ON THE AVERAGE CROSSCAP NUMBER OF A GRAPH

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ABSTRACT. The average crosscap number of a graph G is the expected value of crosscap number random variable, over all labeled 2-cell non-orientable embeddings of G. In this study, some experiment results for average crosscap number are obtained. We calculate all average crosscap number of graphs with Betti number less that 5. As a special case, the smallest ten values of average crosscap number are determined. The distribution of average crosscap number of all graphs in R is sparse. Some structure theorems for average crosscap number with a given or bounded value are provided. The exact values of average crosscap number of Cacti and Necklaces are determined. The crosscap number distributions of Cacti and Necklaces of type $(r, 0)$ are proved to be strongly unimodal, and the mode of embedding distribution sequence is upper-rounding or lower-rounding of its average crosscap number. Some open problems are also proposed.

1. INTRODUCTION

The *average genus* $\gamma_{avg}(G)$ of a graph G is the expected value of genus random variable, over all labeled 2-cell orientable embeddings of G, using the uniform distribution. This concept is investigated by various authors, the contributions include [1, 2, 3, 4, 6, 11, 16, 19, 20, 23, 24, 25] and [25]. In [16], Gross, Klein and Rieper investigated average genus of an individual graph and asked for an analogous theory about non-orientable embeddings of G. The average crosscap number is little known, compared to the average genus. In this paper we will investigate the average crosscap number of individual graphs.

It is assumed that the reader is at least somewhat familiar with the basics of topological graph theory, as found in Gross and Tucker [18]. A *graph* $G = (V(G), E(G))$ is permitted to have loops and multiple edges.

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A *surface* is a compact 2-dimensional manifold, without boundary. In topology, surfaces are classified into the *orientable surfaces* O_g, with g handles ($g \geq 0$), and the *non-orientable surfaces* N_k, with k crosscaps ($k > 0$). The graph embeddings under discussion here are *cellular embeddings.* For any spanning tree of a graph G, the number of co-tree edges is called the *Betti number* of G, or the *cycle rank* of G, and is denoted by $\beta(G)$.

1.1. **General rotation system.** A *rotation at a vertex v* of a graph G is a cyclic ordering of all the edge-ends (or equivalently, the half-edges) incident with v. A *pure rotation system* ρ of a graph G is the collection of rotations at all the vertices of G. An embedding of G into an orientable surface S induces a pure rotation system as follows: the rotation of the edge-ends at v is the cyclic permutation corresponding to the order in which the edge-ends are encountered in an orientation-preserving tour around v. Conversely, by the *Heffter-Edmonds principle*, every rotation system induces a unique embedding (up to homeomorphism) of G into some oriented surface S. The bijectivity of this correspondence implies that the total number of oriented embeddings is $\prod_{v \in G}(d_v - 1)!$, where d_v is the degree of vertex v.

A *general rotation system* for a graph G is a pair (ρ, λ), where ρ is a pure rotation system and λ is a mapping $E(G) \to \{0,1\}$. The edge e is said to be *twisted* (respectively, *untwisted*) if $\lambda(e) = 1$ (respectively, $\lambda(e) = 0$). It is well-known that every oriented embedding of a graph G can be described by a general rotation system (ρ, λ) with $\lambda(e) = 0$, for all $e \in E(G)$.

1.2. **Average crosscap number.** By allowing λ to take non-zero values, we can describe the non-orientable embeddings of G. For any fixed spanning tree T, a *T-rotation system* (ρ, λ) of G is a general rotation system (ρ, λ) such that $\lambda(e) = 0$, for all $e \in E(T)$. Any two embeddings of G are considered to be the *same* if their T-rotation systems are combinatorially equivalent.

Let Φ_G^T denote the set of all T-rotation systems of G. It is known that

$$|\Phi_G^T| = 2^{\beta(G)} \prod_{v \in V(G)} (d_v - 1)!$$

This implies that the number of non-orientable embeddings of G is

$$(2^{\beta(G)} - 1) \prod_{v \in V(G)} (d_v - 1)!$$

Suppose that among these $|\Phi_G^T|$ embeddings of G, there are a_i embeddings, for $i = 0, 1, \ldots$, into the orientable surface S_i, and there are b_j embeddings, for $j = 1, 2, \ldots$, into the non-orientable surface N_j. We call the bivariate polynomial

$$\mathbb{I}_G^T(x,y) = \sum_{i=0}^{\infty} a_i x^i + \sum_{j=1}^{\infty} b_j y^j$$

the *T-distribution polynomial* of G.

By the *total embedding distribution polynomial* of G, we mean the bivariate polynomial $\mathbb{I}_G(x,y) = \mathbb{I}_G^T(x,y)$, for any given spanning tree T. We call the first and second parts of $\mathbb{I}_G(x,y)$ the *genus polynomial* and the *crosscap number polynomial* of G, respectively, and we denote them by $g_G(x) = \sum_{i=0}^{\infty} a_i x^i$ and $f_G(y) = \sum_{i=1}^{\infty} b_i y^i$, respectively. Thus, $\mathbb{I}_G(x,y) = g_G(x) + f_G(y)$.

Definition 1.1. The *average crosscap number* ((or ACN in short)$\widetilde{\gamma}_{avg}(G)$ of a graph G is the ratio of the sum of all crosscap numbers of the non-orientable embeddings over the number of the non-orientable embeddings of G.

1.3. **Joint tree method.** By a polygon with r edges, we shall mean a 2-cell which has its circumference divided into r arcs by r vertices. In fact, a surface can be obtained by pairing the edges of a polygon and identifying the two edges in each pair. The following three operations [21] on a cyclic string representing such a polygon do not change genus of such a surface.

Operation 1: $Aaa^- \sim A$,

Operation 2: $AabBab \sim AcBc$,

Operation 3: $AB \sim \{(Aa),(a^-B)\}$,

where A and B are all linear order of letters.

We have the following relations [21].

Relation 1: $AaBbCa^-Db^-E \sim ADCBEaba^-b^-$.

Relation 2: $AxBxC \sim AB^-Cxx$,

Relation 3: $Axxyzy^-z^- \sim Axxyyzz$.

Relation 1 is also called *handle normalization*, Relation 2 and Relation 3 are called crosscap normalization. In three relations, A, B and C are permitted to be empty. B^- is the inverse of B. By Relations 1, 2 and 3, we can obtain the normal form of a surface as one, and only one, of: $O_0 = aa^-, O_m = \prod_{i=1}^{m} a_i b_i a_i^- b_i^- (m > 0), N_n = \prod_{i=1}^{n} a_i a_i (n > 0)$.

The joint-tree approach [21] is an alternative to the Heffter-Edmonds algorithm for calculating the genus of the surface associated with a given rotation system. The rotation system is what combinatorializes the topological problem; a joint tree can be regarded as the combination of a spanning tree and a rotation system. Given a T-rotation system (P,λ) of G, the associated joint tree, denoted by G_T, which is obtained by the following two cases: If $\lambda(e) = 0$ for a co-tree edge e, we split e into two semi-edges e and e^-, otherwise $\lambda(e) = 1$, we split e into two semi-edges e and e. Then we travel along the rotations, all semi-edges of G_T form a polygon A with $\beta(G)$ pairs of edges. Finally, we apply Relations 1, 2 and 3 and Operations 1, 2 and 3 to normalize the polygon A and get the genus or crosscap number of the embedding. Based on joint trees, the topological problem for determining embeddings of a graph is transformed into a combinatorial problem. For more details, we can also refer to [27, 28].

