OPTIMAL ORIENTATIONS OF G-VERTEX MULTIPLICATIONS OF BIPARTITE GRAPHS

R. Lakshmi
Department of Mathematics
Annamalai University
Annamalainagar - 608 002
Tamilnadu, India.

Abstract. For a graph G, let $\mathcal{D}(G)$ be the set of all strong orientations of G. Define the *orientation number* of G, $\vec{d}(G) = \min\{d(D) \mid D \in \mathcal{D}(G)\}$, where $d(D)$ denotes the diameter of the digraph D. In this paper, it has been shown that $\vec{d}(G(n_1, n_2, \ldots, n_p)) = d(G)$, where $G(n_1, n_2, \ldots, n_p)$ is a G-vertex multiplication ([2]) of a connected bipartite graph G of order $p \geq 3$ with diameter $d(G) \geq 5$ and any finite sequence $\{n_1, n_2, \ldots, n_p\}$ with $n_i \geq 3$.

1 Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. For $v \in V(G)$, the *eccentricity*, denoted by $e_G(v)$, of v is defined as $e_G(v) = \max\ \{d_G(v, x) \mid x \in V(G)\}$, where $d_G(v, x)$ denotes the distance from v to x in G. The *diameter* of G, denoted by $d(G)$, is defined as $d(G) = \max\ \{e_G(v) \mid v \in V(G)\}$.

Let D be a digraph with vertex set $V(D)$ and arc set $A(D)$ which has neither loops nor multiple arcs (that is, arcs with same tail and same head). For $v \in V(D)$, the notions $e_D(v)$ and $d(D)$ are defined as in the undirected graph. For $x, y \in V(D)$, we write $x \rightarrow y$ or $y \leftarrow x$ if $(x, y) \in A(D)$. For sets $X, Y \subseteq V(D)$, $X \rightarrow Y$ denotes $\{(x, y) \in A(D) : x \in X$ and $y \in Y\}$.

An *orientation* of a graph G is a digraph D obtained from G by assigning a direction to each of its edge. By abuse of notation, by D we mean an orientation of G and also the digraph arising out of an orientation of G.

A vertex v is *reachable* from a vertex u of a digraph D if there is a directed path in D from u to v. An orientation D of G is *strong* if any pair of vertices in D are mutually reachable in D. Robbins' celebrated one-way street theorem [5] states that a connected graph G has a strong orientation if and only if G is 2-edge-connected. For a 2-edge-connected graph G, let $\mathcal{D}(G)$ denote the set of all strong orientations of G. The *orientation number* of G is defined to be $\vec{d}(G) = \min \{ d(D) \mid D \in \mathcal{D}(G) \}$. In [3], $\vec{d}(G) - d(G)$ is defined as $\rho(G)$. Any orientation D in $\mathcal{D}(G)$ with $d(D) = \vec{d}(G)$ is called an *optimal orientation* of G. For results on orientations of graphs, see [3], a survey by Koh and Tay.

Let G be a connected graph with $V(G) = \{v_1, v_2, \ldots, v_p\}$. For any finite sequence $\{n_1, n_2, \ldots, n_p\}$ of p positive integers, let $G(n_1, n_2, \ldots, n_p)$ denote the graph with vertex set $V^* = \bigcup_{i=1}^{p} V_i$ and edge set E^*, such that V_i's are pairwise disjoint sets with $|V_i| = n_i$, $i \in \{1, 2, \ldots, p\}$, and for any two distinct vertices x, y in V^*, $xy \in E^*$ if and only if $x \in V_i$ and $y \in V_j$ for some $i, j \in \{1, 2, \ldots, p\}$ with $i \neq j$ and $v_iv_j \in E(G)$. The graph $G(n_1, n_2, \ldots, n_p)$ is called an *extension* of G. It is also called a G-vertex multiplication. If $n_i = n$, $i \in \{1, 2, \ldots, p\}$, then $G(n, n, \ldots, n)$ is denoted by $G^{(n)}$. When $G = K_p$, the graph $K_p(n_1, n_2, \ldots, n_p)$ is a complete p-partite graph with partite sets containing $n_1, n_2, \ldots, n_p$ vertices. In [2], Koh and Tay have extended the results on the optimal orientations of the complete p-partite graphs to $G(n_1, n_2, \ldots, n_p)$. We next list some of the results in [2] for $G(n_1, n_2, \ldots, n_p)$.

Theorem 1.1. (Koh and Tay [2]). *Given $n_i \geq 2$ for each $i \in \{1, 2, \ldots, p\}$, where $p \geq 3$, $d(G) \leq \vec{d}(G(n_1, n_2, \ldots, n_p)) \leq d(G) + 2$.*

Theorem 1.2. (Koh and Tay [2]). *If $d(G) \geq 4$ and $n_i \geq 4$ for each $i \in \{1, 2, \ldots, p\}$, then $\vec{d}(G(n_1, n_2, \ldots, n_p)) = d(G)$.*

Theorem 1.3. (Koh and Tay [2]). *If $d(G) = 3$ and $n_i \geq 4$ for each $i \in \{1, 2, \ldots, p\}$, then $\vec{d}(G(n_1, n_2, \ldots, n_p)) \leq d(G) + 1$.*

The following conjecture was posed by Koh and Tay [2].

