# BICIRCULAR MATROID DESIGNS

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ABSTRACT. Murty characterized the connected binary matroids with all circuits having the same size. Here we characterize the connected bicircular matroids with all circuits having the same size.

## 1. INTRODUCTION

Young [10] reports that Murty [5] was the first to study matroids with all hyperplanes having the same size. Murty called such a matroid an "Equicardinal Matroid". Young renamed such a matroid a "Matroid Design". Further work on determining properties of these matroids was done by Edmonds, Murty, and Young [6, 11, 12]. These authors were able to connect the problem of determining the matroid designs with specified parameters with results on balanced incomplete block designs. The dual of a matroid design is one in which all circuits have the same size. Murty [6] restricted his attention to binary matroids and was able to characterize all connected binary matroids having circuits of a single size. Lemos, Reid, and Wu [3] provided partial information on the class of connected binary matroids having circuits of two different sizes. They solved this problem when the largest of the two circuit sizes is odd and they also showed that there are many such matroids when the largest of the two circuit sizes is even. In general, there are not many results that specify the matroids with circuits of just a few different sizes. Cordovil, Lemos, and Maia [1, 4] provided such results on matroids with small circumference. Here we determine the connected bicircular matroids with all circuits having the same size. The bicircular matroids considered are in general non-binary. Hence these results are a start on extending Murty's characterization of binary matroid designs to non-binary matroids.

Some terminology is given next before the statement of Murty's seminal result on matroid designs. The *circuit-spectrum* of a matroid $M$ is $\text{spec}(M) = \{|C| : C \in \mathcal{C}(M)\}$. A *k-subdivision* of a matroid is obtained by replacing each element by a series class of size $k$. A *connected* matroid is one in which each pair of distinct elements is contained in some circuit. We use $PG(r,2)$ and $AG(r,2)$, respectively, to denote the binary projective and affine geometries of rank $r+1$. This terminology used here mostly follows [8].

**Theorem 1.1.** *Let $M$ be a connected binary matroid. For $\eta \in \mathbb{Z}^+$, $\mathrm{spec}(M) = \{\eta\}$ if and only if $M$ is isomorphic to one of the following matroids:*

(i) *an $\eta$-subdivision of $U_{0,1}$,*
(ii) *a $k$-subdivision of $U_{1,n}$, where $\eta = 2k$ and $n \geq 3$,*
(iii) *an $l$-subdivision of $PG(r,2)^*$, where $\eta = 2^r l$ and $r \geq 2$,*
(iv) *an $l$-subdivision of $AG(r+1,2)^*$, where $\eta = 2^r l$ and $r \geq 2$.*

In Theorem 1.2 we provide a generalization of Theorem 1.1 to bicircular matroids. This is a class of matroids which are in general non-binary. The proof of this result is given in Section 3 of the paper.

**Theorem 1.2.** *Let $M$ be a connected bicircular matroid. For $\eta \geq 2$, $\mathrm{spec}(M) = \{\eta\}$ if and only if $M$ is isomorphic to one of the following matroids:*

(i) *a $k$-subdivision of $U_{1,n}$ where $\eta = 2k$ and $n \geq 2$,*
(ii) *a $k$-subdivision of $U_{2,n}$ where $\eta = 3k$ and $n \geq 3$,*
(iii) *a $k$-subdivision of $U_{3,5}$ or $U_{3,6}$ where $\eta = 4k$,*
(iv) *a $k$-subdivision of $U_{4,6}$ where $\eta = 5k$.*

A uniform matroid has circuits of at most one size. Hence we obtain the following immediate corollary of the above result.

