# Connectivity of lexicographic product and direct product of graphs*

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**Abstract**

In this paper, we prove that the connectivity and the edge connectivity of the lexicographic product of two graphs $G_1$ and $G_2$ are equal to $\kappa_1 v_2$ and $\min\{\lambda_1 v_2^2, \delta_2 + \delta_1 v_2\}$, respectively, where $\delta_i, \kappa_i, \lambda_i$ and $v_i$ denote the minimum degree, the connectivity, the edge-connectivity and the number of vertices of $G_i$, respectively. We also obtain that the edge-connectivity of the direct product of $K_2$ and a graph $H$ is equal to $\min\{2\lambda, 2\beta, \min_{j=\lambda}^{\delta}\{j + 2\beta_j\}\}$, where $\beta$ is the minimum size of a subset $F \subset E(H)$ such that $H - F$ is bipartite and $\beta_j = \min\{\beta(C)\}$, where $C$ takes over all components of $H - B$ for all edge-cuts $B$ of size $j \geqslant \lambda = \lambda(H)$.

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# 1 Introduction

Throughout this paper, a graph $G = (V, E)$ always means a finite undirected graph without self-loops or multiple edges, where $V = V(G)$ is the vertex-set and $E = E(G)$ is the edge-set. The symbol $K_n$ denotes a complete graph with $n$ vertices. For two disjoint subsets $X$ and $Y$ in $E(G)$, the symbol $E_G(X, Y)$ (sometimes $[X, Y]$ for short) denotes the set of edges in $G$ with one end-vertex in $X$ and the other in $Y$. For the graph theoretical terminology and notation not defined here, we refer the reader to [15].

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It is well-known that when the underlying topology of an interconnection network is modelled by a connected graph $G = (V, E)$, where $V$ is the set of processors and $E$ is the set of communication links in the network, the connectivity $\kappa(G)$ and the edge-connectivity $\lambda(G)$ of $G$ are two important measurements for fault-tolerance of the network. In general, the larger $\kappa(G)$ or $\lambda(G)$ is, the more reliable the network is. It is well-known that $\kappa(G) \leq \lambda(G) \leq \delta(G)$, where $\delta(G)$ is the minimum degree of $G$. A connected graph $G$ is called to be *$\kappa$-maximal* and *$\lambda$-maximal* if $\kappa(G) = \delta(G)$ and $\lambda(G) = \delta(G)$, respectively.

Product graphs have always been a good method to construct large graphs from small ones, thus it also has many applications in the design of interconnection networks (see [14]). There are many ways to define products of two graphs, the most widely used one may be the Cartesian product, first introduced by Sabidussi [9]. In the same paper, Sabidussi also proposed another kind of product, the strong product. It has been known for a long time that the connectivity and the edge-connectivity of the Cartesian product of two graphs are at least the sum of the connectivity and the edge-connectivity of the two factor graphs, respectively (see [1, 10, 13]). Recently, the authors [16, 17] have determined the connectivity and edge-connectivity of the Cartesian product of two graphs in terms of the minimum degree, connectivity, edge-connectivity and vertex number of the factor graphs. The connectivity of the strong product of graphs has been studied by Sun and Xu in [11].

In this paper, we study the connectivity of another two kinds of product graphs, the lexicographic product and the direct product. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. The *lexicographic* product $G_1 \circ G_2$ has $V_1 \times V_2$ as its vertex-set, and two vertices $x_1x_2$ and $y_1y_2$ are adjacent if and only if either $x_1y_1 \in E_1$, or $x_1 = y_1$ and $x_2y_2 \in E_2$. According to [6], the lexicographic product is first defined by Hausdorff [4]. Many graph theoretical invariants of lexicographic product of graphs have been studied in the literature, see [7, 8] for example. The *direct product* $G_1 \times G_2$ also has the vertex-set $V_1 \times V_2$. Two vertices $x_1x_2$ and $y_1y_2$ are adjacent if and only if $x_1y_1 \in E_1$ and $x_2y_2 \in E_2$. The direct product sometimes appears in the literature with other names, such as the cross product [2, 3], the categorical product [12], the cardinal product [5] and so on.

