Some C_4 - supermagic graphs

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Abstract: A simple graph $G = (V,E)$ admits an H-covering if every edge in E belongs to a subgraph of G isomorphic to H, we say that G is H-magic if there is a total labeling $f: V \cup E \rightarrow \{1,2,...|V|+|E|\}$ such that for each subgraph $H'=(V',E')$ of G isomorphic to H, $\sum_{v \in V'} f(v) + \sum_{e \in E'} f(e)$ is constant. When $f(V)=\{1,2,\ldots,|V|\}$, then G is said to be H-supermagic. In this paper we show that all prism graphs $C_n \times P_m$ except for $n = 4$, the ladder graph $P_2 \times P_n$ and the grid $P_3 \times P_n$ are C_4-supermagic.

Keywords: prism graph, ladder graph, grid graph, magic covering.
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1. Introduction

The concept of H-magic graphs was introduced in [2]. An edge-covering of a graph G is a family of different subgraphs $H_1, H_2, ..., H_k$ such that each edge of E belongs to at least one of the subgraphs H_i, $1 \leq i \leq k$. Then, it is said that G admits an $(H_1,H_2,\ldots,H_k)$ - edge covering. If every H_i is isomorphic to a given graph H, then we say that G admits an H-covering.

Suppose that $G = (V, E)$ admits an H-covering. We say that a bijective function $f: V \cup E \rightarrow \{1,2,...,|V|+|E|\}$ is an H-magic labeling of G if there is a positive integer $m(f)$, which we call magic sum, such that for each subgraph $H' = (V', E')$ of G isomorphic to H, we have, $f(H') = \sum_{v \in V'} f(v) + \sum_{e \in E'} f(e) = m(f)$. In this case we say that the graph G is H-magic. When $f(V)=\{1,2,\ldots,|V|\}$, we say that G is H-supermagic and we denote its supermagic-sum by $s(f)$.

In [2], A. Gutierrez, and A. Llado studied the families of complete and complete bipartite graphs with respect to the star-magic and star-supermagic properties and proved the following results.

The star $K_{1,n}$ is $K_{1,h}$-supermagic for any $1 \leq h \leq n$.

- Let G be a d-regular graph. Then G is not $K_{1,h}$-magic for any $1 < h < d$.
- (a) The complete graph K_n is not $K_{1,h}$-magic for any $1 < h < n$-1.
  (b) The complete bipartite graph $K_{n,n}$ is not $K_{1,h}$-magic for any $1 < h < n$.
- The complete bipartite graph $K_{n,n}$ is $K_{1,n}$-magic for $n \geq 1$.
- The complete bipartite graph $K_{n,n}$ is not $K_{1,n}$-supermagic for any integer $n > 1$.
- For any pair of integers $1 < r < s$, the complete bipartite graph $K_{r,s}$ is $K_{1,h}$ -supermagic if and only if $h = s$.

The following results regarding path-magic and path-supermagic coverings are also proved in [2].

- The path P_n is P_h-supermagic for any integer $2 \leq h \leq n$.
- Let G be a P_h-magic graph, $h > 2$. Then G is C_h-free.
- The complete graph K_n is not P_h-magic for any $2 < h \leq n$.
- The cycle C_n is P_h-supermagic for any integer $2 \leq h < n$ such that gcd(n, $h(h$-1))=1.

Also in [2], the authors constructed some families of H-magic graphs for a given graph H by proving the following results.

- Let H be any graph with $|V(H)|+|E(H)|$ even. Then the disjoint union $G = kH$ of k copies of H is H-magic.

Let G and H be two graphs and $e \in E(H)$ a distinguished edge in H. We denote by $G \bullet eH$ the graph obtained from G by gluing a copy of H to each edge of G by the distinguished edge $e \in E(H)$.

- Let H be a 2-connected graph and G an H-free supermagic graph. Let k be the size of G and $h = |V(H)|+|E(H)|$. Assume that h and k are not both even. Then, for each edge $e \in E(H)$, the graph $G \bullet eH$ is H-magic.

In [3], P. Jeyanthi and P. Selvagopal proved that for any positive integer n, k - polygonal snake of length n is C_k-supermagic.

In this paper we prove the prism graph $C_n \times P_m$ is C_4-supermagic for all $m \geq 2$ and $n \geq 3$ except for $n = 4$. And also we prove that the ladder graph $P_2 \times P_n$ and the grid graph $P_3 \times P_n$ are C_4-supermagic for all $n \geq 2$.