**Corollary 1.3.** *Let $M$ be a connected bicircular matroid with at least 2 elements. If $M$ is uniform, then it is one of the five types listed in Theorem 1.2*

## 2. Bicircular Matroids

We review the definition and some properties of bicircular matroids in this section of the paper. Let $G$ be a graph on edge set $E$. The bicircular matroid of $G$, denoted by $B(G)$, has ground set $E$ and circuits being the edge sets of a subdivision of one of the following three graphs: (i) two loops that share a vertex, (ii) two loops with distinct vertices that are joined by an edge, (iii) three edges joining the same pair of vertices. The circuits of $B(G)$ are called the *bicycles* of $G$. A bicycle of type (i), (ii), and (iii) is referred to as a bow-tie, a barbell, or a theta, respectively (see Figure 1 for some examples). Graphs whose bicircular matroids are isomorphic to the matroids $U_{1,n}$ and $U_{2,n}$, for $n \geq 2$, are given in Figure 2. Note that two non-isomorphic graphs in that figure have the same bicircular matroid $U_{4,6}$. Coullard, del Greco, and Wagner [2] determined precisely when this phenomenon can occur using certain graph operations (see also [7]). We next describe two of these operations that will be of particular interest here after first giving some graph terminology.

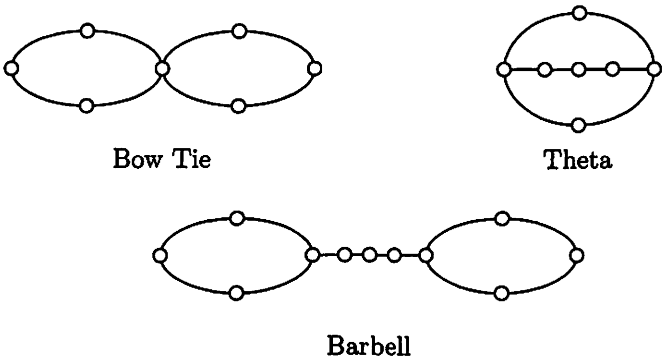


FIGURE 1. Bicycles

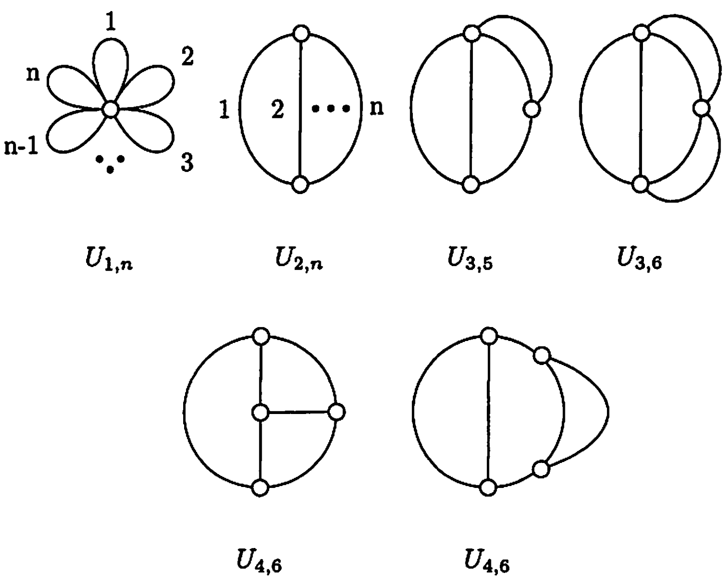


FIGURE 2. Some Bicircular Matroids

Let $G$ be a graph with $F$ a non-empty proper subset of the edge set $E$. The *vertex-boundary* of $F$ consists of those vertices of $G$ that are in both of the subgraphs induced by $F$ and by $E-F$. A *block* is a maximal connected subgraph without a cutvertex. An *end-block* of $G$ is a block whose vertex-boundary contains exactly one vertex. A *balloon* of $G$ is subgraph of $G$

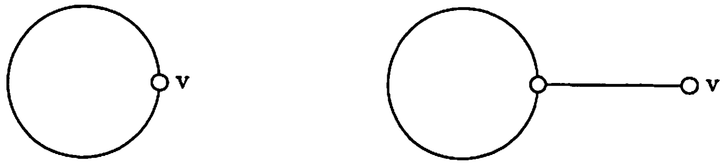


FIGURE 3. Balloons

which is a subdivision one of the two graphs of Figure 3, whose vertex-boundary contains exactly one vertex. The vertex boundary (the vertex $v$ pictured there) is called the *tip* of the balloon.