Conjecture 1.1. (Koh and Tay [2]). *If G is a graph such that $d(G) \geq 3$ and $n_i \geq 2$ for each $i \in \{1, 2, \ldots, p\}$, then $\vec{d}(G(n_1, n_2, \ldots, n_p)) \leq d(G) + 1$.*

In this paper we assume stronger conditions on G, namely, $d(G) \geq 5$, G is bipartite and $n_i \geq 3$ and prove a stronger result, namely, $\vec{d}(G(n_1, n_2, \ldots, n_p)) = d(G)$.

Let C_n and K_n denote the cycle and complete graph of order n, respectively. Notations and terminology not defined here can be seen in [1].

2 Results

We shall now establish our main result.

Theorem 2.1. *Let G be a connected bipartite graph with $d(G) \geq 5$. If $n_i \geq 3$ for each $i \in \{1, 2, \ldots, p\}$, then $\vec{d}(G(n_1, n_2, \ldots, n_p)) = d(G)$.*

For our convenience, let $V(G) = \{1, 2, \ldots, p\}$. For $i \in \{1, 2, \ldots, p\}$, let $V_i = \{(i,1), (i,2), \ldots, (i, n_i)\}$ and call (i, x) the xth vertex of V_i.

For the proof of Theorem 2.1, we will use the following Lemma.

Lemma 2.1. (Koh and Tay [2]). *Let t_i, n_i be integers such that $t_i \leq n_i$ for $i \in \{1, 2, \ldots, p\}$. If the graph $G(t_1, t_2, \ldots, t_p)$ admits an orientation F in which every vertex v lies on a cycle of length not exceeding m, then $\vec{d}(G(n_1, n_2, \ldots, n_p)) \leq max\{m, d(F)\}$.* ■

Proof of Theorem 2.1. Let (X, Y) be the bipartition of G. We first show that $\vec{d}(G^{(3)}) = d(G)$. Orient $G^{(3)}$ so that for every edge xy of G with $x \in X$ and $y \in Y$, $(x, i) \to (y, i)$ for $i \in \{1, 2, 3\}$ and $(x, i) \leftarrow (y, j)$ for $i, j \in \{1, 2, 3\}$ with $i \neq j$. Let D be the resulting digraph.

Let $u = (a, i)$ and $v = (b, j)$ be any two vertices in D. We shall now prove that $d(D) = d(G)$ by showing that $d_D(u, v) \leq \max\{5, d_G(a, b)\}$. By the nature of the orientation, assume that $u = (a, 1)$. Let P be a shortest (a, b)-path of length ℓ in G. We shall split our proof into several cases according as $a \in X$ or Y and ℓ, the length of P.

Case 1. $a \in X$ and $\ell \geq 4$.

We shall prove Case 1 by induction on ℓ. For $\ell = 4$, let $P = x_1 y_1 x_2 y_2 x_3$ and let $a = x_1$ and $b = x_3$. The existence of the paths $(x_1, 1) \to (y_1, 1) \to (x_2, 3) \to (y_2, 3) \to (x_3, 1)$, $(x_1, 1) \to (y_1, 1) \to (x_2, 3) \to (y_2, 3) \to (x_3, 2)$ and $(x_1, 1) \to (y_1, 1) \to (x_2, 2) \to (y_2, 2) \to (x_3, 3)$ in D shows that $d_D(u, v) \leq 4$. Therefore, the result is true for $\ell = 4$. Hence we assume that $\ell = m + 1 \geq 5$. Let c be an internal vertex of P such that c is adjacent to b. By the induction hypothesis, $d_D(u, \{(c, 1), (c, 2), (c, 3)\}) \leq m$. Hence $d_D(u, \{(b, 1), (b, 2), (b, 3)\}) \leq m + 1 = \ell$.

Case 2. $a \in X$ and $\ell = 3$.

Let $P = x_1 y_1 x_2 y_2$ and let $a = x_1$ and $b = y_2$. The existence of the paths $(x_1, 1) \to (y_1, 1) \to (x_2, 2) \to (y_2, 2) \to (x_2, 1) \to (y_2, 1)$ and $(x_1, 1) \to (y_1, 1) \to (x_2, 3) \to (y_2, 3)$ in D proves that $d_D(u, v) \leq 5$.

Case 3. $a \in X$ and $\ell = 2$.

Let $P = x_1 y_1 x_2$ and let $a = x_1$ and $b = x_2$. The existence of the paths $(x_1, 1) \to (y_1, 1) \to (x_2, 2) \to (y_1, 2) \to (x_2, 1)$ and $(x_1, 1) \to (y_1, 1) \to (x_2, 3)$ in D shows that $d_D(u, v) \leq 4$.