2. Main Results

Theorem 1: The prism graph $C_n \times P_m$ is C_4-supermagic for all $m \geq 2$ and $n \geq 3$ (except $n = 4$) with supermagic number $10mn$-n+4.

Proof:

Let $C_n \times P_m$ be a prism graph with vertex set $V = \{v_{i,j} : 1 \le i \le m, 1 \le j \le n\}$ and edge set $E = \{v_{i,j}v_{i,j+1} : 1 \le i \le m, 1 \le j \le n-1\} \cup \{v_{i,n}v_{i,1} : 1 \le i \le m\} \cup \{v_{i,j}v_{i+1,j} : 1 \le i \le m-1, 1 \le j \le n\}$.

Let $H_{i,j}$ be the 4-cycle $v_{i,j}v_{i,j+1}v_{i+1,j+1}v_{i+1,j}$ for $1 \le i \le m$-1 and $1 \le j \le n$-1 and $H_{i,n}$ be the 4-cycle $v_{i,n}v_{i,1}v_{i+1,1}v_{i+1,n}$. Define a total labeling $f : V \cup E \to [1, (3m-1)n]$ as follows.

Case 1: n is odd.

For $1 \le i \le m$, $1 \le j \le n$

$$f(v_{i,j}) = \begin{cases} n(i-1) + \dfrac{j+1}{2} & \text{if } i \text{ is odd, } j \text{ is odd} \\ n(i-1) + \dfrac{n+j+1}{2} & \text{if } i \text{ is odd, } j \text{ is even} \\ ni - j + 1 & \text{if } i \text{ is even} \end{cases}$$

For $1 \le i \le m$

$$f(v_{i,j}v_{i,j+1}) = \begin{cases} n(2m-i) + j & \text{if } i \text{ is odd}, 1 \le j \le n \\ n(2m-i) + j + 1 & \text{if } i \text{ is even}, 1 \le j \le n-1 \\ n(2m-i) + 1 & \text{if } i \text{ is even}, j = n \end{cases}$$

where the second subscript j is taken modulo n. (That is, $v_{i,n+1} = v_{i,1}$)

For $1 \le i \le m$-1,

$$f(v_{i,j}v_{i+1,j}) = \begin{cases} n(3m-1) - \dfrac{j-1}{2} - n(i-1) & \text{if } j \text{ is odd} \\ n(3m-1) - \dfrac{n+j-1}{2} - n(i-1) & \text{if } j \text{ is even} \end{cases}$$

For $1 \le i \le m$-1 and $1 \le j \le n$-1,

$f(H_{i,j}) = f(v_{i,j}) + f(v_{i,j+1}) + f(v_{i+1,j+1}) + f(v_{i+1,j}) + f(v_{i,j}v_{i,j+1}) + f(v_{i+1,j}v_{i+1,j+1}) + f(v_{i,j}v_{i+1,j}) + f(v_{i,j+1}v_{i+1,j+1})$ and $f(H_{i,n}) = f(v_{i,n}) + f(v_{i,1}) + f(v_{i+1,1}) + f(v_{i+1,n}) + f(v_{i,n}v_{i,1}) + f(v_{i+1,n}v_{i+1,1}) + f(v_{i,n}v_{i+1,n}) + f(v_{i,1}v_{i+1,1})$.

Now, $f(v_{i,j}) + f(v_{i,j+1}) + f(v_{i+1,j+1}) + f(v_{i+1,j}) = 4ni + \dfrac{n+3}{2} - j + 1$ for $1 \le i \le m$-1and$1 \le j \le n$-1.

$$f(v_{i,j}v_{i,j+1}) + f(v_{i+1,j}v_{i+1,j+1}) = \begin{cases} 4mn - 2ni + 2j - n + 1 & \text{for } 1 \le i \le m-1 \text{ and } 1 \le j \le n-1 \\ 4mn - 2ni + 1 & \text{for } 1 \le i \le m-1 \text{ and } j = n \end{cases}$$

$$f(v_{i,j}v_{i+1,j})+f(v_{i,j+1}v_{i+1,j+1})=\begin{cases}6mn-2ni-j-\dfrac{n-1}{2} & \text{for } 1\le i\le m-1 \text{ and } 1\le j\le n-1\\ 6mn-2ni-\dfrac{n-1}{2} & \text{for } 1\le i\le m-1 \text{ and } j=n\end{cases}$$

Then, it can be easily verified that $f(H_{i,j}) = 10mn-n+4$ for $1 \le i \le m-1$ and $1 \le j \le n$.

