Alternating Subsets and Successions

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Abstract

We present a unified extension of alternating subsets to k-combinations of $\{1, 2, \ldots, n\}$ containing a prescribed number of sequences of elements of the same parity. This is achieved by shifting attention from parity-alternating elements to pairs of adjacent elements of the same parity. Enumeration formulas for both linear and circular combinations are obtained by direct combinatorial arguments. The results are applied to the enumeration of bit strings.

1 Introduction

A totally ordered finite set of positive integer numbers is called *alternating* if any pair of adjacent elements have opposite parities (cf. [5]). The empty set and the single-element set are also alternating by convention. Such sets are known as *alternating subsets of integers* (see for example [2, 12, 8]).

In 1975, Tanny [10] showed that the number $h_0(n, k)$ of alternating k-subsets of $\{1, 2, \ldots, n\}$ is given by:

$$h_0(n,k) = \binom{\lfloor \frac{n+k}{2} \rfloor}{k} + \binom{\lfloor \frac{n+k-1}{2} \rfloor}{k}, \tag{1}$$

where $\lfloor N \rfloor$ is the floor function.

The enumerative function $h_0(n, k)$ has been studied by several authors (see for example [5, 12]), and it is known that $\sum_{k>0} h_0(n, k) = F_{n+3} - 2$, where F_N is the Nth Fibonacci number. It has since been extended to the enumerator of (α, β)-alternating subsets, that is, combinations consisting of a sequence of blocks of lengths $\alpha, \beta, \alpha, \beta, \ldots$, in which the first α elements have the same parity, the next β elements have opposite parity, and so on [11, 8]. A more recent paper on the subject [4] describes how certain restricted classes of alternating subsets may be applied to the study of faces of the polytope of tridiagonal doubly stochastic matrices.

However, there was no known result on the enumeration of the fundamental superset of k-combinations of $[n] = \{1, 2, \ldots, n\}$ containing a prescribed number of sequences of elements of the same parity, of which the class of (α, β) subsets is a specialization. In order to fill this gap we approach the study of alternating subsets by concentrating on pairs of adjacent elements of the same parity since they are analogous to pairs of consecutive integers (also known as *successions*) which are used in the enumeration of general subsets (see for example [1, 7]).

A *parity succession* (or simply *succession*) is a pair of integers x, y satisfying $x \equiv y \pmod 2$, and we define a *succession block* as a finite sequence of elements of the same parity. Let $H(n, k, r)$ denote the set of k-combinations of $[n]$ containing r successions, with cardinality $|H(n, k, r)| = h(n, k, r)$. Thus $h_0(n, k) = h(n, k, 0)$.

Example 1 *Elements of* $H(9, 5, 2)$ *include*
$(1, 2, 3, 5, 7), (1, 3, 4, 5, 9), (2, 4, 5, 7, 8), (3, 4, 6, 8, 9)$.

In what follows we show that the function $h(n, k, r)$ provides the natural setting for obtaining the most complete results.

In Section 2 we give a relatively simple derivation of Equation (1), followed by its recurrence relation. Then we employ combinatorial arguments to obtain both recursive and exact formulas for $h(n, k, r)$, which lead to the general result on k-combinations of $[n]$ containing a prescribed number of succession blocks (Corollary 2). The section closes with a discussion of the subset of $H(n, k, r)$ consisting of succession blocks of length at most 2.

Section 3 is devoted to the enumeration of circular combinations, where the previous questions are posed for combinations of $[n]$ whose elements are arranged on a circle. Lastly, Section 4 deals with the application of our results to the enumeration of bit strings.

2 Enumeration Formulas

We start with a relatively simple derivation of Equation (1).

Associate each member of $H(n, k, 0)$ with a succession block of lenght k with elements in $[n+k-1]$. Notice that to each $(x_1, \dots, x_k) \in H(n, k, 0)$ there corresponds a unique succession block $(x_1, x_2+1, \dots, x_k+k-1) \subset [n+k-1]$, and conversely. For instance, $(1, 4, 5, 6, 7) \in H(9, 5, 0)$ corresponds to $(1, 5, 7, 9, 11) \subset [13]$.

