On equality of restricted edge connectivity and minimum edge degree of graphs[1]

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Abstract Restricted edge connectivity is a more refined network reliability index than edge connectivity. It is known that communication networks with lager restricted edge connectivity are more locally reliable. This work presents a distance condition for graphs to be maximally restricted edge connected, which generalizes Plesnik's corresponding result.

Keywords Restricted edge connectivity; triangle-free graph; distance condition

AMS Classification 05C40

1 Introduction

The restricted edge connectivity $\lambda'(G)$ of a graph G is the minimum cardinality over all its restricted edge cuts, where an edge cut S of G is restricted if $G - S$ contains no isolated vertices. Let $\xi(G) = \min\{d(x) + d(y) - 2 : xy$ is an edge of graph $G\}$, where $d(xy) = d(x) + d(y) - 2$ is called the degree of edge xy. If G is a connected graph of order at least four that is not isomorphic to any star $K_{1,n}$ then $\lambda'(G) \le \xi(G)$ [3], graph G is called maximally restricted edge connected if the equality holds. It is known that communication networks are locally more reliable if they have greater restricted edge connectivity [6], [14]. And so, the optimization of restricted edge connectivity draws a lot of attentions. For advances in this field, the readers are suggested to refer to [4], [8-13],[15] and a survey [1].

Conditions involving the diameter and the girth for a graph to be maximally restricted edge connected are widely studied, resent advances on this subject are as follows. A connected graph G that contains restricted edge cut is maximally restricted edge connected if one of the following conditions holds: (1). $d(u) + d(v) \ge |V(G)| + 1$ holds for all pairs u, v of nonadjacent vertices [16]; (2). $|N(u) \cap N(v)| \ge 3$ for all pairs u, v of nonadjacent vertices and the lower bound can be decrease to 2 if G is triangle-free [2]; (3). minimum degree $\delta \ge 2$ and diameter $d(G) \le g - 2$ [4]; (4). $\delta \ge 2$, girth g is odd, $d(G) = g - 1$ and $|N_{(g-1)/2}(u) \cap N_{(g-1)/2}(v)| \ge 3$ holds for all pairs u, v of vertices at distance $d(u, v) = g - 1$ [4]; (5). $\delta \ge 2$, g is even, $d(G) = g - 1$ and only $\delta - 1$ vertices are mutually at distance $g - 1$ apart [5].

In [11], Plesnik shows that if a connected graph G contains no such four vertices u_1, v_1, u_2, v_2 that $d(u_1, u_2), d(u_1, v_2), d(v_1, u_2), d(v_1, v_2) \ge 3$ then its edge

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connectivity $\lambda(G)$ is equal to its minimum degree $\delta(G)$, where $d(u_1, v_1)$ represents the distance between vertices u_1 and v_1. In this work, we generalize this observation and show that if the same condition holds for a triangle-free graph then it is maximally restricted edge connected.

Let $N_G(x)$, or simply $N(x)$, indicate the neighborhood of a vertex x in graph G. For any two disjoint subsets X, Y of the vertex set $V(G)$ of graph G or two subgraphs, let $[X, Y]$ denote the set of edges of G with one end in X and the other in Y. For other symbols and terminologies not specified, we follow that of [7].

2 Optimization of restricted edge connectivity

Theorem 2.1 Let G be a connected triangle-free graph that contains restricted edge cuts. If G contains no such four vertices u_1, v_1, u_2, v_2 that $d(u_1, u_2)$, $d(u_1, v_2)$, $d(v_1, u_2)$, $d(v_1, v_2) \geq 3$, then $\lambda'(G) = \xi(G)$.