A *line* of $G$ is a set of edges that forms a path with the internal vertices having degree two and two end-vertices having degree at least three. We further require that the line is not contained in any balloon. Now let $L$ be a line of $G$ with end vertices $u$ and $v$ and $e$ be the edge of $L$ that is incident with $v$. Let $H$ be a graph obtained from $G$ by redefining the incidence relation of $e$ so that $e$ is adjacent to a vertex $w \neq v$ of $L$ instead of v. Then $H$ is said to be obtained from $G$ by *rolling* $L$ away from v. Likewise, $G$ is said to be obtained from $H$ by *unrolling* of $L$ to $v$. Note that $L$ is a balloon of $H$ (see, for example, the top right graph of Figure 4). Hence the operation of unrolling reduces the number of balloons of a graph. The following useful results can be found in [9] and [2] respectively.

**Lemma 2.1.** *Suppose that $G$ and $H$ are graphs with $B(H)$ connected and $H$ is obtained from $G$ by rolling a line $L$ away from a vertex $v$. Then $B(G) = B(H)$ if and only $v$ is the tip of an end-block of $G$ that contains $L$ and every cycle of the end-block contains $v$.*

**Lemma 2.2.** *If $H$ is a graph obtained from a graph $G$ replacing a balloon with another balloon on the same edge set and with the same vertex-boundary, then $B(G) = B(H)$.*

## 3. THE PROOF

We present the proof of Theorem 1.2 in this section of the paper after first giving some graph terminology. Let $G$ be a graph. When $X$ and $Y$ are subgraphs of $G$, an *$X$-$Y$ path* is a path which intersects each of $X$ and $Y$ in exactly one vertex. A path is said to be *internally disjoint* from a subgraph $X$ if it intersects $X$ only in its end vertices, if at all. Each *block* of $G$ is either a maximal 2-connected subgraph, a cut-edge (bridge) or cut-set of parallel edges, a loop, or an isolated vertex (i.e. a vertex with no incident edges). We will call a block a *t-block* if it is not a vertex, a single edge, or a cycle. Note that in any t-block, there must be some pair of vertices $\{u, v\}$ for which the block contains at least three internally disjoint $u$-$v$ paths. We

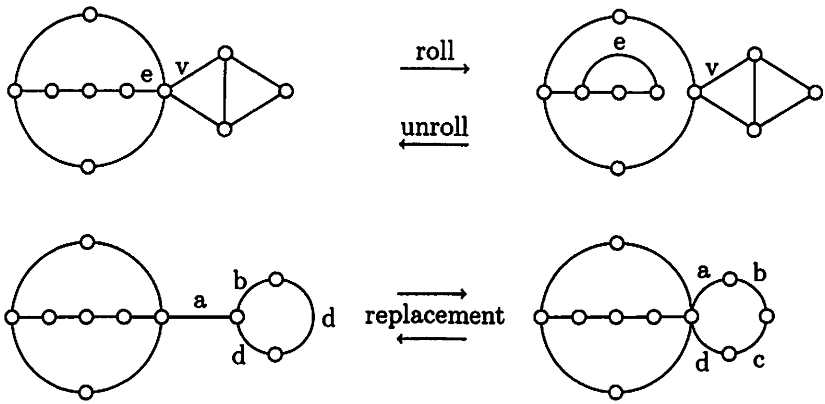


FIGURE 4. Rolling and Replacement

call any such pair a *branching pair* of the block. A set of internally disjoint $u$-$v$ paths is called a set of *arms* of the block. When $P$ is a path in a graph $G$, and $u, v \in V(P)$, we let $P[u, v]$ denote the subpath of $P$ between $u$ and $v$, inclusive. Let $P(u, v) := P[u, v] - \{u, v\}$, $P(u, v] := P[u, v] - u$, and $P[u, v) := P[u, v] - v$. We use similar notation to indicate subpaths in cycles. We will use the convention that an uppercase letter refers to a subgraph, while the corresponding lowercase letter refers to the number of edges in that subgraph. So where $P_1$ is a path, for example, $p_1$ is the number of edges in that path. A graph is said to be a *bundle of balloons* if its edge set can be partitioned into disjoint balloons whose vertex boundaries share a single common vertex (see, for example, the first graph of Figure 2).