Example 1. Given graph G=(V,E), $V = \{v_1, v_2, v_3, v_4\}, E = \{a, b, c, d, e, f\}$, a, b and d are edges on T, c, e and f are co-tree edges. The rotation system R at each vertex is counterclockwise: $v_1(dea)$, $v_2(afb)$, $v_3(bec)$, $v_4(cfd)$. We travel along on G_T according to the rotation system and obtain the polygon $c^{-}cfef^{-}e^{-} \sim fef^{-}e^{-}$, which is an embedding of G into torus (See Figure 1).

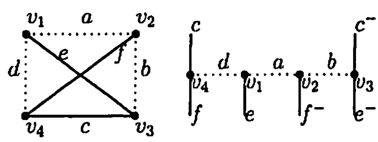

FIGURE 1. The graph G and it's joint tree G_T.

If we associate the graph G with the T-rotations (P,λ) as follows: $\lambda(c) = 1$, and $\lambda(e) = 0$, for all $e \in E - \{c\}$. In this case, we have $ccfef^{-}e^{-} \sim ccffee$ (*Relation* 3) $\sim N_3$. In other words, the T-rotations (P,λ) corresponding to an embedding into N_3.

The outline of this study is as follows. In Section 2, a theorem on the ACN of a bar-amalgamation of two graphs G and a splitting theorem for ACN are provided. In Section 3, all ACN for graphs with Betti number less that 5 are given, and structures for the ACN of G with a given or bounded value are provided. In Section 4, we show that the distribution of ACN of all graphs is sparse in R. Different examples are constructed to demonstrate that a single value of ACN can be shared by arbitrarily many different graphs are discussed in Section 5. In Section 6, we investigate the relationship between the ACN and the mode of embedding distribution of general graphs. Finally, in Section 7 we mention some problems which further research on this topic might follow and a table of the average genus and ACN is present.

2. STRUCTURE FOR A GRAPH WITH ACN

2.1. The ACN of bar-amalgamation of two disjoint graphs.

A *bar-amalgamation* of two disjoint graphs H and G is obtained by running an edge between a vertex of G and a vertex of H. Gross and Frust [17] proved the genus polynomial for the bar-amalgamation of two disjoint rooted graphs (G, u) and (H, v) is a constant multiple of the product of the genus polynomial for graph G and H. The constant factor equals the valence of u in G times the valence of v in H.

Theorem 2.1. *(See [17])* $g_{G\oplus_e H}(x) = d_G(u)d_H(v)g_G(x)g_H(x)$.

By Theorem 2.1, we have:

Corollary 2.2. *(See [4])* $\gamma_{\text{avg}}(G \oplus_e H) = \gamma_{\text{avg}}(G) + \gamma_{\text{avg}}(H)$.

We have:

Theorem 2.3. $f_{G\oplus_e H}(y) = d_G(u)d_H(v)\Big[f_G(y)f_H(y) + f_G(y)g_H(y^2) + g_G(y^2)f_H(y)\Big]$, *where* $d_G(u)$ *is the vertex degree of* u *in* G *and* $d_G(v)$ *is the vertex degree of* v *in* H.

Proof. Let P_G (P_H) be the pure rotation systems of the graph G (H). Then every pure rotation system $P_{G\oplus_e H}$ can be obtained by inserting the edge uv and taking the union of the two adjusted systems.

Since the total genus polynomial of a graph is independent of the choice of its spanning tree, we arbitrarily choose a spanning tree T_G (T_H) of G (H). For a general rotation system (P_G, λ) of G (H) such that $\lambda(e) = 0, \forall e \in E(T_G)$ $((P_H, \lambda)$ *such that* $\lambda(e) = 0, \forall e \in E(T_H))$. Then, replace each edge not of T_G (T_H) by two semi-edges with the same letter. we label the same letter with the same or distinct indices according to $\lambda(e) = 1$ or $\lambda(e) = 0$. So we get the joint tree $G_1(H_1)$ of $G(H)$. It is evident that the tree $T_G \oplus_e T_H$ is a spanning tree of $G \oplus_e H$. So we get the general rotation system $(P_{G\oplus_e H}, \lambda)$ of $G \oplus_e H$ where $\lambda(e) = 0, \forall e \in E(T_G \oplus_e T_H)$ and $\lambda(e) = 1$,if $\lambda(e) = 1$ in (P_G, λ) or (P_H, λ). Similarly, we replace each edge not of $T_G \oplus_e T_H$ by two semi-edges with the same letter. we label the same letter with the same or distinct indices according to $\lambda(e) = 1$ or $\lambda(e) = 0$. So we get a joint tree $(G \oplus_e H)_1$ of $G \oplus_e H$.

From above discussion and according to that rotation, all lettered semi-edges of $G_1(H_1)$ form a polygon $A(B)$ with $\beta(G)(\beta(H))$ pairs of edges. From the definition of $(G \oplus_e H)_1$, we similarly get the polygon AB with $\beta(G \oplus_e H)$ pairs of edges. Now let's consider how to construct the embedding into a non-orientable surface of the bar-amalgamation that corresponds to that of general rotation system.

Case 1 The original G general rotation system corresponds to an embedding of G into the surface N_k and that the original H general rotation system corresponds to an N_h. Or equivalently

$$A \sim \prod_{i=1}^{k} a_i a_i \text{ and } B \sim \prod_{i=1}^{h} b_i b_i$$

By Relation 2

$$AB \sim \prod_{i=1}^{k} a_i a_i \prod_{i=1}^{h} b_i b_i \sim \prod_{i=1}^{k+h} a_i a_i$$

So the crosscap number of this resulting surface on which the graph $G \oplus_e H$ embeds is $k+h$.

Case 2 The original G general rotation system corresponds to an embedding of G into the surface O_k and the original H general rotation system corresponds to an embedding of H into the surface N_h. Or equivalently

$$A \sim \prod_{i=1}^{k} a_i b_i a_i^- b_i^- (k > 0) \text{ or } A \sim aa^- (k = 0) \text{ and } B \sim \prod_{i=1}^{h} b_i b_i$$

By Relations 3 and Operation 1

$$AB \sim \prod_{i=1}^{k} a_i b_i a_i^- b_i^- \prod_{i=1}^{h} b_i b_i \sim \prod_{i=1}^{2k+h} a_i a_i$$

or

$$AB \sim aa^- \prod_{i=1}^{h} b_i b_i \sim \prod_{i=1}^{2\times 0+h} a_i a_i$$

So The crosscap number of this resulting surface on which the graph $G \oplus_e H$ embeds is $2k+h$.

Case 3 The original G general rotation system corresponds to an embedding of G into the surface N_k and the original H general rotation system corresponds to an embedding of H into the surface O_h. By symmetry and Case 2, we know the crosscap number of this resulting surface on which the graph $G \oplus_e H$ embeds is $k+2h$.