Case 4. $a \in X$ and $\ell = 1$.

As $\ell = 1$, ab is an edge of G. The existence of the paths $(a,1) \to (b,1)$, $(a,1) \to (b,1) \to (a,2) \to (b,2)$ and $(a,1) \to (b,1) \to (a,3) \to (b,3)$ in D proves that $d_D(u,v) \leq 3$.

Case 5. $a \in X$ and $\ell = 0$.

There is a vertex y in Y with $ay \in E(G)$. The existence of the paths $(a,1) \to (y,1) \to (a,2)$ and $(a,1) \to (y,1) \to (a,3)$ in D shows that $d_D(u,v) \leq 2$.

Case 6. $a \in Y$ and $\ell \geq 3$.

We shall prove Case 6 by induction on ℓ. For $\ell = 3$, let $P = y_1x_1y_2x_2$ and let $a = y_1$ and $b = x_2$. The existence of the paths $(y_1,1) \to (x_1,2) \to (y_2,2) \to (x_2,1)$, $(y_1,1) \to (x_1,3) \to (y_2,3) \to (x_2,2)$ and $(y_1,1) \to (x_1,2) \to (y_2,2) \to (x_2,3)$ in D proves that $d_D(u,v) \leq 3$. Therefore, the result is true for $\ell = 3$. Hence we assume that $\ell = m+1 \geq 4$. Let c be an internal vertex of P such that c is adjacent to b. By the induction hypothesis, $d_D(u,\{(c,1),(c,2),(c,3)\}) \leq m$. Hence $d_D(u,\{(b,1),(b,2),(b,3)\}) \leq m+1 = \ell$.

Case 7. $a \in Y$ and $\ell = 2$.

Let $P = y_1x_1y_2$ and let $a = y_1$ and $b = y_2$. The existence of the paths $(y_1,1) \to (x_1,2) \to (y_2,2) \to (x_1,1) \to (y_2,1)$ and $(y_1,1) \to (x_1,3) \to (y_2,3)$ in D shows that $d_D(u,v) \leq 4$.

Case 8. $a \in Y$ and $\ell = 1$.

As $\ell = 1$, ab is an edge of G. The existence of the paths $(a,1) \to (b,2) \to (a,2) \to (b,1)$ and $(a,1) \to (b,3)$ in D proves that $d_D(u,v) \leq 3$.

Case 9. $a \in Y$ and $\ell = 0$.

There is a vertex x in X with $ax \in E(G)$. The existence of the paths $(a,1) \to (x,2) \to (a,2)$ and $(a,1) \to (x,3) \to (a,3)$ in D shows that $d_D(u,v) \leq 2$.

This completes the proof of $\vec{d}(G^{(3)}) = d(G)$.

By the nature of the orientation, every vertex of D lies on a directed cycle of length 4 in D. The proof now follows from Lemma 2.1. ■

Corollary 2.1. *Let G be a bipartite graph with $d(G) \geq 5$. If $n_i \geq 3$ for each $i \in \{1,2,\ldots,p\}$, then $\rho(G(n_1,n_2,\ldots,n_p)) = 0$.* ■

Corollary 2.2. *$\vec{d}(C_p(n_1,n_2,\ldots,n_p)) = \frac{p}{2}$ for all $p \geq 10$ is even and $n_i \geq 3$ for each $i \in \{1,2,\ldots,p\}$.* ■

We now state a result obtained by Ng and Koh [4] which is partially implied by the above corollary.

Theorem 2.2. (Ng and Koh [4]).
(i) $\vec{d}(C_p(n_1,n_2,\ldots,n_p)) = d(C_p)$ for all $p \geq 10$ and $n_i \geq 3$ for each $i \in \{1,2,\ldots,p\}$;

(ii) $\vec{d}(C_p^3) = d(C_p) + 1$ *for* $6 \leq p \leq 9$;
(iii) $\vec{d}(C_p^4) = d(C_p)$ *for* $p = 6, 7$. ∎

Acknowledgement. The author expresses her sincere thanks to Dr. P. Paulraja for several useful discussions.

References

[1] J. Bang-Jensen and G. Gutin, *Digraphs: Theory, Algorithms and Applications*, Springer, London, 2000.

[2] K.M. Koh and E.G. Tay, On optimal orientations of G vertex-multiplications, *Discrete Math.* 219 (2000) 153-171.

[3] K.M. Koh and E.G. Tay, Optimal orientations of graphs and digraphs: a survey, *Graphs and Combin.* 18 (2002) 745-756.

[4] K.L. Ng and K.M. Koh, On optimal orientation of cycle vertex multiplications, Discrete Math. 297 (2005) 104-118.

[5] H.E. Robbins, A theorem on graphs with an application to a problem of traffic control, *Amer. Math. Monthly* 46 (1939) 281-283.