A Sequential Locating Game on Graphs

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Abstract. Consider the game of locating a marked vertex on a connected graph, where the player repeatedly chooses a vertex of the graph as a probe, and is given the distance from the probe to the marked vertex, until she can uniquely locate the hidden vertex. The goal is to minimize the number of probes. The static version of this game is the well-known problem of finding the *metric dimension* (or *location number*) of the graph. We study the sequential version of this game, and the corresponding *sequential location number.*

1 Introduction

"Where is Minou Hiding" [9] is an Internet game intended for young children learning to count. Minou the cat is randomly hidden on one cell of a 9×5 grid. The child clicks on any cell to see if Minou is hiding there. If so, she wins the game. If not, that cell is labelled with its distance to Minou (moving horizontally and/or vertically from cell to cell). The game continues until the child finds Minou. For a child player guessing semi-randomly, it could take many guesses to locate Minou; however the mathematician player will quickly discover that after two well-chosen guesses the location of Minou is uniquely determined.

In this paper, we generalize this game from the 9×5 grid to any connected graph, and search for strategies requiring the minimum number of guesses in the worst case. Cáceres et al. [1] note that this problem is related to two well-known sequential games: coin weighing (described by Sőderberg and Shapiro[8]) and Mastermind (first analyzed by Knuth [5]), corresponding to hypercubes and Hamming graphs respectively. However, Cáceres et al. focus primarily on the *static* version of these games, in which the player is required to make all but the last guess at the start. The static version of the Minou game dates from 1975: Slater [7] introduced it as the problem of locating an intruder on a graph by placing distance-detecting devices on a minimum size set of vertices, while Harary and Melter [3] introduced the same concept in the context of finding a *metric basis* for a graph by analogy with metric spaces. We review some of the results for the static Minou game in the next section, then return to the sequential game for the rest of the paper.

2 The Static Game

Assume throughout that $G = (V, E)$ is a simple connected graph with $n \geq 2$ vertices. A *resolving set* is a set $S = \{v_1, v_2, ..., v_m\} \subseteq V$ such that for all $u, v \in V$, if $d(v_i, u) = d(v_i, v)$ for $i = 1, 2, ..., m$, then $u = v$. The *metric dimension* $dim(G)$ (also known as the *location number*) is the minimum number of vertices in a resolving set. A resolving set with $dim(G)$ elements is a *metric basis*. Many people have investigated these concepts; a good bibliography is given in [1]. We summarize here results for some well-known classes of graphs.

Theorem 1 [3,7] *Let G be a graph with $n \geq 2$ vertices.*
(a) $dim(G) = 1$ if and only if $G = P_n$ is a path.
(b) For cycles, $dim(C_n) = 2$.
(c) For complete graphs, $dim(K_n) = n - 1$.
(d) For complete bipartite graphs,
$dim(K_{r,s}) = r + s - 2$ for $n = r + s \geq 3$.
(e) For grids, $dim(P_r \times P_s) = 2$ for $r, s \geq 2$.

Theorem 2 [6] *Let W_m be a wheel with $n = m + 1$ vertices, $m \geq 3$. Then $dim(W_m) = \lfloor \frac{2m+4}{5} \rfloor$.*

Let T be a tree which is not itself a path. A *leg* at a vertex v is a component of $T - v$ which is a path, and ℓ_v is the number of legs at v. An *exterior major vertex* is a vertex v such that $deg(v) \geq 3$ and $\ell_v > 0$. For example, in Figure 1, T_1 has exterior

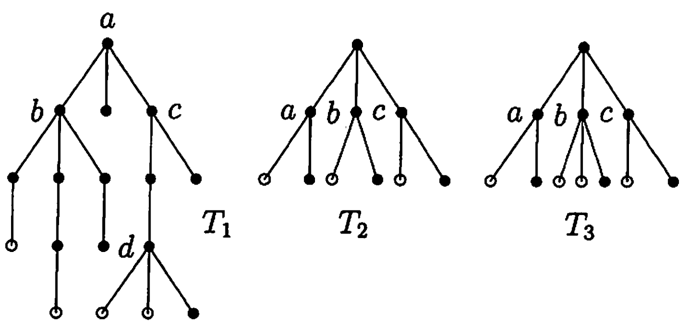

Figure 1: Trees with 4, 3, and 3 exterior major vertices.

major vertices a, b, c and d, while T_2 and T_3 have exterior major vertices a, b and c.