Thus if we define $E_{N,k}$ as the set of k-combinations of $\{2, 4, \dots\} \subset [N]$, and $O_{N,k}$ as the set of k-combinations of $\{1, 3, \dots\} \subset [N]$, we have a bijection

$$H(n, k, 0) \to E_{n+k-1,k} \cup O_{n+k-1,k} : (x_1, x_2, \dots, x_k) \mapsto (x_1, x_2+1, \dots, x_k+k-1).$$

Since $[n+k-1]$ contains $\lfloor (n+k-1)/2 \rfloor$ even numbers and $\lfloor (n+k)/2 \rfloor$ odd numbers, it follows that

$$|E_{n+k-1,k}| = \binom{\lfloor \frac{n+k-1}{2} \rfloor}{k} \quad \text{and} \quad |O_{n+k-1,k}| = \binom{\lfloor \frac{n+k}{2} \rfloor}{k}.$$

Thus the proof of (1) is complete.

The recurrence relation for $h_0(n,k)$ is found by considering the the least element $min(X)$ of each $X \in H(n,k,0)$. Let $h_0(n,k)_1$, $h_0(n,k)_{>1}$ and $h_0(n,k)_P$, denote the number of objects $X \in H(n,k,0)$ such that $min(X) = 1$, $min(X) > 1$ and $min(X)$ has parity P, respectively. Then we have

$$\begin{aligned} h_0(n,k) &= h_0(n,k)_1 + h_0(n,k)_{>1} = h_0(n,k-1)_{even} + h_0(n,k)_{>1} \\ &= h_0(n-1,k-1)_{\text{odd}} + h_0(n-1,k), \end{aligned}$$

since we can delete 1 from each object enumerated by $h_0(n,k)_1$, and then subtract 1 from every term of each (resulting) object enumerated by $h_0(n,k-1)_{even}$ or by $h_0(n,k)_{>1}$.

Expanding $h_0(n-1,k)$ by the last relation, we have

$$\begin{aligned} h_0(n,k) &= h_0(n-1,k-1)_{\text{odd}} + h_0(n-2,k-1)_{\text{odd}} + h_0(n-2,k) \\ &= h_0(n-1,k-1)_{\text{odd}} + h_0(n-1,k-1)_{\text{even}} + h_0(n-2,k), \end{aligned}$$

which gives the following result (the boundary condition $h_0(n,0) = 2$ is conventional).

$$h_0(n,k) = h_0(n-1,k-1) + h_0(n-2,k), \quad 2 \le k \le n, \tag{2}$$
$$h_0(n,1) = n, \; h_0(n,0) = 2.$$

By extending Equation (2) we can prove the following theorem (δ_{ij} is the Kronecker delta).

Theorem 2 *If n,k,r are integers such that $n \ge k > r \ge 1$, then*

$$h(n,k,r) = h(n-1,k-1,r) + h(n-2,k,r) + h(n-2,k-1,r-1), \tag{3}$$

$$h(n,0,r) = 2\delta_{0r}, \; h(n,1,r) = n\delta_{0r}, \; h(n,k,0) = h_0(n,k), \; k > 1.$$

Proof. By the proof of (2) the number of elements of $H(n,k,r)$ in which n or $n-1$ do not belong to a succession is given by $h(n-1,k-1,r) + h(n-2,k,r)$.

An element $A \in H(n,k,r)$ in which n or $n-1$ belongs to a succession is obtained as follows: for each $X \in H(n-2,k-1,r-1)$, if $max(X) \equiv n-2 \pmod 2$, then put n into X, else put $n-1$ into X. This gives

$h(n-2,k-1,r-1)$ elements of $H(n,k,r)$. The boundary values are clear. $\square$

The following identity can be established by showing that it satisfies (3). But we give a direct combinatorial proof.

Theorem 3

$$h(n,k,r) = \binom{k-1}{r} h_0(n-r,k). \tag{4}$$

Proof. Let $A \in H(n,k,r)$. We transform A into a member of $H(n-r,k,0)$. If A has m succession blocks $u_1,\ldots,u_m$, then each u_j has the form $u_j = 2q_1, 2q_2, \ldots, 2q_{t_j}$ or $u_j = 2q_1-1, 2q_2-1, \ldots, 2q_{t_j}-1$, where $1 \le q_1 < q_2 < \cdots < q_{t_j}$, $1 \le t_j \le r+1$ and $(t_1-1)+\cdots+(t_m-1) = r$.