*Proof of Theorem 1.2.* First note that if $M$ is isomorphic to one of the matroids listed in the theorem statement, then $\text{spec}(M) = \{\eta\}$. Conversely, let $G$ be a graph without isolated vertices whose bicircular matroid represents $M$ and suppose that $\text{spec}(M) = \{\eta\}$. We begin by showing that we may assume that $G$ satisfies the following conditions.

(1) $G$ is connected, with minimum degree at least two, and each balloon of $G$ is a cycle.
(2) $G$ includes at most one t-block.
(3) Any t-block of $G$ is isomorphic to a $p$-subdivision of one of the five loopless graphs in Figure 2.
(4) There is a block $B$ of $G$ whose vertex-boundary meets all blocks of $G$.
(5) If $G$ has no t-block, then it is a bundle of balloons.

*Proof of (1).* The matroid $M$ is connected with at least two elements so that each pair of edges of $G$ is contained in a bicycle. Thus $G$ is connected.

Suppose that $G$ has a vertex $v$ of degree one. Then the unique edge of $G$ that meets $v$ is in no bicycle; a contradiction. Hence the minimum degree of $G$ is at least two. It follows from Lemma 2.2 that each balloon $G$ may be replaced by a cycle with its tip being the unique vertex in its vertex-boundary. □

*Proof of (2).* It follows from Lemma 2.1 that we may assume that $G$ has the fewest number of balloons among all representations for $M$ that satisfy condition (1) (no unrolling of a balloon is possible). Suppose $G$ includes two t-blocks $B$ and $B'$. Let $u, v$ be a branching pair of $B$ with arms $P_1$, $P_2$, and $P_3$ and let $x, y$ be a branching pair of $B'$ with arms $Q_1$, $Q_2$, and $Q_3$. Since $G$ is connected there is some $B$-$B'$ path $R$. Without loss of generality, assume that $R$ intersects paths $P_1$ and $Q_1$. Consider the following bicycles of $G$.

$P_1 \cup P_2 \cup P_3 \qquad P_1 \cup P_2 \cup R \cup Q_1 \cup Q_2 \qquad Q_1 \cup Q_2 \cup Q_3 \qquad P_1 \cup P_3 \cup R \cup Q_1 \cup Q_3$

From the first two bicycles we obtain $p_3 = r + q_1 + q_2$, and from the last two bicycles we obtain

$$\begin{aligned} \eta &= p_1 + p_3 + r + q_1 + q_3 \\ &= p_1 + (r + q_1 + q_2) + r + q_1 + q_3 \\ &= q_1 + q_2 + q_3 + q_1 + p_1 + 2r \\ &= \eta + q_1 + p_1 + 2r \end{aligned}$$

Thus $q_1 + p_1 + 2r = 0$; a contradiction. Therefore $G$ includes at most one t-block. □

*Proof of (3).* Let $B$ be a t-block of $G$ with branch vertices $\{u, v\}$ and arms $B_1, \ldots B_n$.

When $n \geq 4$, any three arms of $B$ form a bicycle. By symmetry, each arm is of the same length, $\eta/3$. Suppose there is some path $P$ from $B_i$ to $B_j$ internally disjoint from the arms of $B$, where $P$ is not itself an arm. (We allow the possibility that $i = j$). Assume without loss that there is a vertex $w \in P \cap B_1$ with $w \notin \{u, v\}$, and that $j \in \{1, 2\}$. Then $B_1 \cup B_2 \cup P$ forms a bicycle, so $p = \eta/3$. But $B_1[w, u] \cup P \cup B_2 \cup B_3$ is contained in a bicycle of size strictly larger than $p + b_2 + b_3 = \eta$. Hence there can be no such path $P$. So if $B$ has at least 4 arms, then $B$ is exactly the union of those arms, and $B$ represents a $p$-subdivision of $U_{2,n}$.