Theorem 3 [2,3,4,7] *Let T be a tree which is not a path and let $x_1, x_2, ..., x_m$ be its exterior major vertices. Then $dim(T) = \sum(\ell_i - 1)$, and any set consisting of the leaves at the end of all but one of the legs of each exterior major vertex of T is a metric basis for T.*

Thus in Figure 1, $dim(T_1) = (1-1)+(3-1)+(1-1)+(3-1) = 4$, and similarly $dim(T_2) = 3$, and $dim(T_3) = 4$. A corresponding metric basis for each graph is indicated by the open circles.

3 The Sequential Game

Consider the game where Player B chooses a vertex $M \in V$ (the vertex where Minou is hidden). Player A then has to locate M by choosing a *probe* v_1 from V. Player B responds with the distance $d(v_1, M)$. Player A then chooses a second probe from V, and this process continues until Player A can uniquely determine the location of M. Player A's objective is to locate M with a minimum number of probes.

A *strategy tree* for G is a rooted tree T, with the vertices labelled with Player A's choices and the edges labelled with Player B's responses. Thus the root is labelled v_1, and from the root there is an edge labelled with each possible distance $d(v_1, v)$ for $v \in V$. For each vertex x at a lower level of T, if the sequence of probe and response labels on the path from the root to x in T uniquely determines a vertex $w \in V$, then x is labelled w and is a leaf in T. Otherwise, x is labelled with the next probe Player A chooses in this sequence, and the edges from x are labelled with each possible response from Player B. Thus, a strategy tree corresponds to a strategy for Player A and vice versa. So the depth of a strategy tree (i.e. the maximum distance from the root to a leaf) is the maximum number of probes it would take Player A in the worst case to locate M by following the corresponding strategy.

For example, let $P_2 \times P_3$ be the 2 by 3 grid with vertices a, b, c, d, e, f as in Figure 2. The first strategy tree in Figure 2 represents the following strategy for Player A: choose e as the first probe, to get $d(e, M)$. If $d(e, M) = 0$, then $M = e$. If $d(e, M) = 1$ or 2, choose c as the second probe to get $d(c, M)$; if this is 3, 0, or 2 then $M = d, c, a$ respectively. Otherwise $d(c, M) = 1$, so choose f as the third probe to get $d(f, M)$. If $d(f, M) = 0$, then $M = f$; otherwise $d(f, M) = 2$ and $M = b$. So in the worst case for this strategy, three probes are required to locate M.

The *sequential location number of* G, $SL(G)$, is the minimum depth of a strategy tree for G. An *optimal* strategy tree is a minimum depth strategy tree, and an *optimal* strategy is a strategy corresponding to an optimal strategy tree. Thus $SL(G)$ represents number of probes required in the worst case by an optimal strategy for Player A. For example, the second strategy tree in Figure 2 is optimal, requiring at worst 2 probes, so $SL(P_2 \times P_3) = 2$.

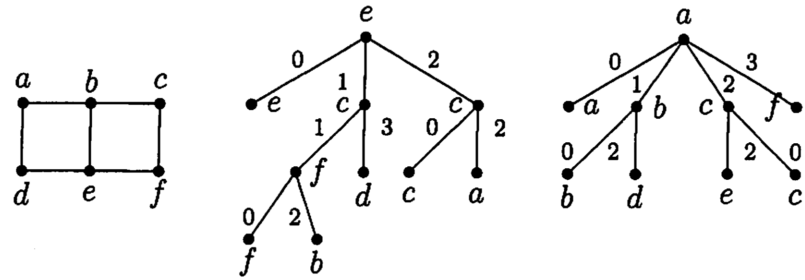

Figure 2: $P_2 \times P_3$ with two possible strategy trees.

4 The Sequential Location Number

We consider the relationship between the sequential location number and the metric dimension, beginning with an analog to Theorem 1(a).

Theorem 4 *$SL(G) = 1$ if and only if G is a path.*

This follows since paths are the only graphs for which there exists a vertex such that all other vertices are a distinct distance from it.

Theorem 5 *For all connected graphs G, $SL(G) \leq dim(G)$. Further, if $dim(G) \leq 2$, then $SL(G) = dim(G)$.*

Proof. Let $S = \{v_1, v_2, ..., v_{dim(G)}\}$ be a metric basis for G. One strategy for Player A is to choose $v_1, v_2, v_3, ...$ as consecutive probes until the location of M is uniquely determined. By the definition of a metric basis, M must be located within $dim(G)$ probes, and so $SL(G) \leq dim(G)$. By Theorem 1 and Theorem 4, $L(G) = 1$ if and only if $SL(G) = 1$; thus if $dim(G) \leq 2$ then $SL(G) = dim(G)$. □

Corollary 6 *$SL(C_n) = dim(C_n) = 2$ for all n, and $SL(P_r \times P_s) = dim(P_r \times P_s) = 2$ for all $r, s \geq 2$.*

Theorem 7 *$SL(K_n) = dim(K_n) = n-1$ for all $n \geq 2$.*

Proof. Since all vertices are at the same distance from each other, it follows that in the worst case Player A must choose $n-1$ probes to guarantee locating M. □

For the remaining classes of graphs considered in Section 2, the sequential location number and the metric dimension need not be equal.