Now apply the following transformation successively to the blocks $u_1,\ldots,u_m$. Assuming that $u_j = 2q_1, 2q_2, \ldots, 2q_{t_j}$, replace u_j by $2q_1, 2q_2-1, \ldots, 2q_{t_j}-t_j+1$, and subtract t_j-1 from each integer larger than $2q_{t_j}$. Note that the first element of the next block u_{j+1} namely $2u+1$, $u \ge q_{t_j}$, is transformed into $2u+2-t_j$ which has opposite parity with the last element of the image of u_j. The result is a t_j-element alternating subset of a k-combination whose last element is $max(A)-t_j+1$. It is clear that this procedure yields the same type of set if we start with a block with odd elements. Hence A corresponds to a unique member of $H(n-(t_1-1)-\cdots-(t_{k-r}-1),k,0) = H(n-r,k,0)$.

Conversely, given $X=(x_1,\ldots,x_k) \in H(n-r,k,0)$, we must reverse the above procedure, but only if we know which elements of X should change parities. To find out we determine the pattern of the succession blocks in some inverse image $A \in H(n,k,r)$ by choosing r of the first $k-1$ elements of X in $\binom{k-1}{r}$ ways. This may be represented as a $(k-1)$-vector of 0's and 1's having r 1's, say $(\epsilon_1,\ldots,\epsilon_{k-1})$. Then to specify A we reverse the procedure described in the previous paragraph by changing the parity of each element x_{i+1} whenever $\epsilon_i = 1$, such that a sequence of ℓ consecutive 1's, starting from position j, corresponds to the succession block of length $\ell+1$, with first member x_j. $\square$

Corollary 1 *The number of k-combinations of $[n]$ consisting of r parity successions (i.e, r pairs of adjacent elements of the same parity) is given by*

$$h(n,k,r) = \binom{k-1}{r}\left(\binom{\lfloor\frac{n+k-r}{2}\rfloor}{k} + \binom{\lfloor\frac{n+k-r-1}{2}\rfloor}{k}\right). \tag{5}$$

Observe that if $X \in H(n,k,r)$ consists of b succession blocks then $b = k-r$ (since if the blocks have lengths $u_1,\ldots,u_b$, then $r = (u_1-1)+\cdots+(u_b-1) = k-b$). Substituting $k-r=b$ in (5), we obtain the alternating subsets analogue of the classical result on combinations and blocks of consecutive integers (see Remark 4 below):

Corollary 2 *The number $f(n,k,b)$ of k-combinations of $[n]$ consisting of b alternating succession blocks (i.e, b separate sequences of elements of the same parity) is given by*

$$f(n,k,b) = \binom{k-1}{b-1}\left(\binom{\lfloor\frac{n+b}{2}\rfloor}{k} + \binom{\lfloor\frac{n+b-1}{2}\rfloor}{k}\right). \tag{6}$$

Remark 4 *The enumeration of combinations according to the number of pairs of consecutive integers, also called successions, originated with the works of Kaplansky and Riordan in the 1940's (see [6, 9]). Abramson and Moser [1] seem to be the first to consider enumeration by the number of arbitrary sequences of consecutive elements, and proved the basic result that the number of k-combinations of $[n]$ with exactly r blocks of consecutive integers is*

$$\binom{k-1}{r-1}\binom{n-k+1}{r}.$$

Note that by the proof of (1) we have the following convenient decomposition according to the parity of least elements.

$$f(n,k,b)_{\text{odd}} = \binom{k-1}{b-1}\binom{\lfloor\frac{n+b}{2}\rfloor}{k}, \quad f(n,k,b)_{\text{even}} = \binom{k-1}{b-1}\binom{\lfloor\frac{n+b-1}{2}\rfloor}{k}.$$

Example 5 $f(8,6,4) = 10$ *enumerates the following combinations:*
$(1,2,3,4,6,8), (1,2,3,5,6,8), (1,2,3,5,7,8), (1,2,4,5,6,8), (1,2,4,5,7,8),$
$(1,3,4,5,6,8), (1,3,4,5,7,8), (1,3,4,6,7,8), (1,3,5,6,7,8), (1,2,4,6,7,8).$

It is clear from (6) and (4) that the enumerator of k-combinations according to b succession blocks of specified type is $f(n,k,b)/\binom{k-1}{b-1}$, denoted by $z(n,k,b)$. The last remark suffices to deduce the next result, stated in our notation for brevity.