We now assume $n = 3$. Suppose there is some path $P$ from $B_1$ to $B_2$ internally disjoint from the arms of $B$. Assume without loss that $P \cap B_1 = w \notin \{u, v\}$, and let $x = P \cap B_2$. (Figure 5)

If $x \notin \{u, v\}$, consider the 6 paths $B_1[u, w]$, $B_1[w, v]$, $B_2[u, x]$, $B_2[x, v]$, $B_3$, and $P$. Any 5 of these together will form a bicycle. By symmetry, each is of the the same size, $p$, and $\eta = 5p$. Note that since this argument applies

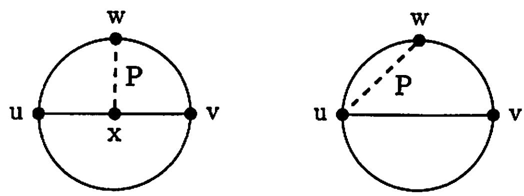


FIGURE 5

to any $B_i$ to $B_j$ paths, it precludes any such paths except those from $w$ to $x$. But if $Q$ is a $w, x$ path internally disjoint from the arms of $B$, and $q = p$, then $B_1[u,w] \cup B_2[u,x] \cup P \cup Q$ is a bicycle of size $4p < \eta$. So we find that no such $Q$ exists. Hence $B$ represents a $p$-subdivision of $U_{4,6}$.

Now suppose $x = u$. Consider the 5 paths $B_1[u,w]$, $B_1[w,v]$, $B_2$, $B_3$, $P$. Any 4 of these together form a bicycle, so each is of size $p$, and $\eta = 4p$. In this case, $B$ represents a $p$-subdivision of $U_{3,5}$. Now consider whether there may be another path $Q$ from $B_1$ to $B_2$ internally disjoint from the arms of $B$. If $Q$ is another $w, x$ path, then $B_1[u,w] \cup P \cup Q$ is a bicycle of size $3p$, a contradiction. If $Q$ is a path from $w$ to $v$, then by symmetry $q = p$ and $B$ represents a $p$-subdivision of $U_{3,6}$.

Finally, suppose there is a non-trivial path $P$ from $B_1$ to $B_1$ internally disjoint from the arms of $B$. Assume that $P \cap B_1 = \{w, x\}$ with $\{w,x\} \cap \{u,v\} = \emptyset$ and such that $R = B_1[u,w]$, $S = B_1[w,x]$, and $T = B_1[x,v]$ partition the edges of $B_1$. By symmetry we see that $b_2 = b_3$, that $p = s$, and that $r = t$. Consider each of the following bicycles of $G$.

$$B_1 \cup B_2 \cup B_3 \qquad B_2 \cup B_3 \cup R \cup S \cup P \qquad B_1 \cup B_2 \cup P$$

The first gives $\eta = b_1 + b_2 + b_3 = r + s + t + 2b_2 = 2r + s + 2b_2$. The second gives $\eta = b_2 + b_3 + r + s + p = r + 2s + 2b_2$. The third gives $\eta = b_1 + b_2 + p = r + s + t + b_2 + p = 2r + 2s + b_2$. Hence $r = s = b_2 (= t = b_1 = b_3 = p)$, and $\eta = 5p$. Here $B$ represents a $p$-subdivision of $U_{4,6}$.

□

*Proof of (4).* If $G$ contains a t-block, call it $B$ and label some set of its arms $B_1, \ldots B_n$ for $n \geq 3$. Otherwise, let $B$ be any cycle of $G$ which is not a balloon if such exists, or any balloon if it does not, and let $B_1$ and $B_2$ be any two paths forming a partition of the edges of $B$. We now proceed to show that every balloon of $G$ must have it's tip in $B$. This will show that every end-block, and hence every block, must meet the vertex boundary of $B$.

Suppose there is some balloon $C$ of $G$ whose tip $v$ is not in $B$. Then there is some cycle $D$ with the vertex-boundary of $C$ and $D$ meeting in $v$. There is a path $P$ from $D$ to $B$ that is internally disjoint from $C$ and $D$.

