M

3 $L(2,1)$-labeling of H_n and G_n

Lemma 3.1. $\lambda(H_3) = 7$.
Proof. Figure 3.1(1) shows a $L(2,1)$-labeling of H_3 in $[0, 7]$. Hence we have $\lambda(H_3) \leq 7$. Now, we prove $\lambda(H_3) \geq 7$. Suppose on the contrary that f be a $L(2,1)$-labeling of H_3 in $[0, 6]$. Let $(f(a_0), f(a_1), f(a_2), f(b_0), f(b_1), f(b_2), f(d_0), f(d_1), f(d_2)) = (x_0, x_1, x_2, x_3, x_4, x_5, y_0, y_1, y_2)$. Since the vertices labeled x induce a subgraph of diameter 2, and each vertex labeled x is at distance at most 2 of a vetex labeld y, then $x_i \neq x_j (0 \leq i < j \leq 5)$ and $x_i \neq y_j (0 \leq i \leq 5, 0 \leq j \leq 2)$. It follows $y_0 = y_1 = y_2 = y$ and one of $\{y-1, y+1\}$ has to be equal to one of $\{x_0, x_1, x_2, x_3, x_4, x_5\}$, i.e. $|y - x_i| = 1$ $(0 \leq i \leq 5)$, a contradiction to $|y - x_i| \geq 2$ (see figure 3.1(2)). □

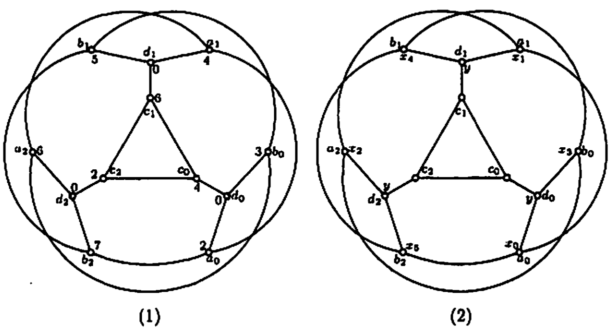

Figure 3.1. The L(2,1)-labeling of H_3

Theorem 3.2. For $n \geq 4$, $\lambda(H_n) = 6$.
Proof. Let f be a function as follows:
Case1. $n \equiv 0 \pmod 3$.

$$f(a_i) = \begin{cases} 2(i \bmod 3) + 2, & i \leq n-3, \\ 5, & i = n-2, \\ 3, & i = n-1, \end{cases}$$
$$f(b_i) = \begin{cases} 2((i+2) \bmod 3) + 2, & i \leq n-3, \\ 3, & i = n-2, \\ 5, & i = n-1, \end{cases}$$

$$\begin{aligned} f(c_i) &= 2((i+1) \bmod 3)+2, \\ f(d_i) &= 0, \end{aligned}$$

Case 2. $n \equiv 1 \pmod{3}$.

$$\begin{aligned} f(a_i) &= 2(i \bmod 3)+2, \\ f(b_i) &= \begin{cases} 2((i+1) \bmod 3)+2, & i \le n-3, \\ 3, & i = n-2, \\ 5, & i = n-1, \end{cases} \\ f(c_i) &= \begin{cases} 2((i+2) \bmod 3)+2, & i \le n-3, \\ 5, & i = n-2, \\ 3, & i = n-1, \end{cases} \\ f(d_i) &= 0, \end{aligned}$$

Case 3. $n \equiv 2 \pmod{3}$.

$$\begin{aligned} f(a_i) &= 6-2((i+2) \bmod 3), \\ f(b_i) &= \begin{cases} 6-2((i+1) \bmod 3), & i \le n-3, \\ 3, & i = n-2, \\ 5, & i = n-1, \end{cases} \\ f(c_i) &= \begin{cases} 6-2(i \bmod 3), & i \le n-3, \\ 5, & i = n-2, \\ 3, & i = n-1, \end{cases} \\ f(d_i) &= 0. \end{aligned}$$

The labeling f has the required properties of a L(2,1)-labeling for $n \ge 4$. We then have $\lambda(H_n) \le 6$ for $n \ge 4$. However, by Corollary 2.2, $\lambda(H_n) \ge 6$. This concludes the proof. □

Figure 3.2 shows L(2,1)-labelings of H_5, H_6 and H_7.