□

Corollary 2.4.

$$\widetilde{\gamma}_{avg}(G \oplus_e H) = \widetilde{\gamma}_{avg}(G) + \widetilde{\gamma}_{avg}(H) + \frac{2^{\beta(H)}-1}{2^{\beta(G)+\beta(H)}-1}\Big(2\gamma_{avg}(G) - \widetilde{\gamma}_{avg}(G)\Big) + \frac{2^{\beta(G)}-1}{2^{\beta(G)+\beta(H)}-1}\Big(2\gamma_{avg}(H) - \widetilde{\gamma}_{avg}(H)\Big).$$

Proof. Since $\widetilde{\gamma}_{avg}(G) = \frac{f_G'(1)}{f_G(1)}$, by theorem 2.1, we have the following formula

$$\widetilde{\gamma}_{avg}(G \oplus_e H) = \frac{2^{\beta(H)}(2^{\beta(G)}-1)}{2^{\beta(G)+\beta(H)}-1}\widetilde{\gamma}_{avg}(G) + \frac{2^{\beta(G)}(2^{\beta(H)}-1)}{2^{\beta(G)+\beta(H)}-1}\widetilde{\gamma}_{avg}(H) + \frac{2(2^{\beta(G)}-1)}{2^{\beta(G)+\beta(H)}-1}\gamma_{avg}(H) + \frac{2(2^{\beta(H)}-1)}{2^{\beta(G)+\beta(H)}-1}\gamma_{avg}(G).$$

which is equivalent to the formula of the corollary. □

2.2. **A splitting theorem for ACN of a graph.**

Definition 2.5. Suppose the graph $G = (V, E)$ is simple. Let u be a vertex of G of valence $d(u) = d+1 > 3$ and $v, v_1, v_2, \ldots, v_d$ be its neighbors ($u \in V$). We denote the edge uv_i by e_i, for $i = 1, 2, \ldots, d$, and the edge uv by f. The graph $G_{i_1,\ldots,i_k}$ is called a *k-degree proper splitting* of G at u if it can be obtained from $G - u$ by adjoining v, $v_{i_1}, \ldots, v_{i_k}$ to a new vertex x, adjoining all the other ex-neighbors of u to a new vertex y ($i_l \in \{1, 2, \ldots, k\}$, for $l = 1, 2, \ldots, k$ and $d > k \geq 1$), and finally adjoining x and y.

The new vertex x is $(k+2)$-valent for each $G_{i_1,\ldots,i_k}$ and the new vertex y is $(d-k+1)$-valent. Let Λ be the set of all graphs $G_{i_1,\ldots,i_k}$, then the number of elements in Λ is $\binom{d}{k}$. It is obvious that each graph $G_{i_1,\ldots,i_k}$ has the same the Betti number as that of G, and they can contract the new edge xy to get the graph G. Figure 2 gives an example of a 2-degree proper splitting of G at u.

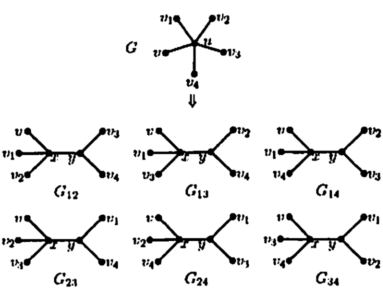

FIGURE 2. The 2-degree proper splitting of G at u with a designate neighbor v

Lemma 2.6. [13] *Let G be a connected graph with a vertex of valence $d+1$, $d \geq 3$, $G_{i_1,i_2,\ldots,i_k}$, $i_j \in \{1, 2, \ldots, d\}$, be a graph obtained by k-degree properly splitting at vertex u, and let Λ be the set of all such graphs $G_{i_1,i_2,\ldots,i_k}$. Then we have*

$$f_G(x) = \frac{1}{k+1} \sum_{G_{i_1,i_2,\ldots,i_k} \in \Lambda} f_{G_{i_1,i_2,\ldots,i_k}}(x).$$

Theorem 2.7. *Let G be a connected graph with a vertex u of valence $d+1(d \geq 3)$, and let $G_{i_1,i_2,\ldots,i_k}$ ($i_j \in \{1, 2, \ldots, d\}$) be graphs obtained by k-degree properly splitting at vertex u, and Λ be the sets of all the graphs $G_{i_1,i_2,\ldots,i_k}$. Then we have $\widetilde{\gamma}_{avg}(G) = \frac{1}{\binom{d}{k}} \sum_{G_{i_1,i_2,\ldots,i_k} \in \Lambda} \widetilde{\gamma}_{avg}(G_{i_1,i_2,\ldots,i_k})$.*

Proof. It is directly from Lemma 2.6. □

3. ACN for graphs with Betti Number less that 5

3.1. 2-edge-connected graph with small average crosscap number. Gross, Klein and Rieper [16] have studied 3-connected graphs with small average genus. They proved K_4 is the only 3-connected graph of graphs less than 1. Chen and Gross [4] characterized all 2-connected graphs of average genus less than 1. They also obtained the Kuratowski type theorems for average genus. In [4], obtained the following result.

Theorem 3.1. *(see [4]) The ACN of a graph is not less than the ACN any of its subgraphs.*

By Theorem 3.1, it is obvious that the number 1 is the smallest one. So we turn to other the smallest positive numbers. It is should be noted that all the computations are aided by a computer program.

Definition 3.2. Let $G = (V, E)$ be a graph. A *linear synthesis* $L = [P_0, P_1, \ldots, P_r]$ of G is a partition of E into an ordered collection of edge disjoint subgraphs $P_0, P_1, \ldots, P_r$ of G, such that P_0 is a subgraph, each P_i is a simple path for, $i = 1, \ldots, r$, and each endpoint of P_i, $i = 1, \ldots, r$, is contained in some P_j, with $j < i$. The $P_i^{\prime}s$, $i = 1, \ldots, r$, are called the paths of L and r is called the length of L. We call L an *open linear synthesis* if all paths P_i, $i = 1, \ldots, r$, are not simple cycles. Otherwise we call L an *closed linear synthesis.*

In [4], proved the following theorem.

Theorem 3.3. *(see [4]) A graph G has a linear synthesis from a simple cycle if and only if G is 2-edge-connected.*

In this subsection, we will calculate the ACN of all 2-edge-connected graphs with Betti number less than 5. By Theorem 3.3, we can begin our process by adding a path to a simple cycle. Since adding a path to a graph increase the cycle rank of the graph by 1 and the length of a linear synthesis of a graph from a simple equals the dimension of its cycle subspace minus 1, we only need to add at most 3 path to a simple cycle in order to obtain a graph of Betti number at most 4. Since the ACN is a topological invariant under graph homeomorphism, we can suppose all the graphs in this section with minimum degree at least 3.

Definition 3.4. We define the class of graphs $\mathbb{C}_k$ to be the classes of 2-edge connected graphs with Betti number k.

It should be note that the class $\mathbb{C}_{k+1}$ can be obtained from $\mathbb{C}_k$ by adding a path to the graphs which contained in $\mathbb{C}_k$. It is obvious that the class $\mathbb{C}_1$ is the graph B_1 which homeomorphic to a simple cycle. In other words, $\mathbb{C}_1 = \left\{ B_1 \right\}$. It is a routine task to compute the ACN of B_1 and we get $\widetilde{\gamma}_{avg}(B_1) = 1$

Now we obtain the classes of graphs which contained in $\mathbb{C}_2$. By Theorem 3.3, there are 2 nonhomeomorphic graphs that can be obtained by adding a path to a simple cycle. One is adding an open path to the simple cycle, the other is adding a closed path to a simple cycle. Actually they are graphs B_2 and D_3. In other words, we have $\mathbb{C}_2 = \left\{ B_2, D_3 \right\}$ (See Figure 3). We

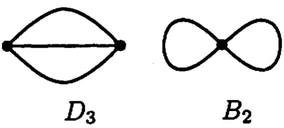

FIGURE 3. Two graphs B_2 and D_3

have:

$$\widetilde{\gamma}_{avg}(B_2) = 1\frac{4}{9}, \quad \widetilde{\gamma}_{avg}(D_3) = 1\frac{1}{2}$$

Remark 3.5. By Theorem 2.7, $\widetilde{\gamma}_{avg}(B_2) = \frac{1}{3}\left(\widetilde{\gamma}_{avg}(B_1 \oplus_e B_1) + 2\widetilde{\gamma}_{avg}(D_2)\right)$.