Note that in the sense of isomorphism the direct product satisfies the commutative law, while the lexicographic product does not. The lexicographic product and the direct product, together with the Cartesian product ($\square$) and the strong product ($\boxtimes$), are the main four standard products of graphs that is being treated in the monograph [6]. The monograph devotes to all aspects related to these products. The graphs shown in Figure 1 illustrate the differences of these four kinds of products.

In Section 2, we determine the connectivity and the edge-connectivity


Figure 1: Four kinds of products of $K_2$ and $P_2$

of the lexicographic product $G_1 \circ G_2$ of two graphs $G_1$ and $G_2$, that is, $\kappa(G_1 \circ G_2) = \kappa_1 v_2$ and $\lambda(G_1 \circ G_2) = \min\{\lambda_1 v_2^2, \delta_2 + \delta_1 v_2\}$. And in Section 3, we study the edge-connectivity of the direct product of $K_2$ and an arbitrary connected graph $H$ and obtain that $\lambda(K_2 \times H) = \min\{2\lambda, 2\beta, \min_{j=\lambda}^{\delta}\{j + 2\beta_j\}\}$. All throughout this paper, $\delta_i, \kappa_i, \lambda_i$ and $v_i$ will denote the minimum degree, the connectivity, the edge-connectivity and the number of vertices of the graph $G_i (i = 1, 2)$, respectively; while the parameters $\beta$ and $\beta_j$ will be defined in Section 3.

## 2 Lexicographic product

**Lemma 1** *Let $G_1$ and $G_2$ be two graphs, then $\delta(G_1 \circ G_2) = \delta_2 + \delta_1 v_2$.*

By simple observation, $G_1 \circ G_2$ is connected if and only if $G_1$ is connected.

**Theorem 1** *Let $G_1$ and $G_2$ be two graphs. If $G_1$ is non-trivial, non-complete and connected, then*

$$\kappa(G_1 \circ G_2) = \kappa_1 v_2.$$

**Proof.** By the hypothesis that $G_1$ is a non-complete graph, there are separating sets in $G_1$ and $G_1 \circ G_2$. Let $S_1$ be a minimum separating set of $G_1$. Then, by the definition, $S_1 \times V_2$ is a separating set of $G_1 \circ G_2$ and so $\kappa(G_1 \circ G_2) \leqslant |S_1 \times V_2| = \kappa_1 v_2$.

Now, let $S$ be any separating set of $G_1 \circ G_2$. We need to show that $|S| \geqslant \kappa_1 v_2$. It is easy to see that there exist two vertices $x_1 y_1$ and $x_2 y_2$ in $G_1 \circ G_2 - S$ such that they are in distinct components of $G_1 \circ G_2 - S$

and $x_1 \neq x_2$. Then $x_1$ and $x_2$ are not adjacent in $G_1$, otherwise $x_1y_1$ and $x_2y_2$ are adjacent in $G_1 \circ G_2$, which means that $S$ can not separates $x_1y_1$ and $x_2y_2$ in $G_1 \circ G_2$, a contradiction. So there are $\kappa_1$ internal-disjoint $(x_1, x_2)$-paths $P_1, P_2, \ldots, P_{\kappa_1}$ in $G_1$.

Let $P_i = (x_1, t_1, t_2, \ldots, t_k, x_2)$. If for each $j = 1, 2, \ldots, k$ there exists a $z_j \in V_2$ such that $t_jz_j \notin S$, then the $(x_1y_1, x_2y_2)$-path $(x_1y_1, t_1z_1, \ldots, t_kz_k, x_2y_2)$ avoids $S$ in $G_1 \circ G_2$, which contradicts to our hypothesis that $S$ separates $x_1y_1$ and $x_2y_2$ in $G_1 \circ G_2$. Thus, for each $i = 1, 2, \ldots, \kappa_1$, there is at least one internal vertex $t^i$ in $P_i$ such that $\{t^i\} \times V_2 \subset S$. It follows that

$$|S| \geqslant \sum_{i=1}^{\kappa_1} |\{t^i\} \times V_2)| = \kappa_1 v_2.$$

The proof is complete. □

By similar argument, it is easy to see that $\kappa(K_n \circ G_2) = (n-1)v_2 + \kappa_2$, where $G_1 = K_n$. So, $G_1 \circ G_2$ is $\kappa$-maximal if and only if $G_1$ is a complete graph and $G_2$ is $\kappa$-maximal.