Proof. Suppose on the contrary that graph G is not maximally restricted edge connected. Choose a minimum restricted edge cut $S = [A, \bar{A}]$ of G such that $|A|$ is as small as possible, where $\bar{A} = V(G) - A$. Let $A_1 \subseteq A$ and $\bar{A_1} \subseteq \bar{A}$ be the sets of vertices that are incident with edges of S, write $A_0 = A - A_1$ and $\bar{A_0} = \bar{A} - \bar{A_1}$. If denote the cardinalities of $A_0, A_1, \bar{A_1}$ and $\bar{A_0}$ by $a_0, a_1, \bar{a_1}$ and $\bar{a_0}$ respectively, then $\lambda' \geq a_1$, $\lambda' \geq \bar{a_1}$.

If $a_0 \geq 2$ and $\bar{a_0} \geq 2$ hold simultaneously, then there are four vertices $u_1, v_1 \in A_0$ and $u_2, v_2 \in \bar{A_0}$ that destroy the distance condition of theorem 2.1. And so, we may assume without loss of generality that $a_0 \leq 1$. Now two different cases appear.

Case 1. $a_0 = 0$.

In this case, $A_0 = \emptyset$ and $A = A_1$. Since G is triangle-free, it follows that

$$\sum_{e \in E(G[A])} d(e) \leq |E(G[A])|(a_1 - 2) + \lambda'(G) \leq |E(G[A])|(\lambda'(G) - 2) + \lambda'(G).$$

Since G contains restricted edge cuts, we can see G is not any star and $\xi(G) \geq \lambda'(G)$, then $\xi(G) \geq \lambda'(G) + 1$, otherwise G is maximally restricted edge connected, which is a contradiction, so

$$\sum_{e \in E(G[A])} d(e) \geq |E(G[A])|\xi(G) \geq |E(G[A])|(\lambda'(G) + 1),$$

the previous two formulas show that $3|E(G[A])| \leq \lambda'(G)$. But we shall show the opposition of this inequality is also true by the following claim 1, and so case 1 cannot occur.

Claim 1. $\lambda'(G) < 3|E(G[A])|$ when $a_0 = 0$.

Since S is a minimum restricted edge cut and $\lambda'(G) < \xi(G)$, it follows that $|A| \geq 3$. Suppose on the contrary that $\lambda'(G) \geq 3|E(G[A])|$. For any vertex $x \in A$, let $d_S(x)$ denote the number of edges in S that are incident with x and $d_A(x)$ denote the degree of x in $G[A]$. If $d_A(x) \geq d_S(x)$ for all $x \in A$, then

$$3|E(G[A])| = \frac{3}{2}\sum_{x\in A} d_A(x) \geq \frac{3}{2}\sum_{x\in A_1} d_S(x) = \frac{3}{2}\lambda'(G) > \lambda'(G).$$

It follows from this contradiction that A contains a vertex x_0 such that $d_A(x_0) < d_S(x_0)$.

If $G[A] - \{x_0\} \neq \phi$ contains no isolated vertices, choose a component of $G[A] - \{x_0\}$ say $G[A']$, easily we can see $G[\bar{A}']$ is also connected and $|\bar{A}'| \geq 2$. let $S' = [A', \bar{A}']$, then S' is a restricted edge cut with $|S'| \leq |S| - |d_S(x_0)| + |d_A(x_0)| < |S|$; if $G[A] - \{x_0\}$ has an isolated vertex y, for any edge $e = x_0y \in E(G[A])$ we have $\xi(G) \leq d(e) = d_A(x_0) - 1 + d_S(x_0) + d_S(y) \leq \lambda'(G)$. These contradictions confirm claim 1.

Case 2. $a_0 = 1$.

In this case, we have

$$\sum_{e\in E(G[A])} d(e) \leq |E(G[A])|(a_1 - 1) + \lambda'(G) \leq |E(G[A])|(\lambda'(G) - 1) + \lambda'(G).$$

Combining this observation with

$$\sum_{e\in E(G[A])} d(e) \geq |E(G[A])|\xi(G) \geq |E(G[A])|(\lambda'(G) + 1),$$

we obtain $2|E(G[A])| \leq \lambda'(G)$. But on the other hand, we shall show by the following claim 2 that the opposition of this inequality is also true when $a_0 = 1$. So, case 2 cannot occur yet and the theorem follows.