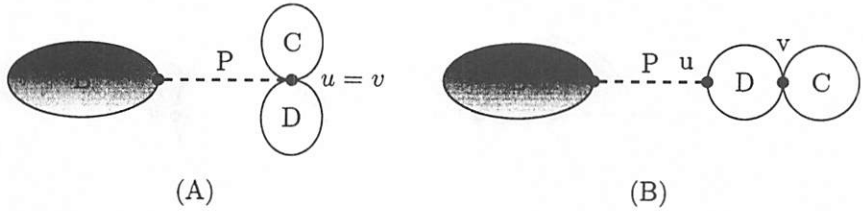


FIGURE 6

Let $u = P \cap D$, $w = P \cap B$, and assume without loss of generality that $w \in B_1$. Note that $C \cup D$ is a bicycle, so $\eta = c + d$

When $u = v$ (Figure 6 (A)), we must have $p > 0$. By symmetry $c = d$, and hence $\eta = 2c$. Since $B_1 \cup B_2 \cup P \cup C$ is a bicycle, we have $b_1 + b_2 + p = c$. Alternatively, when $u \neq v$ (Figure 6 (B)), fix an orientation of the cycle $D$ and let $D_1 = D[u, v]$, $D_2 = D[v, u]$. Then each of the following is a bicycle.

$$B_1 \cup B_2 \cup P \cup D \qquad B_1 \cup B_2 \cup P \cup D_1 \cup C \qquad B_1 \cup B_2 \cup P \cup D_2 \cup C$$

So we have $d_1 = d_2$ and $d_1 + c = d$. Hence $d_1 = d_2 = c$, and $\eta = 3c$. Again we find $b_1 + b_2 + p = c$.

Consider the case that $B$ is a t-block. Then $B_1 \cup B_2 \cup B_3$ is a bicycle, so $b_1 + b_2 + b_3 = \eta = c + d$. By symmetry, we have $b_2 = b_3$. From the arguments above we know $b_1 + b_2 + p = c$. So

$$2c = (b_1 + b_2 + p) + (b_1 + b_3 + p) = (b_1 + b_2 + b_3) + b_1 + 2p = c + d + b_1 + 2p$$

which gives $c = d + b_1 + 2p$. But from above we have either $d = c$ (when $u = v$) or $d = 2c$ (when $u \neq v$), and hence this is impossible.

Now consider the case that $B$ is a cycle. Suppose $B$ intersects exactly one other block. Then $B$ is a balloon. By our choice of $B$, every cycle must be a balloon, and $G$ is a bundle of two balloons (i.e. a bowtie). We are left with the case that the cycle $B$ intersects at least two other blocks. We may choose some subgraph $A$ which contains exactly one cycle, is disjoint from $C \cup D \cup P$, and intersects $B$ at exactly one vertex. (See Figure 7.) Then $B_1 \cup B_2 \cup A$ is a bicycle, so $b_1 + b_2 + a = \eta$. Since $b_1 + b_2 + p = c$, we have $a + c = \eta + p$. But there is some bicycle which strictly contains $A \cup C$, so $a + c < \eta$; a contradiction. □

*Proof of (5).* Here $B$ is a cycle. We wish to show that $G$ is a bundle of balloons. From property (3), $G$ must consist of $B$ and balloons with tips in $B$. We proceed to show that the vertex boundaries of those balloons share a single common vertex.

Suppose $B$ is a cycle and $A_1$ and $A_2$ are balloons with distinct tips $u$ and $v$. Fix an orientation of $B$ and let $B_1 = B[u, v]$ and $B_2 = B[v, u]$. Note