Now we consider the graphs in $\mathbb{C}_3$. We first adding a path to the graph D_3. There are three nonhomeomorphic graphs that can be obtained by adding an open path e to D_3: if one end of e is connected to a vertex v of D_3, then the other end of e could be (1) connected to another vertex u of D_3, (2) connected to the middle of any edge of D_3. If neither end of e is connected to a vertex of D_3, then either (3) two ends of e are connected to middle of any two edges of three edges of D_3, or (4) two ends of e are connected to the middle of an edge of three edges of D_3. There are two nonhomeomorphic graphs that can be obtained by adding an closed path f to D_3: (1) the vertex of f connected to the vertex of D_3, (2) the vertex of f connected to the middle of an edge of D_3 (Figure 4 lists all these five graphs). We call these graphs $D_{31}, D_{32}, D_{33}, D_{34}, D_{35}$, and D_{36}, respectively.

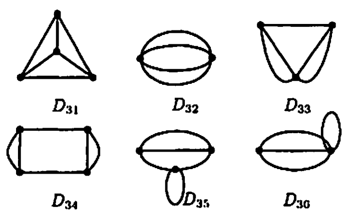

FIGURE 4. Six graphs $D_{31}, D_{32}, D_{33}, D_{34}, D_{35}$, and D_{36}

It is routine task to compute the ACN of graphs $D_{31}, D_{32}, \ldots, D_{36}$ and we get

$$\begin{array}{llllll} \widetilde{\gamma}_{avg}(D_{31}) = & 2\frac{3}{8} & \widetilde{\gamma}_{avg}(D_{32}) = & 2\frac{1}{3} & \widetilde{\gamma}_{avg}(D_{33}) = & 2\frac{1}{3} \\ \widetilde{\gamma}_{avg}(D_{34}) = & 2\frac{1}{4} & \widetilde{\gamma}_{avg}(D_{35}) = & 2\frac{1}{6} & \widetilde{\gamma}_{avg}(D_{36}) = & 2\frac{1}{4} \end{array}$$

Now we consider the graph B_2 and adding path to B_2. There are three nonhomeomorphic graphs that can be obtained by adding an open path e to B_2: (1) one end of e is connected to the vertex v of B_2, the other end of e the middle of an edge of any two edges of B_2, (2) one end of e is connected to the middle of the edge e of B_2, the other end of e is connected to the middle of the other edge f of B_2. (3) the two ends of e are connected the middle of the same edge of B_2. There are two nonhomeomorphic graphs that can be obtained by adding an closed path f to B_2: (1) the vertex of f connected to the vertex of B_2, (2) the vertex of f connected to the middle of an edge of B_2. Figure 6 lists all these five graphs. We call these graphs $B_{21}, B_{22}, B_{23}, B_{24}$, and B_{25}, respectively.

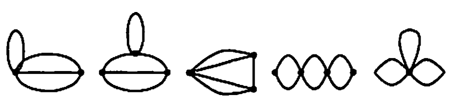

FIGURE 5

It should be note that the graph B_{21} is homeomorphic to the graph D_{36} which has ACN $2\frac{1}{4}$. The graph B_{22} is homeomorphic to the graph D_{35} which has ACN $2\frac{1}{6}$. B_{23} is homeomorphic to the graph D_{33} which has ACN $2\frac{1}{3}$. We have:

$$\widetilde{\gamma}_{avg}(B_{24}) = 2\frac{5}{63}, \quad \widetilde{\gamma}_{avg}(B_{25}) = 2\frac{19}{105}$$

In other words, we have $\mathbb{C}_3 = \left\{ D_{31}, D_{32}, \ldots, D_{36}, B_{24}, B_{25} \right\}$

Remark 3.6. By Theorem 2.7, we have $\widetilde{\gamma}_{avg}(D_{32}) = \frac{1}{3}\left(\widetilde{\gamma}_{avg}(B_1 \oplus_e D_{34}) + 2\widetilde{\gamma}_{avg}(D_{31})\right)$, $\widetilde{\gamma}_{avg}(D_{35}) = \frac{1}{3}\left(\widetilde{\gamma}_{avg}(B_1 \oplus_e D_3) + 2\widetilde{\gamma}_{avg}(D_{34})\right)$, etc.

By Theorem 3.1, we have the following result.

Theorem 3.7. *Let G be a 2-edge connected graph. Then G has ACN less than or equal to 2 if and only if G is homeomorphic to B_1, B_2 or D_3.*

Theorem 3.8. *A real number r in the half open interval $[1, 2)$ is a value of ACN of a 2-edge connected graph if and only if r is equal to* $1, 1\frac{4}{9}$, *or* $1\frac{1}{2}$.

In a similar discussion like above, we obtain all 2-edge connected graphs with Betti number 4. See Figure 6.