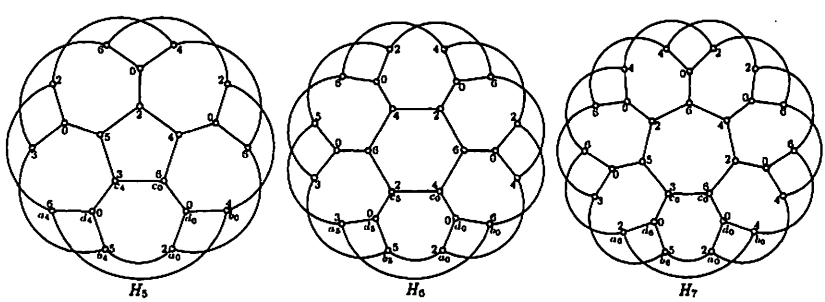

Figure 3.2. L(2,1)-labelings of H_5, H_6 and H_7

Theorem 3.3. For $n \ge 3$, $\lambda(G_n) = 6$.
Proof. Let f be a function as follows:
Case 1. $n \equiv 0 \pmod{3}$.

$$\begin{aligned} f(a_i) &= 2(i \bmod 3)+2, \\ f(b_i) &= 2((i+1) \bmod 3)+2, \\ f(c_i) &= 2((i+2) \bmod 3)+2, \\ f(d_i) &= 0, \end{aligned}$$

Case 2. $n \equiv 1 \pmod 3$.

$$f(a_i) = \begin{cases} 2(i \bmod 3) + 2, & i \leq n-4 \\ 5, & i = n-3, \\ 3, & i = n-2, \\ 6, & i = n-1, \end{cases}$$

$$f(b_i) = \begin{cases} 2((i+2) \bmod 3) + 2, & i \leq n-3, \\ 5, & i = n-2, \\ 3, & i = n-1, \end{cases}$$

$$f(c_i) = \begin{cases} 2((i+1) \bmod 3) + 2, & i \leq n-5, \\ 5, & i = n-4, \\ 3, & i = n-3, \\ 6, & i = n-2, \\ 2, & i = n-1, \end{cases}$$

$$f(d_i) = 0,$$

Case 3. $n \equiv 2 \pmod 3$.

$$f(a_i) = \begin{cases} 2(i \bmod 3) + 2, & i \leq n-3, \\ 3, & i = n-2, \\ 5, & i = n-1, \end{cases}$$

$$f(b_i) = \begin{cases} 2((i+1) \bmod 3) + 2, & i \leq n-4, \\ 3, & i = n-3, \\ 5, & i = n-2, \\ 2, & i = n-1, \end{cases}$$

$$f(c_i) = \begin{cases} 2((i+2) \bmod 3) + 2, & i \leq n-5, \\ 3, & i = n-4, \\ 5, & i = n-3, \\ 2, & i = n-2, \\ 4, & i = n-1, \end{cases}$$

$$f(d_i) = 0.$$

The labeling f has the required properties of a $L(2,1)$-labeling. We then have $\lambda(G_n) \leq 6$. However, by Corollary 2.2, $\lambda(G_n) \geq 6$. This concludes the proof. □

Figure 3.3 shows L(2,1)-labelings of G_5, G_6 and G_7.

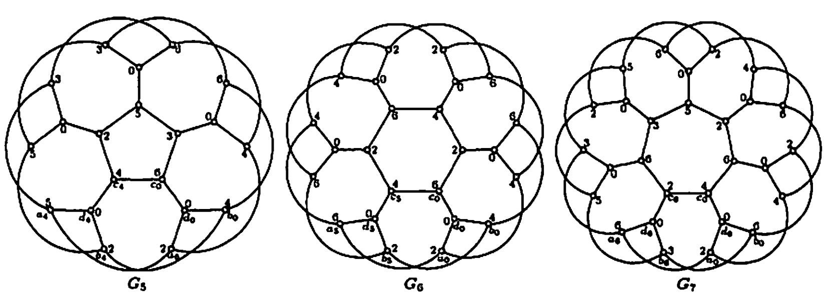

Figure 3.3. L(2,1)-labelings of G_5, G_6 and G_7

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