**Theorem 2** *Let $G_1$ and $G_2$ be two non-trivial graphs, and $G_1$ is connected, then*

$$\lambda(G_1 \circ G_2) = \min\{\lambda_1 v_2^2, \delta_2 + \delta_1 v_2\}.$$

**Proof.** We only need to prove that $\lambda(G_1 \circ G_2) \geqslant \min\{\lambda_1 v_2^2, \delta_2 + \delta_1 v_2\}$ since the reversed inequality is obvious by finding two edge-cuts of size $\lambda_1 v_2^2$ and $\delta_2 + \delta_1 v_2$, respectively. Let $G = G_1 \circ G_2$. For $x \in V_1$, let $G_2^x$ denote the subgraph of $G$ induced by $\{x\} \times V_2$. It is clear that $G_2^x$ is isomorphic to $G_2$. Let $B$ be a minimum edge-cut in $G$. Then $G - B$ has exactly two components (see, for example, the exercise 4.3.2 in [15]), denoted by $C_1$ and $C_2$.

Let $X = \{x \in V(G_1) : xy \in V(C_1)$ for some $y \in V(G_2)\}$ and $Y = \{x \in V(G_1) : xy \in V(C_2)$ for some $y \in V(G_2)\}$. Then $X \neq \emptyset$ and $Y \neq \emptyset$, clearly.

It $X \cap Y = \emptyset$, then $\{X, Y\}$ is a partition of $V(G_1)$. Thus

$$|B| \geqslant \sum_{xy \in E_{G_1}(X,Y)} |E_G(V(G_2^x), V(G_2^y))| = |E_{G_1}[X, Y]| v_2^2 \geqslant \lambda_1 v_2^2.$$

We assume $X \cap Y \neq \emptyset$ below and let $x_0 \in X \cap Y$. Note that for each neighbor $x$ of $x_0$, the graph that consists of the vertex-set $V(G_2^{x_0}) \cup V(G_2^x)$ and the edge-set $E_G(V(G_2^{x_0}), V(G_2^x))$ is isomorphic to a complete bipartite $K_{v_2,v_2}$, denoted by $G_2^{[x_0,x]}$, has edge-connectivity $v_2$. Let $B_{x_0x} = B \cap E_G(V(G_2^{x_0}), V(G_2^x))$. Then $|B_{x_0x}| \geqslant v_2$ for each neighbor $x$ of $x_0$, otherwise

$G_2^{x_0} - B$ is connected through $G_2^x$, a contradiction. Next we claim that

$$|B_{x_0x}| + |B_{x_0}| \geqslant \delta_2 + v_2, \tag{1}$$

where $B_{x_0} = B \cap E(G_2^{x_0})$ and $x$ is a neighbor of $x_0$. Let $D = V(G_2^{x_0}) \cap V(C_1)$, $F = V(G_2^{x_0}) \cap V(C_2)$, and assume that $|D| \leqslant |F|$. If $|D| = 1$, then (1) holds since $|B_{x_0}| \geqslant \delta_2$. If $|D| \geqslant 2$, we will find $|D|v_2$ edge-disjoint $(D, F)$-paths in $G_2^{[x_0,x]}$. Let $D = \{u_1, u_2, \ldots, u_t\}$ and $\{w_1, w_2, \ldots, w_t\} \subseteq F$. Then for each $i(1 \leqslant i \leqslant t)$, there are $v_2$ edge-disjoint $(u_i, w_i)$-paths in $G_2^{[x_0,x]}$: $(u_i, xz, w_i)$ with $z \in V(G_2)$. So all together, we find $|D|v_2$ edge-disjoint $(D, F)$-paths. In order to disconnect $D$ from $F$, we must have $|B_{x_0,x}| \geqslant |D|v_2 \geqslant 2v_2 > \delta_2 + v_2$ and (1) also holds. Let $x_1, x_2, \ldots, x_{\delta_1}$ be $\delta_1$ neighbors of $x_0$ in $G_1$, then