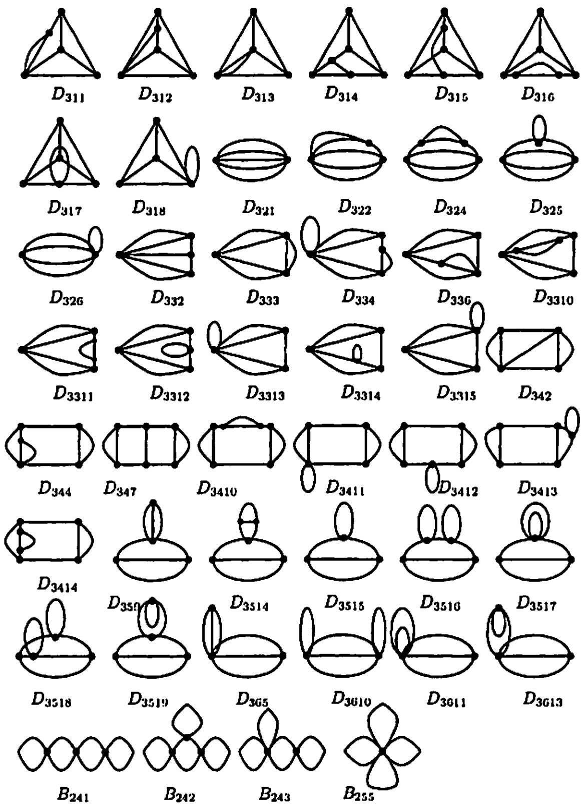

FIGURE 6. 2-edge connected graphs with Betti number 4

The ACN of the graphs D_{311}, D_{312}, $\ldots$, D_{318} be $3\frac{7}{36}$, $3\frac{91}{360}$, $3\frac{7}{30}$, $3\frac{113}{480}$ $3\frac{23}{80}$, $3\frac{9}{80}$, $3\frac{11}{360}$, $3\frac{9}{80}$ respectively. The ACN of the graphs D_{321}, D_{322}, D_{324}, D_{325} and D_{326} be $3\frac{7}{36}$, $3\frac{7}{36}$, $3\frac{7}{90}$, $2\frac{134}{135}$, $3\frac{3}{25}$ respectively. The ACN of the graphs D_{332}, D_{333}, D_{334}, D_{336}, D_{3310}, D_{3311}, $\ldots$, D_{3315} be $3\frac{7}{36}$, $3\frac{83}{405}, 3\frac{4}{27}, 3\frac{7}{45}, 3\frac{7}{90}, 3\frac{1}{18}, 2\frac{44}{45}, 2\frac{134}{135}, 3\frac{19}{270}$, respectively. The ACN of the graphs $D_{342}, D_{344}, D_{347}, D_{3410}, D_{3411}, \ldots, D_{3414}$ be $3\frac{7}{45}$, $3\frac{7}{90}$, $3\frac{1}{120}$, $2\frac{113}{120}$, $2\frac{71}{72}$, $2\frac{157}{180}$, $2\frac{11}{12}, 3\frac{1}{120}$. The ACN of the graphs $D_{359}, D_{3514}, D_{3515}, \ldots, D_{3519}$ be $3\frac{4}{27}, 2\frac{169}{180}$, $2\frac{487}{540}, 2\frac{43}{54}, 2\frac{139}{150}, 2\frac{223}{270}, 2\frac{223}{270}$, respectively. The ACN of the graphs D_{365}, D_{3610},

D_{3611}, D_{3613} be $3\frac{3}{25}, 2\frac{71}{72}, 3\frac{1}{18}, 2\frac{11}{12}$, respectively. The ACN of the graphs $B_{241}, B_{242}, B_{243}, B_{255}$ be $2\frac{301}{405}, 2\frac{289}{405}, 2\frac{113}{135}, 2\frac{1573}{1575}$.

Remark 3.9. By Theorem 2.7, we have $\widetilde{\gamma}_{avg}(D_{311}) = \frac{1}{3}\Big(\widetilde{\gamma}_{avg}(D_{316}) + 2\widetilde{\gamma}_{avg}(D_{314})\Big)$, $\widetilde{\gamma}_{avg}(D_{312}) = \frac{1}{3}\Big(\widetilde{\gamma}_{avg}(D_{315}) + 2\widetilde{\gamma}_{avg}(D_{314})\Big)$, $\widetilde{\gamma}_{avg}(D_{313}) = \frac{1}{3}\Big(\widetilde{\gamma}_{avg}(D_{311}) + 2\widetilde{\gamma}_{avg}(D_{312})\Big)$, etc.

By Theorem 3.1 and the above discussion, we have the following result.

Theorem 3.10. *Let G be a 2-edge connected graph. Then G has ACN less than or equal to 3 if and only if G is homeomorphic to B_1, B_2, D_2, $D_{31}, \ldots, D_{36}$, B_{24}, B_{25}, D_{325}, D_{3312}, D_{3314}, D_{3410}, D_{3411}, D_{3412}, D_{3413}, D_{357}, D_{3610}, D_{3613}, B_{243}, or B_{255}.*

Theorem 3.11. *A real number r in the open interval $(2,3)$ is a value of ACN of a 2-edge connected graph if and only if r is equal to $2\frac{5}{65}$, $2\frac{19}{105}$, $2\frac{1}{6}$, $2\frac{1}{4}$, $2\frac{1}{3}$, $2\frac{3}{8}$, $2\frac{289}{405}$, $2\frac{301}{405}$, $2\frac{43}{54}$, $2\frac{223}{270}$, $2\frac{113}{135}$, $2\frac{157}{180}$, $2\frac{11}{12}$, $2\frac{139}{150}$, $2\frac{113}{120}$, $2\frac{44}{55}$, $2\frac{71}{72}$, $2\frac{134}{135}$, or $2\frac{1573}{1575}$.*

3.2. **1-edge-connected graphs with small average crosscap number.** In this subsection, we will calculate the graphs with ACN not larger than 3.

Definition 3.12. Defined $\mathbb{E}_i$ to be the set of all 1-edge-connected graphs with Betti number i $(i \geq 2)$.

The graphs of $\mathbb{E}_i$ can be obtained by the bar-amalgamation of two disjoint graphs with smaller Betti number. Since $\mathbb{C}_1 = \Big\{B_1\Big\}$, $\mathbb{C}_2 = \Big\{B_2, D_3\Big\}$ and $\mathbb{C}_3 = \Big\{D_{31}, D_{32}, \ldots, D_{36}, B_{24}, B_{25}\Big\}$, we have $\mathbb{E}_2 = \mathbb{C}_1 \oplus_e \mathbb{C}_1$, $\mathbb{E}_3 = \mathbb{C}_1 \oplus_e \mathbb{C}_2$ and $\mathbb{E}_4 = \mathbb{C}_1 \oplus_e \mathbb{C}_3 \bigcup \mathbb{C}_2 \oplus_e \mathbb{C}_2$

In [4], calculated the following:

$$\begin{array}{llll} \gamma_{avg}(B_1) = 0 & \gamma_{avg}(B_2) = \frac{1}{3} & \gamma_{avg}(D_3) = \frac{1}{2} & \gamma_{avg}(D_{31}) = \frac{7}{8} \\ \gamma_{avg}(D_{32}) = \frac{5}{6} & \gamma_{avg}(D_{33}) = \frac{5}{6} & \gamma_{avg}(D_{34}) = \frac{3}{4} & \gamma_{avg}(D_{35}) = \frac{2}{3} \\ \gamma_{avg}(D_{36}) = \frac{9}{4} & \gamma_{avg}(B_{24}) = \frac{5}{9} & \gamma_{avg}(B_{25}) = \frac{2}{3} & \end{array}$$

Let $G^2_{\oplus_e}$ be the graph of bar-amalgamation of G and G. For $n \geq 3$, we define $G^n_{\oplus_e}$ be the graph of bar-amalgamation of G and $G^{n-1}_{\oplus_e}$. By Corollary 2.4, we have

$$\widetilde{\gamma}_{avg}(B_1 \oplus_e B_1) = 1\frac{1}{3} \qquad \widetilde{\gamma}_{avg}(B_1 \oplus_e B_2) = 1\frac{19}{21}$$

$$\widetilde{\gamma}_{avg}(B_1 \oplus_e D_3) = 2 \qquad \widetilde{\gamma}_{avg}(B_1 \oplus_e D_{31}) = 2\frac{13}{15}$$

$$\widetilde{\gamma}_{avg}(B_1 \oplus_e D_{32}) = 2\frac{37}{45} \qquad \widetilde{\gamma}_{avg}(B_1 \oplus_e D_{33}) = 2\frac{37}{45}$$

$$\widetilde{\gamma}_{avg}(B_1 \oplus_e D_{34}) = 2\frac{11}{15} \qquad \widetilde{\gamma}_{avg}(B_1 \oplus_e D_{35}) = 2\frac{29}{45}$$

$$\widetilde{\gamma}_{avg}(B_1 \oplus_e D_{36}) = 2\frac{11}{15} \qquad \widetilde{\gamma}_{avg}(B_1 \oplus_e B_{24}) = 2\frac{74}{135}$$