Corollary 3 *((α,β)-alternating subsets)*
The number of (α,β)-alternating k-combinations of $[n]$ is given by

$$\begin{cases} z(n,\, k,\, 2(k-c)/(\alpha+\beta)+1), & 0 < c \le \alpha, \\ z(n,\, k,\, 2(k-c)/(\alpha+\beta)+2), & \alpha < c < \alpha+\beta. \end{cases} \tag{7}$$

Proof. By modifying Euclid's algorithm we have $k = (\alpha + \beta)q + c$, where $0 < c < \alpha + \beta$ and q is a positive integer. So the number of blocks $b \in \{2q+1, 2q+2\}$. If $c \leq \alpha$, then $b = 2q+1 = 2(k-c)/(\alpha+\beta)+1$; otherwise $b = 2(k-c)/(\alpha+\beta)+2$ for $\alpha < c < \alpha + \beta$. □

We conclude this section with a discussion of the parity analogue of *detached* successions which was introduced in [7] in connection with pairs of consecutive integers as successions. A combination is said to consist of detached successions if it contains only sequences of u consecutive elements, where $u \in \{1, 2\}$. More generally, detached t-successions, when $u \in \{1, t\}$, $t \geq 2$, were considered. However, the former case suffices to understand the idea.

A parity succession x_i, x_{i+1} in $(x_1, \ldots, x_k)$ will be called detached if neither x_{i-1}, x_i nor x_{i+1}, x_{i+2} is a parity succession in $(x_i, \ldots, x_k)$.

Let $D(n,k,r)$ denote the subset of $H(n,k,r)$ containing r detached parity successions, and $|D(n,k,r)| = d(n,k,r)$. For example, of the three objects $(1,3,4,5,9), (2,4,5,7,8), (1,2,3,5,7) \in H(9,5,2)$, the first two belong to $D(9,5,2)$ while $(1,2,3,5,7) \notin D(9,5,2)$.

Proposition 1 *We have*

$$d(n,k,r) = \binom{k-r}{r} h_0(n-r,k). \tag{8}$$

Proof. This is proved the same way as (4) except that, in the reverse transformation, we select only nonconsecutive r of the first $k-1$ elements, obtaining $\binom{(k-1)-r+1}{r} = \binom{k-r}{r}$, where we have used the fact that the number of nonconsecutive k-combinations of $[n]$ is given by $\binom{n-k+1}{k}$. □

The following relation holds.

$$(k-1)_r d(n,k,r) = (k-r)_r h(n,k,r),$$

where $(N)_r = N(N-1)\cdots(N-r+1)$, $(N)_0 = 1$.

The detached version of (5) may now be written, and we have

Corollary 4 *The number $g(n,k,b)$ of k-combinations of $[n]$ consisting of b alternating succession blocks, each of length* 1 *or* 2, *is given by*

$$g(n,k,b) = \binom{b}{k-b}\left(\binom{\lfloor\frac{n+b}{2}\rfloor}{k} + \binom{\lfloor\frac{n+b-1}{2}\rfloor}{k}\right). \tag{9}$$

We state a useful extension of (8).

Theorem 6 *The number $d_t(n,k,b)$ of k-combinations of $[n]$ consisting of r succession blocks of length t, and $k-tr$ units, is given by*

$$d_t(n,k,r) = \binom{k-(t-1)r}{r} h_0(n-(t-1)r,k), \quad 2 \le t \le k. \tag{10}$$

Proof. Suitably modify the proof of (8); choose r tuples $(y_1,\ldots,y_r)$ from $[k-t+1]$ such that $y_i - y_{i-1} \ge t$, $2 \le i \le r$, in as many ways as $\binom{k-t+1-(t-1)(r-1)}{r} = \binom{k-(t-1)r}{r}$. □

3 Circular Combinations

We consider analogous results for *circular combinations* $(y_1,\ldots,y_k)$, when the elements are arrayed on a circle and y_k, y_1 are treated as adjacent. A bar is placed over each previous notation to distinguish corresponding enumerators of circular combinations.