Theorem 8 *Let $1 \leq r \leq s$, with $s > 1$. Then $SL(K_{r,s}) = max\{r, s-1\}$.*

Proof. Let $U = \{u_1, u_2, ..., u_r\}$ and $W = \{w_1, w_2, ...w_s\}$ be the bipartition of $K_{r,s}$. Since $d(w_i, v) = d(w_j, v)$ for $1 \leq i, j \leq s$ and for all $v \in V - \{w_i, w_j\}$, it follows that if $M \in W$, then in the worst case $s-1$ probes from W will be required. Similarly, if $M \in U$, then in the worst case $r-1$ probes from U will be required. So if the first probe is chosen from U, then either $M \in W$ and at worst $1+(s-1) = s$ probes will be required, or $M \in U$ and at worst $r-1$ probes will be required. And if the first probe is chosen from W, then either $M \in W$ and at worst $s-1$ probes will be required, or $M \in U$ and at worst $1+(r-1) = r$ probes will be required. Thus $SL(K_{r,s}) \geq max\{r, s-1\}$.

We now construct a strategy with $max\{r, s-1\}$ probes in the worst case. Choose w_1 as the first probe. If $d(w_1, M) = 0$, then $M = w_1$. If $d(w_1, M) = 1$, then choose probes $u_1, u_2, ...$ until M is located, requiring at worst $1+(r-1) = r$ probes. If $d(w_1, M)$ $= 2$ then choose probes $w_2, w_3, ...$ until M is located, requiring at worst $s-1$ probes. Thus $SL(K_{r,s}) = max\{r, s-1\}$.□

Corollary 9 *$SL(K_{1,s}) = dim(K_{1,s}) = s-1$ for $s > 1$, and $SL(K_{r,s}) < dim(K_{r,s}) = r+s-2$ for $r = 2, s > r$ and $2 < r \leq s$.*

Theorem 10 *For a wheel W_m with $n = m+1$, $SL(W_3) = 3$, $SL(W_4) = 2$, and $SL(W_m) = \lfloor \frac{m+1}{3} \rfloor$ for $m \geq 5$.*

Proof. This is true for $m = 3$ by Theorem 7, and for $m = 4$ by Theorem 5 and Theorem 2, so assume $m \geq 5$. Let w_0 be the centre vertex, with vertices $w_1, w_2, ..., w_m$ on the rim of W_m. From w_0 all other vertices are at distance 1. From any rim vertex, one vertex is at distance 0, three vertices (including w_0) are at distance 1, and $m - 3$ rim vertices are at distance 2. By induction, after any choice of i probes where $1 \leq i \leq \lfloor \frac{m-2}{3} \rfloor$, at least $m - 3i \geq 2$ rim vertices are at distance 2 from all i probes. Thus $SL(W_m) \geq \lfloor \frac{m-2}{3} \rfloor + 1 = \lfloor \frac{m+1}{3} \rfloor$.

We now construct a strategy requiring $\lfloor \frac{m+1}{3} \rfloor$ probes in the worst case. Let the first probe be w_2. If $d(w_2, M) = 0$, then $M = w_2$ has been located in one probe. If $d(w_2, M) = 1$, let the second probe be w_1, locating M as either w_1, w_0 or w_3 in two probes. So assume $d(w_2, M) = 2$ and let the second probe be w_5. Continue with the sequence of probes $w_2, w_5, w_8, ..., w_{3k-1}$ until either $d(w_{3k-1}, M) = 0$ (so M is located in k probes), or $d(w_{3k-1}, M) = 1$ (so M can be located in $k+1$ probes by choosing w_{3k-2} as the last probe), or $m - 5 \leq 3k - 1 \leq m - 3$. The worst case occurs when $m-5 \leq 3k-1 \leq m-3$; i.e., $m = 3k+2, 3k+3$, or $3k + 4$. This can occur either with $d(w_{3k-1}, M) = 1$ (which we have seen requires $k + 1$ probes), or with $d(w_{3k-1}, M) = 2$. In the latter case, $d(w_{3i-1}, M) = 2$ for $i = 1, 2, ..., k - 1$, so $M \in \{w_{3k+1}, w_{3k+2}, ..., w_m\}$, a set with only 2, 3, or 4 elements. This means M can be located in $k+1$ probes by choosing w_{3k+2} as the last probe. Thus $m + 1 = 3(k + 1), 3(k + 1) + 1$, or $3(k + 1) + 2$, and the number $k + 1$ of probes is $\lfloor \frac{m+1}{3} \rfloor$. □