$$\widetilde{\gamma}_{avg}(B_1 \oplus_e B_{25}) = 2\frac{148}{225} \qquad \widetilde{\gamma}_{avg}(B_2 \oplus B_2) = 2\frac{416}{705}$$

$$\widetilde{\gamma}_{avg}(B_2 \oplus D_3) = 2\frac{31}{45} \qquad \widetilde{\gamma}_{avg}(D_3 \oplus D_3) = 2\frac{4}{5}$$

$$\widetilde{\gamma}_{avg}(B^3_{1\oplus_e}) = 1\frac{5}{7} \qquad \widetilde{\gamma}_{avg}(B^2_{1\oplus_e} \oplus_e B_2) = 2\frac{16}{45}$$

$$\widetilde{\gamma}_{avg}(B^2_{1\oplus_e} \oplus_e B_2) = 2\frac{228}{279} \qquad \widetilde{\gamma}_{avg}(B^2_{1\oplus_e} \oplus_e D_2) = 2\frac{7}{15}$$

$$\widetilde{\gamma}_{avg}(B^3_{1\oplus_e} \oplus_e B_2) = 2\frac{29}{31} \qquad \widetilde{\gamma}_{avg}(B^4_{1\oplus_e}) = 2\frac{2}{15}$$

$$\widetilde{\gamma}_{avg}(B^5_{1\oplus_e}) = 2\frac{18}{31}$$

By the above discussion, we have

Theorem 3.13. *Let G be a 1-edge connected graph. Then G has ACN less than or equal to 2 if and only if G is homeomorphic to $B^2_{1\oplus_e}$, $B^3_{1\oplus_e}$ or $B_1 \oplus_e B_2$.*

Theorem 3.14. *A real number r in the closed interval $[1, 2]$ is a value of ACN of a 1-edge connected graph if and only if r is equal to $1\frac{1}{3}$, $1\frac{5}{7}$, $1\frac{19}{21}$ or 2.*

Theorem 3.15. *A real number r in the open interval $(2, 3)$ is a value of ACN of a 1-edge connected graph if and only if r is equal to $2\frac{2}{15}$, $2\frac{16}{45}$, $2\frac{7}{15}$, $2\frac{18}{31}$, $2\frac{74}{135}$, $2\frac{416}{705}$, $2\frac{29}{45}$, $2\frac{148}{225}$, $2\frac{31}{45}$, $2\frac{11}{15}$, $2\frac{4}{5}$, $2\frac{228}{279}$, $2\frac{37}{45}$, $2\frac{13}{15}$, $2\frac{29}{31}$.*

Theorem 3.16. *The smallest ten values of ACN are 1, $1\frac{1}{3}$, $1\frac{4}{9}$, $1\frac{1}{2}$, $1\frac{5}{7}$, $1\frac{19}{21}$, 2, $2\frac{5}{63}$, $2\frac{2}{15}$, and $2\frac{19}{105}$.*

4. The distribution of ACN

By Corollary 2.2, for every positive integer k, there exists a graph G which has average genus equals to k. We know that $\widetilde{\gamma}_{avg}(B_1) = 1$, and $\widetilde{\gamma}_{avg}(B_1 \oplus_e D_3) = 2$. However, by Section 3, we know that 3 is not a value of ACN for a graph.

Theorem 4.1. *Not all integers are values of ACN for graphs.*

Theorem 4.2. *(see [7]) Let G be a connected graph which is not a tree, then*

$$\frac{2^{\beta(G)-1}}{2^{\beta(G)}-1}\beta(G) \leq \tilde{\gamma}_{avg}(G) \leq \beta(G).$$

Furthermore the bounds are best possible.

Corollary 4.3. *Let $G=(V,E)$ be a connected graph with minimum degree at least 3, then $\tilde{\gamma}_{\text{avg}}(G) > \frac{|E|+2}{6}$.*

Proof. Since $\delta(G) \geq 3$, we have $2|E| = \sum_{v\in V} d(v) \geq 3|V|$. Thus,

$$\tilde{\gamma}_{\text{avg}}(G) > \frac{\beta(G)}{2} = \frac{|E|-|V|+1}{2} \geq \frac{|E|-\frac{2}{3}|E|+1}{3} = \frac{|E|+2}{6}.$$

□

By Corollary 4.3, we have:

Theorem 4.4. *Let r be a finite number. the number of non-homeomorphic graphs whose ACN is bounded by or equal to r is finite.*

By the above theorem, the distribution of ACN is sparse in R: *within any finite real interval, there are at most finitely many different numbers are values of average crosscap number of graphs.*

A sequence $\{G_i\}$ $(i = 1, 2, \ldots)$ of graphs is called strictly monotone sequence if each G_i is homeomorphic to a subgraph of G_{i+1} and no graphs in the sequence are homeomorphic. For average genus of a graph, Chen and Gross [4] have proved that adding to a single edge many ears is essentially the only way to obtain a limit point of average genus. While for the ACN of a graph, it is may not true.

Theorem 4.5. *Let $\{G_i\}$ $(i = 1, 2, \ldots)$ be a strictly monotone sequence of graphs, Then the ACN of the sequence $\{G_i\}$ does not exist limit points.*

Proof. By Theorem 3.1, $\tilde{\gamma}_{avg}(G_i) > \tilde{\gamma}_{avg}(G_{i-1})$ $(i \geq 2)$. By Theorem 4.2, the theorem follows □

5. Shared Values of ACN

We know that ACN is an homeomorphic invariance of a graph. Thus, two isomorphic graph may share the same values of ACN. But a natural question is posed: Whether two non-homeomorphic graphs have the same ACN or not. Of course, the answer is affirmative. We can easily find some graphs in section 3 and section 4. For example: the graph D_{32} and D_{33} have ACN $2\frac{1}{3}$. D_{34} and D_{36} have ACN $2\frac{1}{4}$, etc. Now, we use three different methods to construct many non-homeomorphic graphs which share the same ACN.

5.1. **Adding ears serially to an edge of a graph.** Let G be a graph and $e \in E(G)$, if we insert two vertices u and v and double the edge between them, we say we attach an open ear to the interior of e. Similarly, if the vertices $u = v$, then we say we attach a closed ear to the interior of e. The two vertices u and v are called the ends of the ear. We say r open ears and s closed ears are *attached serially* to the edge e, if all ends of the ears are distinct. Now we add r closed ears and s open ears to an edge of a graph G. we denote the set of result graphs by $G_{r,s}$. It is easy to see that any two non-homeomorphic graphs G_1 and G_2 of $G_{r,s}$ have the same total genus polynomial. Of course, they share the same ACN.

Example 2. A cactus and a necklace can be obtained by adding a series of closed ears and open ears to a single edge or a cycle graph, respectively. All cacti (necklaces) have the same ACN.

5.2. **Using the operation of bar-amalgamation.** Let G_i, for i=1,2, and H_i, for i=1,2, be connected graphs. By Theorem 4.3, if $\beta(G_1) = \beta(G_2)$, $\beta(H_1) = \beta(H_2)$, $\widetilde{\gamma}_{avg}(G_1) = \widetilde{\gamma}_{avg}(G_2)$ and $\widetilde{\gamma}_{avg}(H_1) = \widetilde{\gamma}_{avg}(H_2)$, we have $\widetilde{\gamma}_{avg}(G_1 \oplus_e H_1) = \widetilde{\gamma}_{avg}(G_2 \oplus_e H_2)$.