Claim 2. $\lambda'(G) < 2|E(G[A])|$ when $a_0 = 1$.

Suppose on the contrary that $\lambda'(G) \geq 2|E(G[A])|$ when $a_0 = 1$. Let $A_0 = \{x'\}$ and take any edge $e = x'y \in E(G[A])$. If $d_A(x) \geq d_S(x)$ holds for every vertex $x \in A_1$, then

$$2|E(G[A])| = \sum_{x\in A_1} d_A(x) + d_A(x') \geq \sum_{x\in A_1} d_S(x) + d_A(x') > \sum_{x\in A_1} d_S(x) = \lambda'(G).$$

This contradiction implies that A_1 contains a vertex x_1 such that $d_S(x_1) > d_A(x_1)$.

If $G[A] - \{x_1\} \neq \phi$ contains no isolated vertices, choose a component of $G[A] - \{x_1\}$, say $G[A']$. Easily we can see $G[\bar{A}']$ is also connected and $|\bar{A}'| \geq 2$. Let $S' = [A', \bar{A}']$, then S' is a restricted-edge-cut with $|S'| \leq |S| - |d_S(x_1)| + |d_A(x_1)| < |S|$. This contradiction shows that the set Y of isolated vertices of

$G[A] - \{x_1\}$ is not empty. If $x' \in Y$, then $e = x_1x' \in E(G[A])$ and $d(e) = d_A(x') - 2 + d_S(x_1) + d_A(x_1) = d_S(x_1) + d_A(x_1) - 1 \leq \lambda'(G)$; if $x' \notin Y$, then $Y \subset A_1$, choose a component of $G[A] - (\{x_1\} \cup Y)$ say $G[A']$, and $G[A']$ and $G[\bar{A}']$ are connected subgraphs of order at least two. Let $S' = [A'_1, \bar{A}'_1]$, then S' is a restricted edge cut. Since $d_A(x_1) < d_S(x_1)$, it follows that

$$|S'| \leq |N_A(x_1) - Y| + \sum_{x \in A_1 - (\{x_1\} \cup Y)} d_S(x) < \sum_{x \in A_1} d_S(x) \leq \lambda'(G).$$

Claim 2 follows from these two contradictions. □

Corollary 2.2 Let G be a connected triangle-free graph that contains restricted edge cuts. If G contains a vertex v such that $d(x,y) \leq 2$ for all $x, y \in V(G) - \{v\}$, then $\lambda'(G) = \xi(G)$.

Proof. Since the condition postulated in this corollary implies the condition of theorem 2.1, corollary 2.2 follows. □

Remark Let G be a connected graph with minimum degree at least two that contains restricted edge cuts. Balbuena shows in [4] that if its diameter $d(G) \leq g - 2$ then G is maximally restricted edge connected, where g is the girth of graph G. Corollary 2.2 strengthens this observation in the case $g = 4$ since it allows that $d(G) = 3$.

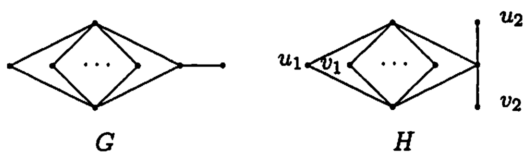

Figure 1. Example of graphs

Figure 1 lists two classes of graphs. Graph G is an example that can be proved to be maximally restricted edge connected by theorem 2.1, since G does not contains two vertices that have distance at least three from other two vertices. But it cannot be shown to be so by other known results such as is listed in the first section one of this paper, including those obtained in [4].

Graph H is an example to show that the condition of theorem 2.1 is best possible to some extent, it contains four vertices u_1, u_2, v_1, v_2 such that $d(u_1, u_2)$, $d(u_1, v_2)$, $d(v_1, u_2)$, $d(v_1, v_2) = 3$ and H is not maximally restricted edge connected. We also remark here that every set of four vertices of H with the property postulated in theorem 2.1 must contains u_2 and v_2. So, the condition of theorem 2.1 is best possible to some extent.

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