$$\begin{aligned} |B| &\geqslant (|B_{x_0,x_1}| + |B_{x_0}|) + \sum_{i=2}^{\delta_1} |B_{x_0,x_i}| \\ &= \delta_2 + v_2 + (\delta_1 - 1)v_2 \\ &= \delta_2 + \delta_1 v_2. \end{aligned}$$

This completes the proof. □

# 3 Direct product

**Lemma 2** *Let $G_1$ and $G_2$ be two graphs, then $\delta(G_1 \times G_2) = \delta_1\delta_2$.*

**Lemma 3** *Let $G_1$ and $G_2$ be two non-trivial connected graphs, then $G_1 \times G_2$ is connected if and only if at least one of $G_1$ and $G_2$ is non-bipartite.*

**Proof.** First assume both $G_1$ and $G_2$ are bipartite graphs with partite sets $V(G_1) = (A, B)$ and $V(G_2) = (C, D)$. Then there are no edges between the sets of vertices $(A \times C) \cup (B \times D)$ and $(B \times C) \cup (A \times D)$ in $G_1 \times G_2$, hence $G_1 \times G_2$ is disconnected.

Conversely, we suppose, without loss of generality, that $G_2$ is non-bipartite. Then $G_2$ contains odd cycles certainly. To show that $G_1 \times G_2$ is connected, it is sufficient to prove $K_2 \times G_2$ is connected since $G_1$ is connected. Let $V(K_2) = \{a, b\}$. Let $u$ and $w$ be two vertices of $K_2 \times G_2$, then we have to show there is a $(u, w)$-path in $K_2 \times G_2$. There are four cases: (i) $u = ax$ and $w = ay$; (ii) $u = bx$ and $w = by$; (iii) $u = ax$ and $w = by$ and (iv) $u = bx$ and $w = ay$, where $x$ and $y$ are two arbitrary vertices in $G_2$. The first two cases are symmetric, and the last two cases are also symmetric. In case (iii) $u$ has a neighbor $u' = bx'$ in $K_2 \times G_2$, where $x' \in N_{G_2}(x)$, thus case (iii) can be reduced to case (ii). So we only need to show that there is an $(ax, ay)$-path in $K_2 \times G_2$, namely case (i). Since $G_2$ is connected, there is an $(x, y)$-path in $G_2$. If there is an $(x, y)$-path $(x, z_1, z_2, \ldots, z_{2k-1}, y)$ of even length in $G_2$, then $(ax, bz_1, az_2, \ldots, bz_{zk-1}, ay)$ is an $(ax, ay)$-path in $K_2 \times G_2$, and so the lemma follows. Suppose below that there is no $(x, y)$-path of even length in $G_2$.

Suppose that at least one of $x$ and $y$, say $x$, lies in an odd cycle $C_0 = (x, w_1, w_2, \ldots, w_{2t}, x)$ in $G_2$ and let $Q = (y, z_1, z_2, \ldots, z_k)$ be a shortest path from $y$ to $C_0$ in $G_2$. Then $V(Q) \cap V(C_0) = z_k$. If $z_k \neq x$, then $Q$ can be extend to an $(x, y)$-path $G_2$ along $C_0$ to $x$ such that it is of even length since $C_0$ is an odd cycle, which contradicts to our hypothesis. Thus, $z_k = x$ and $Q$ is of odd length. Let $k = 2m + 1$, then $Q$ can be extended to an $(x, y)$-trail $Q^* = (x, w_{2t}, \ldots, w_2, w_1, x, z_{2m}, \ldots, z_2, z_1, y)$ in $G_2$. Therefore, $(ax, bw_{2t}, \ldots, bw_2, aw_1, bx, az_{2m}, \ldots, az_2, bz_1, ay)$ is an $(ax, ay)$-path in $K_2 \times G_2$.