5.3. **Using the technique of vertex-splitting.** A *fan graph* $F_{(1,n)}$ is defined as the graph $K_1 + P_n$, where K_1 is the empty graph on one vertex and P_n is the path graph on n vertices. A fan-type graph $F_{t_1,t_2,\ldots,t_n}$ is defined as the graph K_1 connect t_j edges to the vertex v_j of P_n, $t_j \geq 1$, $j = 1, 2, \ldots, n$. A dipole graph D_n is a multigraph consisting of two vertices connected with n edges. Figure 7 presents the graphs $F_{(1,n)}$, $F_{2,2,\ldots,2}$ and D_n. By Theorem 2.7, $\widetilde{\gamma}_{avg}(D_n) = \widetilde{\gamma}_{avg}(F_{(1,n)}) = \widetilde{\gamma}_{avg}(F_{2,2,\ldots,2})$.

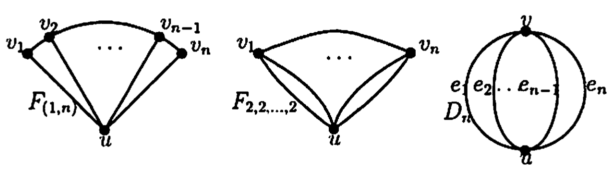

FIGURE 7

6. THE ACN AND THE MODE OF THE NON-ORIENTABLE EMBEDDING DISTRIBUTION

A real polynomial $f(x) = \sum_{i=0}^{n} a_i x^i$ is unimodal if its sequence of coefficients $\{a_i\}$ satisfies that for some $0 \leq j \leq n$, there exists $a_0 \leq a_1 \leq \ldots \leq a_j \geq a_{j+1} \geq \ldots \geq a_n$, such j is called a mode of the sequence and if the sequence of coefficients satisfies that for any $1 \leq j \leq n-1$, there exists

${a_j}^2 \geq a_{j-1}a_{j+1}$, then the polynomial (the sequence) is strongly unimodal. Obviously, a strongly unimodal sequence is unimodal. The objection of this section is to study the unimodality of the non-orientable embedding distribution .

Many papers have been written in last twenty years concerning the genus distribution of the embeddings of graphs. Most of the papers supported the following conjecture.

Conjecture 6.1. (J. Gross) The genus distribution sequence of a graph is strongly unimodal.

Remark 6.2. In [26], Stahl posed a stronger conjecture which states that the zeros of genus polynomial are real. In [9, 10], the authors presented some counter examples, however Gross's conjecture is still open. In [8, 12], obtained explicit formula for the total embedding distributions of necklaces, closed-end ladders and cobblestone path. So far, none of the crosscap number distribution for the classes of graphs are proved to be strongly unimodal.

6.1. **The ACN and the mode of the cacti.** *A cactus* is the graph obtained by the following way: start with a tree T, then replace some of the vertices in T by simple cycle and connect the edges incident on each such vertex to the corresponding cycle in an arbitrary way.

Theorem 6.3. *Let $G = (V, E)$ be a cactus with minimum degree at least 2. Then, the total embedding distribution polynomial of the cactus G is*

$$I(x,y) = \prod_{v \in V(G)} (d_v - 1)!(1+y)^{\beta(G)}$$

Proof. We prove the theorem by induction on number $\beta(G)$. If $\beta(G) = 0$, or 1, it is easy to check that the theorem is true. Since the cactus G is 1-edge connected, we may suppose $G = G_1 \oplus_e G_2$ where $e = uv$ and G_i is a cactus, for $i = 1, 2$. By Theorem 2.1, we have the crosscap number polynomial of G is

$$\begin{aligned} f_G(y) &= d_{G_1}(u)d_{G_2}(v)\Big(f_{G_1}(y)f_{G_2}(y) + f_{G_1}(y)g_{G_2}(y^2) + g_{G_1}(y^2)f_{G_2}(y)\Big) \\ &= d_{G_1}(u)d_{G_2}(v)\Bigg[\prod_{v \in V(G_1)} (d_v - 1)!\Big((1+y)^{\beta(G_1)} - 1\Big) \times \\ &\prod_{v \in V(G_2)} (d_v - 1)!\Big((1+y)^{\beta(G_2)} - 1\Big) \\ &+ \prod_{v \in V(G_1)} (d_v - 1)!\Big((1+y)^{\beta(G_1)} - 1\Big) \prod_{v \in V(G_2)} (d_v - 1)! \end{aligned}$$

$$+\prod_{v\in V(G_2)}(d_v-1)!\left((1+y)^{\beta(G_2)}-1\right)\prod_{v\in V(G_1)}(d_v-1)!\Bigg]$$

$$=\prod_{v\in V(G)}(d_v-1)!\left[(1+y)^{\beta(G)}-1\right]$$

□

By Theorem 6.3, we have the following result.

Corollary 6.4. *Let $G=(V,E)$ be a cactus. Then, the ACN of the cactus G is*

$$\widetilde{\gamma}_{avg}(G)=\frac{2^{\beta(G)-1}}{2^{\beta(G)}-1}\beta(G).$$

Theorem 6.5. *The crosscap number distribution of a cactus is strongly unimodal.*

Proof. It is directly from the theorem 6.3. □

Theorem 6.6. *The mode of the non-orientable embedding distribution sequence for a cactus G equals to $\lfloor\widetilde{\gamma}_{avg}(G)\rfloor$ or $\lceil\widetilde{\gamma}_{avg}(G)\rceil$*

Proof. By theorem 6.3, the following two cases can be easily derived.

Case 1: $\beta(G)$ is even. The mode of the cactus is $\frac{\beta(G)}{2}$ which equals to $\lfloor\widetilde{\gamma}_{avg}(G)\rfloor$.

Case 2: $\beta(G)$ is odd. The mode of the cactus is $\frac{\beta(G)-1}{2}$ or $\frac{\beta(G)+1}{2}$ which equals to $\lfloor\widetilde{\gamma}_{avg}(G)\rfloor$ or $\lceil\widetilde{\gamma}_{avg}(G)\rceil$, respectively. □

6.2. **The ACN and the mode of the necklaces.** A necklace $N_{r,s}$ of type (r,s) is a cycle where r disjoint edges are doubled and s self-loops are added to s vertices which are not endpoints of a doubled edges. In [8], obtained the following result.

Theorem 6.7. *(see [8]) The total embedding distribution polynomial of $N_{r,s}$ is given by*

$$I_{N_{r,s}}(x,y)=2^{2r+s+1}3^s y^2(1+y)^{r+s}+$$
$$2^{r+s}(2+3y)^s(1-y)(1+2y)^{r+1}+(4^r6^s-2^r4^s)(x-y^2).$$

By Theorem 6.7, we have the following result.