Theorem 7 *The number $\bar{f}(n,k,b)$ of circular k-combinations of $[n]$ consisting of b alternating succession blocks is given by*

$$\bar{f}(n,k,b) = \begin{cases} f(n,k,b)+f(n,k,b+1), & \text{if } b \text{ is even,} \\ 0, & \text{if } b \text{ is odd.} \end{cases} \tag{11}$$

Proof. Let $X = (x_1,\ldots,x_k) = (B_1,\ldots,B_b)$ be an object enumerated by $\bar{f}(n,k,b)$. If b is even, then $\bar{f}(n,k,b)$ corresponds to two summands namely $f(n,k,b)$, since the blocks B_b and B_1 have opposite parities, and $f(n,k,b+1)$ since the odd number $b+1$ of blocks implies the merging of B_{b+1} and B_1, when the elements are arranged on a circle. On the other hand, an odd number of blocks is impossible since the block parities must alternate around the circle. □

Example 8 $\bar{f}(8,6,4) = 20$, *the enumerated objects being the* 10 *in Example 5 and the following:*
(1,2,3,4,5,7), (1,2,3,4,6,7), (1,2,3,5,6,7), (1,2,4,5,6,7), (1,3,4,5,6,7), (2,3,4,5,7,8), (2,3,4,6,7,8), (2,3,5,6,7,8), (2,4,5,6,7,8), (2,3,4,5,6,8).

Substituting (6) in Theorem 7 and simplifying give

Corollary 5 *If b is an even number, then*

$$\bar{f}(n,k,b) = \binom{k}{b}\left(\binom{\lfloor\frac{n+b}{2}\rfloor}{k} + \binom{\lfloor\frac{n+b+1}{2}\rfloor}{k} - \frac{b}{k}\binom{\lfloor\frac{n+b-1}{2}\rfloor}{k-1}\right).$$

Theorem 9 *The number $\bar{g}(n,k,b)$ of circular k-combinations of $[n]$ consisting of b alternating succession blocks, each of length* 1 *or* 2, *is given by*

$$\bar{g}(n,k,b) = \begin{cases} g(n,k,b) + \frac{k-b}{b} g(n+1,k,b), & \text{if } b \text{ is even,} \\ 0, & \text{if } b \text{ is odd.} \end{cases} \tag{12}$$

Proof. As in the proof of Theorem 7, we find $\bar{g}(n,k,b)$ only when b is even. The contribution from $g(n,k,b)$ is clear. It remains to account for the contribution from objects enumerated by $g(n,k,b+1)$. We want to isolate combinations $(x_1, x_2, \ldots, x_{k-1}, x_k)$ in which $x_1 \not\equiv x_2$, $x_{k-1} \not\equiv x_k$ and $x_1 \equiv x_k \pmod 2$, such that each contains b circular blocks or r detached circular successions ($b = k-r$). Thus, referring to the proof of (8), we must select $r-1$ nonconsecutive of the $(k-1)-2 = k-3$ eligible elements in as many ways as $\binom{(k-3)-(r-1)+1}{r-1} = \binom{k-1-r}{r-1}$. So the total number of objects is $\binom{k-1-r}{r-1} h_0(n-r+1,k) = \frac{r}{k-r}\binom{k-r}{r} h_0(n-r+1,k) = \frac{r}{k-r} d(n+1,k,r)$, which is equivalent to $\frac{k-b}{b} g(n+1,k,b)$. □

Example 10 $\bar{g}(8,6,4) = 12$ *enumerates the following objects (cf. Example 8):* $(1,2,3,5,6,8), (1,2,4,5,6,8), (1,2,4,5,7,8), (1,3,4,5,6,8),$ $(1,3,4,5,7,8), (1,3,4,6,7,8), (2,3,4,6,7,8), (1,2,3,4,6,7), (1,2,4,5,6,7),$ $(2,3,4,5,7,8), (1,2,3,5,6,7), (2,3,5,6,7,8)$.