Suppose now that neither $x$ nor $y$ lies in any odd cycle in $G_2$. Let $C'_0$ be an arbitrary odd cycle in $G_2$. Choose a shortest path $P_x$ from $x$ to $C'_0$ and a shortest path $P_y$ from $y$ to $C'_0$ in $G_2$ such that they have as many common vertices as possible. If $P_x$ and $P_y$ have no vertices in common, then they can be joint through $C'_0$ to form an even $(x, y)$-path in $G_2$, which contradicts to our hypothesis. Thus, $P_x$ and $P_y$ have vertices in common. We assume that $z_1$ is the first common vertex of $P_x$ and $P_y$. Let $(x, x_1, \ldots, x_r, z_1)$ be the section of $P_x$ from $x$ to $z_1$ and $(y, y_1, \ldots, y_s, z_1)$ be the section of $P_y$ from $y$ to $z_1$. By our hypothesis, $r + s$ is odd certainly. By the choice of $P_x$ and $P_y$, we can suppose that a common section of $P_x$ and $P_y$ from $z_1$ to $z_k$ is $(z_1, z_2, \ldots, z_k)$. So, without loss of generality, assume $r = 2m$, $s = 2h + 1$ and $k = 2n$ (the case that $k$ is odd is similar). Let $C'_0 = (w_1, w_2, \ldots, w_{2t+1}, w_1)$ where $w_1 = z_k$. Then $(ax, bx_1, \ldots, ax_{2m}, bz_1, \ldots, az_{2n}, bw_2, \ldots, aw_{2t+1}, bz_{2n}, \ldots, az_1, by_{2h+1}, \ldots, by_1, ay)$ is an $(ax, ay)$-path in $K_2 \times G_2$. The proof of the lemma is complete. $\square$

**Lemma 4** *Let $G$ be a connected graph, and $H$ be a spanning bipartite subgraph of $G$ with maximum number of edges, then $H$ is connected.*

**Proof.** Let $\{X, Y\}$ be a bipartition of $H$. Suppose to the contrary that $H$ is not connected. Then $H$ can be view as the union of two disjoint bipartite graphs $H_1$ and $H_2$ with partitions $\{X_1, Y_1\}$ and $\{X_2, Y_2\}$, respectively, such that $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$. Then there is neither $(X_1, Y_2)$-edges nor $(X_2, Y_1)$-edges in $G$ since $H$ has maximum number of edges. But $G$ is connected, so there is at least one edge $e$ in $G$ but not in $H$, linking $H_1$ and $H_2$. So $e$ must be an $(X_1, X_2)$-edge or a $(Y_1, Y_2)$-edge. Let $H'$ be the spanning bipartite graph of $G$ induced by the bipartition $\{X_1 \cup Y_2, X_2 \cup Y_1\}$. Note that all edges of $H$ still lie in $H'$, and $H'$ has at least one more edge $e$, a contradiction. $\square$

**Lemma 5** *Let $H$ be a connected bipartite graph and $K_2$ be a complete graph with vertex-set $\{a, b\}$, then $K_2 \times H$ has exactly two components. Moreover, for each $x \in V(H)$, $ax$ and $bx$ are in distinct components of $K_2 \times H$.*

**Proof.** Let $\{X, Y\}$ be a bipartition of $H$. By Lemma 3, $K_2 \times H$ is not connected. The subgraph induced by $(\{a\} \times X) \cup (\{b\} \times Y)$ is isomorphic to $H$, hence is connected and is one component of $K_2 \times H$. The other component is the subgraph induced $(\{b\} \times X) \cup (\{a\} \times Y)$. Thus the lemma follows. □