Corollary 6.8. *The ACN of $N_{r,s}$ is*

$$\widetilde{\gamma}_{avg}(N_{r,s})=2+\frac{r+s}{2}+\frac{(r+s+4)2^r3^s+2^{s+1}-3^{r+1}5^s-2^{r+1}3^s}{2^r3^s(2^{r+s+1}-1)}.$$

Lemma 6.9. *The polynomial $I_{N_{r,0}}(y)$ is strongly unimodal and for $2\le r\le 14$ or r is even, the mode is $\left[\frac{r}{2}+2\right]$, otherwise, the mode is $\left[\frac{r}{2}+3\right]$.*

Proof. From Theorem 6.7, we have $I_{N_{r,0}}(y) = 2^{2r+1}y^2(1+y)^r + 2^r(1+y-2y^2)(1+2y)^r - (4^r-2^r)y^2$. Let $I_{N_{r,0}}(y) = \sum_{i=1}^{r+2} C(i)y^i$, then we obtain that $C(1) = 2^r(2r+1)$, $C(2) = 2^r(2r^2+2^r-1)$, and for $3 \le k \le r+1$, $C(k) = 2^{r+k}\binom{r+1}{k} - 2^{r+k-1}\binom{r+1}{k-1} + 2^{2r+1}\binom{r}{k-2}$.

A short computation shows that $(C(k)^2 - C(k-1)C(k+1) \ge 0, k \ge 2$ which means that the non-orientable embedding distributions of the necklace $N_{r,0}$ is strongly unimodal.

Case 1: For $r = 2n$, where $n = 2, 3, \ldots$, we have that

$$\frac{(n+1)!(n+2)!}{2^{3n}(2n)!}\Big[C(n+1) - C(n+2)\Big]$$
$$= (2n+1)(n+1)(-n+6) - (n+1)(n+2)2^{n+1} \le 0.$$

Consequently, we obtain $C(n+1) \le C(n+2)$. By the same consideration,

$$\frac{(n+3)!(n+2)!}{2^{3n+1}(2n)!}\Big[C(n+2) - C(n+3)\Big] \ge 0.$$

Thus $C(n+2) \ge C(n+3)$, from the unimodality of the sequence $\{C(i)\}$, we obtain the mode $[\frac{r}{2}]+2$ when r is even.

Case 2: For $r = 2n+1$, where $n = 1, 2, \ldots$, we have

$$\frac{(n+1)!(n+2)!}{2^{3n+1}(2n+1)!}\Big[C(n+2) - C(n+1)\Big] = (2n+2)(n-3) + 2^{n+3}(n+1) \ge 0.$$

Consequently, we obtain $C(n+2) \ge C(n+1)$. By the same consideration,

$$\frac{(n+2)!(n+3)!}{2^{3n+2}(2n+1)!}\Big[C(n+3) - C(n+2)\Big] = (2n+2)(n^2-7n-6)$$

It is easy to check that for $1 \le n \le 7$, $n^2-7n-6 < 0$, then $C(n+3) < C(n+2)$ and for $n \ge 8$, $n^2-7n-6 > 0$, then $C(n+3) > C(n+2)$.

In addition, by a similar computation, we can verify that for $\forall n \in N^+$,

$$\frac{(n+4)!(n+3)!}{2^{3n+3}(2n+1)!}\Big[C(n+3) - C(n+4)\Big] \ge 0$$

which means that $C(n+3) \ge C(n+4)$. From the unimodality of the sequence $\{C(i)\}$ and the above result, we obtain the mode $[\frac{r}{2}]+3$ when r is odd with $r \ge 17$, and the mode $[\frac{r}{2}]+2$ when r is odd with $3 \le r \le 15$. Combining the results discussed above, which leads to the corollary. □

Theorem 6.10. *The mode for the crosscap number distribution of a necklace $N_{r,0}$ is the upper-rounding or lower-rounding of its ACN.*

Proof. According Corollary 6.8 and Lemma 6.9, the theorem follows. □

By using Mathematic, we show that for small values of (r, s), the polynomial $I_{N_{r,s}}(y)$ is unimodal. And if $r+s$ is even, the mode of $I_{N_{r,s}}(y)$ equals $[\frac{r+s}{2}+2]$; otherwise $r+s$ is odd, the mode of $I_{N_{r,s}}(y)$ equals $[\frac{r+s}{2}+3]$.

Conjecture 6.11. The non-orientable genus distribution of $N_{r,s}$ is strong unimodal. Furthermore, If $r+s$ is even, the mode of $I_{N_{r,s}}(y)$ equals $[\frac{r+s}{2}+2]$. Otherwise the mode equals $[\frac{r+s}{2}+3]$.

7. Conclusions

Now we give a picture of average genus and ACN. See Table 1

TABLE 1

	Average genus	ACN
Heredity	$\gamma_{avg}(H) \leq \gamma_{avg}(G)$ if $H \subset G$ [2-4,15]	$\widetilde{\gamma}_{avg}(H) \leq \widetilde{\gamma}_{avg}(G)$ if $H \subset G$
Distribution	Sparse in R for CF-graphs [2,11]	Sparse in R for all graphs
Limit points	Exists for non-simple graphs [3]	No
The smallest values	$0, \frac{1}{3}, \frac{1}{2}, \ldots,$[9]	$1, \frac{4}{3}, \frac{13}{9}, \ldots,$
Lower bound	$\frac{\gamma_M(G)}{\Delta(G)-1}$ [6,11]	$\frac{2^{\beta(G)-1}}{2^{\beta(G)}-1}\beta(G)$ [7]
Random graph	Close to $\gamma_M(G)$ [11]	?

By Corollary 2.4, we know that $\widetilde{\gamma}_{avg}(G \oplus_e H) \leq \widetilde{\gamma}_{avg}(G) + \widetilde{\gamma}_{avg}(H)$ if and only if $2\gamma_{avg}(G) \leq \widetilde{\gamma}_{avg}(G)$ and $2\gamma_{avg}(H) \leq \widetilde{\gamma}_{avg}(H)$. By some sample calculations, we verified that $\widetilde{\gamma}_{avg}(G \oplus_e H) \leq \widetilde{\gamma}_{avg}(G) + \widetilde{\gamma}_{avg}(H)$. Thus we pose the following problem.

Problem 7.1. Let G be a connected graph, do we have $2\gamma_{avg}(G) \leq \widetilde{\gamma}_{avg}(G)$?

In Section 6, we showed that the mode of the embedding distributions for G is connection with its ACN. With the help of a computer program, we verified this is true for all graphs with Betti number less than 5.

Problem 7.2. Let G be a graph and $m(G)$ be a mode of it's non-orientable embedding distribution, do we have $m(G) = \lceil \widetilde{\gamma}_{avg}(G) \rceil$ or $m(G) = \lfloor \widetilde{\gamma}_{avg}(G) \rfloor$?

The calculation of ACN seems more difficult than average genus. With a compute program, we have computed the ACN of graphs with Betti number less than 5. In [13, 14], the authors obtained explicit formula for the total embedding distribution for some types of graphs like closed-end ladders, Ringle ladders etc, it seems that the exact value or asymptotic value of ACN can be obtained for these type of graphs.

Problem 7.3. Calculate the exact value or asymptotic value of ACN of other interesting classes of graphs like closed-end ladders, Ringel ladders, complete graph K_n, complete bipartie graph $K_{m,n}$, etc.

In [7], obtained tight bounds for the ACN. It seems that the ACN of graph is more closer to its maximum non-orientable genus than to its non-orientable genus for most graphs. Using stahl's result [24], Lee [20] proved that the average genus of a graph G is asymptotic to the maximum genus if $|E(G)|$ asymptotic to $c|V(G)|^{1+a}$ (where c and a are positive constants).

Problem 7.4. Does the average crosscap number of the complete graph K_n (complete bipartite graph $K_{m,n}$, etc.) asymptotic to its maximum non-orientable genus. Moreover, Do we have Lee type result for ACN?

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