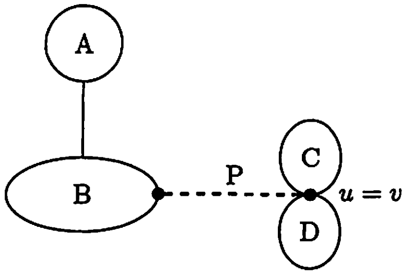


FIGURE 7

that $A_1 \cup A_2 \cup B_i$ is a bicycle for $i \in \{1, 2\}$, so $b_1 = b_2$. Since this argument applies to any two balloons, we can see that there is no balloon $A_3$ attached at $w \notin \{u, v\}$. Note also that $B \cup A_i$ is a bicycle for each $i \in \{1, 2\}$, so $a_1 = a_2$. We now have $\eta = 2a_1 + b_1 = a_1 + 2b_1$, and hence $a_1 = b_1$, $\eta = 3a_1$. If there is a third balloon $A_3$ attached at $u$, then by the argument above $a_3 = a_1$. The bicycle $A_1 \cup A_3$ has length $2a_1$ but $\eta = 3a_1$; a contradiction. Thus there is at most one balloon attached at $u$, and similarly at $v$. Now $A_2$ can be unrolled to $u$, leaving a graph $G'$ representing $M$ which has fewer balloons than $G$. This contradicts our choice of $G$. Hence there is no balloon $A_2$ attached at $v$. This shows that all balloons share a single common tip, and hence $G$ is a bundle of balloons. □

In the case that $G$ has no t-block, we have shown that it is a bundle of balloons. To complete the proof of Theorem 1.2, we now deal with the case that $G$ does have a t-block $B$ with branch vertices $\{u, v\}$ and arms $B_1, \ldots B_n$. We will show that $G$ consists of only the block $B$. Hence $G$ is a subdivision of one of the graphs shown in Figure 2, and $M$ is one of the matroids specified in the theorem.

Applying the arguments from the proof of (4) above to any cycle $B_i \cup B_j$ in $B$, we find that the vertex boundary of $B$ can meet balloons in at most two vertices, which must be equidistant along the cycle. Suppose there are balloons $A_1$ and $A_2$ with tips $x_1$ and $x_2$, respectively, and assume without loss that $x_i \in V(B_i)$. Note that each of the following is a bicycle.

$B_1 \cup B_2 \cup B_3 \qquad A_1 \cup B_1 \cup B_2 \qquad A_1 \cup B_1 \cup B_3 \qquad A_2 \cup B_1 \cup B_2 \qquad A_2 \cup B_2 \cup B_3$

So we have $a_1 = a_2 = b_1 = b_2 = b_3$ and $\eta = 3a_1$. Now the subgraph $A_1 \cup A_2 \cup B_3$ is contained in some bicycle. Since this subgraph includes $\eta$ edges, it must be a bicycle. So each of $A_1$ and $A_2$ must meet $B_3$, and we find $\{x_1, x_2\} = \{u, v\}$. Suppose without loss that $x_1 = u$. There must at least one other balloon $A_3$ with tip $u$. Otherwise $A_1$ can be unrolled to $v$ to obtain a representation with fewer balloons. From symmetry $a_1 = a_3$, and since $A_1 \cup A_3$ is a bicycle we have $\eta = 2a_1$; a contradiction. So $G$ does not have balloons with two distinct tips.

Now consider the case that exactly one vertex in the boundary of $B$ is a balloon tip. If this vertex is $u$, then each balloon can be unrolled to $v$, leaving a graph $G'$ representing $M$ with fewer balloons than $G$. Similarly if the tip is $v$ they may be unrolled to $u$. If exactly one balloon has tip $x \notin \{u, v\}$, it can be unrolled to $v$. So we are left with the case that there are at least two balloons, $A_1$ and $A_2$, with shared tip $x \notin \{u, v\}$. Assume $x \in B_1$. By symmetry $a_1 = a_2$, and since $A_1 \cup A_2$ is a bicycle we have $\eta = 2a_1$. Also by symmetry $b_2 = b_3$. Since $B_1 \cup B_2 \cup B_3$ is a bicycle we have $b_1 + 2b_2 = \eta = 2a_1$. The bicycles $B_1 \cup B_2 \cup A_1$ and $B_1 \cup B_2 \cup B_3$ show $b_3 = a_1$, so we're left with $2a_1 = b_1 + 2b_2 = b_1 + 2b_3 = b_1 + 2a_1$; a contradiction. Hence there are no balloons attached to $B$.

□

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