Note that especially, Lemma 5 is true for $H = K_1$, which is a degenerated bipartite. Let $\beta(G)$ be the minimum number of edges in a subset $F \subset E(G)$ such that $G - F$ is bipartite (including the degenerated bipartite $K_1$). It follows immediately from the definition that $\beta(G) = 0$ if and only if $G$ is bipartite. For each $j \geqslant \lambda$, let $\beta_j(G) = \min\{\beta(C) :$ $C$ is a component of $G - B$ for an edge-cut $B$ consisting of $j$ edges in $G\}$, where the minimum is taken over all components of $G - B$ for any edge-cut $B$ consisting of $j$ edges in $G$. We omit the graph $G$ in the parenthesis of $\beta$ and $\beta_j$ when the underlying graph $G$ is clear by context. Obviously, for a given graph $G$, $\beta_{j+1} \leqslant \max\{\beta_j - 1, 0\}$, $\beta_j \leqslant \beta$ for all $j \geqslant \lambda$, and $\beta_\delta = 0$ (we view $K_1$ as a degenerated bipartite, so $\beta(K_1) = 0$).

**Theorem 3** *Let $H$ be a non-trivial connected graph of edge-connectivity $\lambda$, minimum degree $\delta$, $\beta = \beta(H)$ and $\beta_j = \beta_j(H)$. Then*

$$\lambda(K_2 \times H) = \min\{2\lambda, 2\beta, \min_{j=\lambda}^{\delta}\{j + 2\beta_j\}\}. \tag{2}$$

**Proof.** If $H$ is bipartite, then the lemma holds by Lemma 3 and the fact that $\beta = 0$. So in the rest of the proof, we assume $H$ is non-bipartite. We first prove

$$\lambda(K_2 \times H) \leqslant \min\{2\lambda, 2\beta, \min_{j=\lambda}^{\delta}\{j + 2\beta_j\}\}. \tag{3}$$

To do this, let $B_H$ be a minimum edge-cut of $H$. Then $B = \{(ax, by), (bx, ay) : xy \in B_H\}$ is an edge-cut of $K_2 \times H$ and $|B| = 2|B_H| = 2\lambda$, which implies $\lambda(K_2 \times H) \leqslant 2\lambda$.

Let $F$ be a set of edges consisting of $\beta$ edges in $H$ such that $H - F$ is bipartite. Then $B = \{(ax, by), (bx, ay) : xy \in F\}$ is an edge-cut of $K_2 \times H$ and $|B| = 2|F| = 2\beta$ since $K_2 \times H - B = K_2 \times (H - F)$ is the direct product of two bipartite graphs. This fact shows $\lambda(K_2 \times H) \leqslant 2\beta$.

Now, for each $\lambda \leqslant j \leqslant \delta$, let $B_j$ be an edge-cut consisting of $j$ edges of $H$, and $C_j$ a component of $H - B_j$ with $\beta(C_j) = \beta_j$. Hence there is a set of edges $F_j$ of $C_j$ such that $|F_j| = \beta_j$ and $C_j - F_j$ is bipartite. Let

$$B' = \{(ax, by), (bx, ay) : xy \in F_j\}.$$

Then $(K_2 \times C_j) - B' = K_2 \times (C_j - F_j)$ is the direct product of two bipartite graphs and, hence, disconnected. By Lemma 4, $C_j - F_j$ is a connected bipartite graph and, hence, by Lemma 5, $K_2 \times (C_j - F_j)$ has exactly two components and $ax$ and $bx$ are in distinct components for each $x \in V(C_j)$. Let $C$ be a component of $(K_2 \times C_j) - B'$. Define a injection mapping $\varphi$ from $B_j$ to $E(K_2 \times H)$ as follows: for each edge $e = xy \in B_j$ with $x \in V(C_j)$, $\varphi(e) = (ax, by)$ if $ax \in V(C)$; and $\varphi(e) = (bx, ay)$ if $ax \notin V(C)$(which implies $bx \in V(C)$). Let

$$B'' = \varphi(B_j).$$

Then $B' \cup B''$ is an edge-cut of $K_2 \times H$ since $C$ is a component of $(K_2 \times H) - (B' \cup B'')$. And $|B' \cup B''| = |B'| + |B''| = 2|F_j| + |B_j| = j + 2\beta_j$, which implies that $\lambda(K_2 \times H) \leqslant \min_{j=\lambda}^{\delta}\{j + 2\beta_j\}$, and so the inequality (3) follows.