Corollary 11 *$SL(W_m) = dim(W_m)$ for $m = 3, 4, 5, 7, 8, 10, 11, 15$ and 16; otherwise $SL(W_m) < dim(W_m)$.*

5 The Sequential Location Number for Trees

Finally we consider trees. Here, however, we have only been able to find an explicit formula for one class of trees.

Theorem 12 *For any tree T with $n \geq 2$, $SL(T) \geq \Delta(T) - 1$.*

Proof. Let w be a vertex of maximum degree Δ, and let $w_1, w_2, ..., w_\Delta$ be the neighbours of w. Let T_i be the subtree of $T - w$ containing w_i, for $i = 1, 2, ..., \Delta$. Then for any sequence of at most $\Delta - 2$ probes for T, there exist at least two subtrees T_i and T_j which do not contain any of these probes. This means all paths from the $\Delta - 2$ probes to w_i and w_j go through w, and thus w_i and w_j are the same distance from each probe. Thus at least one more probe is required in the worst case, so $SL(T) \geq \Delta - 1$. □

Theorem 13 *For any tree T which is not a path, if there exists a path P in T such that all vertices of degree at least 3 lie on P, then $SL(T) = \Delta - 1$, and there is an optimal strategy in which all probes are leaves.*

Proof. As a result of Theorem 12, it suffices to give a strategy which locates any vertex of T within $\Delta - 1$ probes, and for which all probes are leaves. Let $x_1, x_2, ..., x_m$ be the vertices of degree at least 3, in the order they occur along P. Since $T - x_i$ must have at least one component with no vertices of degree more than 2, each x_i must have at least one leg. Let $s_i = d(x_1, x_i)$ for $i = 1, 2, ..., m$.

Choose the first probe v_1 to be a leaf at the end of a leg at x_1, and the second probe v_2 to be a leaf at the end of a leg at x_m. Let $t_1 = d(v_1, x_1)$, $t_2 = d(v_2, x_m)$, $d_1 = d(v_1, M)$ and $d_2 = d(v_2, M)$. If $d_i < t_i$ for $i = 1$ or $i = 2$, then M is on one of the two legs, so is located in at most two probes. Thus we may assume $d_i \geq t_i$ for $i = 1, 2$.

Suppose first that $d_1 + d_2 = d(v_1, v_2)$. Then M must be a vertex on the path from x_j to x_{j+1}, for some j with $1 \leq j < m$. So $t_1 + s_j \leq d(v_1, M) \leq t_1 + s_{j+1}$, which uniquely determines j and thus M. Thus M can be located in two probes.

So we may assume $d_1 + d_2 > d(v_1, v_2)$. Then M must be on a leg at x_j for some $j, 1 \leq j \leq m$. So $d_1 = t_1 + s_j + d(x_j, M)$ and $d_m = t_m + (s_m - s_j) + d(x_j, M)$. Solving these two equations for s_j and $d(x_j, M)$ gives $s_j = \frac{1}{2}(d_1 - d_2 - s_m - t_1 + t_2)$, which uniquely determines j, and $d(x_j, M) = \frac{1}{2}(d_1 + d_2 + s_m - t_1 - t_2)$. So choosing the remaining probes from the leaves at the ends of the legs at x_j until M is located will require at worst a total of $deg(x_j) - 1 \leq \Delta - 1$ probes. □

For example, consider the trees in Figure 1. T_1 meets the requirements of Theorem 13, so $SL(T_1) = \Delta(T_1) - 1 = 3$. T_2 and T_3 do not meet the requirements, and we have $SL(T_2) = \Delta(T_2) = 3$ (T_2 is the smallest tree with $SL(T) \geq \Delta(T)$), but $SL(T_3) = \Delta(T_3) - 1 = 3$. Note that T_2 is formed from $K_{1,3}$ by adding two leaves to each leaf of $K_{1,3}$. The tree $T(s)$ formed by adding $s - 1$ leaves to each leaf of $K_{1,s}$ satisfies $\Delta(T(s)) = s < SL(T(s)) = 2(s-1) < dim(T(s)) = s(s-1)$ for all $s \geq 3$. Thus the sequential location number of a tree can be arbitrarily greater than the maximum degree, and arbitrarily less than the metric dimension.

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