Corollary 6 *If b is an even number, then*

$$\bar{g}(n,k,b) = \binom{b}{k-b}\left(\frac{k}{b}\binom{\lfloor\frac{n+b}{2}\rfloor}{k} + \binom{\lfloor\frac{n+b-1}{2}\rfloor}{k} + \frac{k-b}{b}\binom{\lfloor\frac{n+b+1}{2}\rfloor}{k}\right).$$

Note the succession versions of the above results:

$$\bar{d}(n,k,r) = \begin{cases} d(n,k,r) + \frac{r}{k-r} d(n+1,k,r), & \text{if } k-r \text{ is even,} \\ 0, & \text{if } k-r \text{ is odd.} \end{cases} \tag{13}$$

$$\bar{h}(n,k,r) = \begin{cases} h(n,k,r) + h(n,k,r-1), & \text{if } k-r \text{ is even,} \\ 0, & \text{if } k-r \text{ is odd.} \end{cases} \tag{14}$$

Thus in particular,

$$\bar{h}_0(n,k) = \begin{cases} h_0(n,k), & \text{if } k \text{ is even,} \\ 0, & \text{if } k \text{ is odd.} \end{cases} \tag{15}$$

Equation (15) clarifies an expression for $\bar{h}_0(n,k)$ in [2], found through a different approach, namely $\bar{h}_0(n,k) = h_0(n,k)$, which is silent on the parity of k.

4 An Application

We apply the above results to the enumeration of bit strings (sequences of 0's and 1's). Denote by $h_0(k)$ the number of k-bit strings with no adjacent 0's and no adjacent 1's, and let the notation for the number of k-bit strings containing r parity successions be $h(k,r)$. Other analogous notations are employed below.

Then it is immediate from (1) that $h_0(k) = 2$. Hence by (5) and (6) we obtain the equivalent results:

$$h(k,r) = 2\binom{k-1}{r}, \qquad f(k,b) = 2\binom{k-1}{b-1}.$$

Similarly, we have:

$$d(k,r) = 2\binom{k-r}{r}, \qquad g(k,b) = 2\binom{b}{k-b}.$$

To enumerate cyclic bit strings it is convenient to mark one bit as the starting point of circular traversion. This way cyclic strings may be displayed on a line and two objects $(\hat{\epsilon_1}, \epsilon_2, \ldots, \epsilon_k)$, $(\hat{\nu_1}, \nu_2, \ldots, \nu_k)$, are distinct provided $\epsilon_i \neq \nu_i$ for some $i \in [k]$.

The following results are deduced from corresponding ones in Section 3. We set the number of blocks $b = 2q$ (q a positive integer). As an illustration, Theorem 7 gives: $\bar{f}(k,2q) = 2\binom{k-1}{2q-1} + 2\binom{k-1}{2q} = 2\binom{k}{2q}$.

The number of cyclic k-bit strings containing $2q$ succession blocks is given by:

$$\bar{f}(k,2q) = 2\binom{k}{2q}, \quad k \geq 2.$$

The number when the length of a succession block ≤ 2 is:

$$\bar{g}(k,2q) = \frac{k}{q}\binom{2q}{k-2q}.$$

Thus all cyclic k-bit strings containing only detached successions are enumerated by $\sum_{q\geq 1} \bar{g}(k,2q)$. This is the number of cyclic k-bit strings that avoid the substrings 0 0 0 and 1 1 1, in the formulation of Argur, Fraenkel and Klein [3], who obtained the same result in connection with a problem of genetic engineering.

Finally, by extending the derivation of the last result, it can be shown that the number of cyclic k-bit strings containing $2q$ succession blocks, each of length 1 or t, is given by:

$$\bar{g}_t(k,2q) = \frac{1}{q}\left(2q + \lfloor\frac{k-2q}{t-1}\rfloor\right)\binom{2q}{\lfloor\frac{k-2q}{t-1}\rfloor}, \quad 2 \leq t \leq k.$$

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