Next, we will show

$$\lambda(K_2 \times H) \geqslant \min\{2\lambda, 2\beta, \min_{j=\lambda}^{\delta}\{j + 2\beta_j\}\}. \tag{4}$$

Let $B = [S, \overline{S}]$ be a minimum edge-cut of $K_2 \times H$. Partition the vertex-set $V(H)$ into four parts:

$$P = \{x \in V(H) : ax \in \overline{S}, bx \in \overline{S}\}, \quad Q = \{x \in V(H) : ax \in S, bx \in S\},$$
$$R = \{x \in V(H) : ax \in S, bx \in \overline{S}\}, \quad T = \{x \in V(H) : ax \in \overline{S}, bx \in S\}.$$

And let $Z = R \cup T$. We prove the inequality (4) by considering the following four cases, respectively.

*Case 1:* $Z = \emptyset$, then $P \neq \emptyset$ and $Q \neq \emptyset$. Hence

$$|B| \geqslant 2|[P, Q]| \geqslant 2\lambda.$$

*Case 2:* $Z \neq \emptyset$, $P \neq \emptyset$ and $Q \neq \emptyset$. Without loss of generality, we may assume $|E_H(P, Z)| \leqslant |E_H(Q, Z)|$. Note that $[P, Q \cup Z]$ is an edge-cut of $H$, so $|[P, Q \cup Z]| \geqslant \lambda$. For each edge $xy \in [P, Q]$, we can see that both the edges $(ax, by)$ and $(ay, bx)$ are in $B$. For each $xy \in [P, Z]$ or $xy \in [Q, Z]$, exactly one of $(ax, by)$ and $(ay, bx)$ is in $B$. Thus,

$$\begin{aligned} |B| &\geqslant 2|[P, Q]| + |[P, Z]| + |[Q, Z]| \\ &\geqslant 2|[P, Q]| + 2|[P, Z]| \\ &= 2|[P, Q \cup Z]| \\ &\geqslant 2\lambda. \end{aligned}$$

*Case 3:* $Z \neq \emptyset$, $P = Q = \emptyset$. Then for each edge $xy \in E(G[R])$ or $xy \in E(G[T])$, both the edges $(ax, by)$ and $(ay, bx)$ are in $B$. Note that

$H - (E(G[R]) \cup E(G[T]))$ is bipartite, hence

$$|B| = 2(|E(G[R])| + |E(G[T])|) \geqslant 2\beta.$$

*Case 4:* $Z \neq \emptyset$, and exactly one of $P$ and $Q$ is empty. By the symmetry, we may assume that $P \neq \emptyset$ and $Q = \emptyset$. Let $C$ be a maximally connected subgraph of $H$ such that $V(C) \subseteq Z$, and let $R' = R \cap V(C)$ and $T' = T \cap V(C)$. Finally let $X = N_H(C)$, then $X \subseteq P$ by the maximality of $C$. Then

$$|B| \geqslant |[X, V(C)]| + 2(|E(G[S'])| + |E(G[T'])|). \tag{5}$$

Let $k = |[X, V(C)]|$. If $k > \delta$, then by (5),

$$|B| > \delta = \delta + 2\beta_\delta \geqslant \min_{j=\lambda}^{\delta}\{j + 2\beta_j\}.$$

If $k \leqslant \delta$, by (5) we have

$$|B| \geqslant k + 2\beta_k \geqslant \min_{j=\lambda}^{\delta}\{j + 2\beta_j\}.$$

Thus, the proof of the theorem is complete. □

We conclude by mention that each item of the right side of equation (2) cannot be omitted, since it is possible to find a graph with one item, say $2\beta$, strictly less than other items. Such examples are easy to construct so we do not give them here.

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