Since each $H_{i,j}$ is isomorphic to C_4, $C_n \times P_m$ is C_4-supermagic.

Case 2: n is even and $n \ne 4$.

For $1\le i\le m$, $1\le j\le n$ $f(v_{i,j})=\begin{cases}(i-1)n+j & \text{if } i \text{ is odd}, 1\le j\le n\\ in-j+1 & \text{if } i \text{ is even}, 1\le j\le n\end{cases}$

For $1\le i\le m$ define $f(v_{i,j}v_{i,j+1})$ as follows.

When i is odd, $$f(v_{i,j}v_{i,j+1})=\begin{cases}n(2m-i+1) & \text{for } j=1\\ n(2m-i)+1 & \text{for } j=\dfrac{n}{2}+1\\ n(2m-i)+\dfrac{n}{2}-j+2 & \text{for } 2\le j\le \dfrac{n}{2}\\ n(2m-i+1)+\dfrac{n}{2}-j+1 & \text{for } \dfrac{n}{2}+2\le j\le n\end{cases}$$

where the second subscript j is taken modulo n.

When i is even, $$f(v_{i,j}v_{i,j+1})=\begin{cases}n(2m-i)+1 & \text{for } j=1\\ n(2m-i+1) & \text{for } j=\dfrac{n}{2}+1\\ n(2m-i)+2j-2 & \text{for } 2\le j\le \dfrac{n}{2}\\ n(2m-i-1)+2j-1 & \text{for } \dfrac{n}{2}+2\le j\le n\end{cases}$$

where the second subscript j is taken modulo n.

and $$f(v_{i,j}v_{i+1,j})=\begin{cases}n(3m-i-1)+1 & \text{for } j=1\\ n(3m-i-1)+\dfrac{n}{2}-\dfrac{j-3}{2} & \text{if } j \text{ is odd}, j\ne 1\\ n(3m-i)-\dfrac{j}{2}+1 & \text{if } j \text{ is even}\end{cases}$$

For $1\le i\le m-1$ and $1\le j\le n-1$,

$f(H_{i,j}) = f(v_{i,j}) + f(v_{i,j+1}) + f(v_{i+1,j+1}) + f(v_{i+1,j}) + f(v_{i,j}v_{i,j+1}) + f(v_{i+1,j}v_{i+1,j+1}) + f(v_{i,j}v_{i+1,j}) + f(v_{i,j+1}v_{i+1,j+1})$ and $f(H_{i,n}) = f(v_{i,n}) + f(v_{i,1}) + f(v_{i+1,1}) + f(v_{i+1,n}) + f(v_{i,n}v_{i,1}) + f(v_{i+1,n}v_{i+1,1}) + f(v_{i,n}v_{i+1,n}) + f(v_{i,1}v_{i+1,1})$.

Now, $f(v_{i,j}) + f(v_{i,j+1}) + f(v_{i+1,j+1}) + f(v_{i+1,j}) = 4ni+2$ for $1 \le i \le m-1$ and $1 \le j \le n-1$.

$$f(v_{i,j}v_{i,j+1}) + f(v_{i+1,j}v_{i+1,j+1}) = \begin{cases} 4mn - 2ni + j - \frac{n}{2} & \text{for } 1 \le i \le m-1 \text{ and } j \ne 1,\ j \ne \frac{n}{2}+1 \\ 4mn - 2ni + 1 & \text{for } 1 \le i \le m-1 \text{ and } j = 1,\ j = \frac{n}{2}+1 \end{cases}$$

$$f(v_{i,j}v_{i+1,j}) + f(v_{i,j+1}v_{i+1,j+1}) = \begin{cases} 6mn - 2ni - j - \frac{n}{2} + 2 & \text{for } 1 \le i \le m-1 \text{ and } j \ne 1 \\ 6mn - 2ni - n + 1 & \text{for } 1 \le i \le m-1 \text{ and } j = 1 \end{cases}$$

Then, it can be easily verified that $f(H_{i,j}) = 10mn-n+4$ for $1 \le i \le m-1$ and $1 \le j \le n$.

Since each $H_{i,j}$ is isomorphic to C_4, $C_n \times P_m$ is C_4-supermagic.

2.2 Remark: The prism graph $C_n \times P_m$ contains $n(m-1)$ 4-cycles C_4 except for $n = 4$. When $n = 4$, $C_n \times P_m$ contains $n(m-1)+1$ 4-cycles and under the labeling we defined in Theorem 1 every cycle C_4 except the inner most cycle has the sum $10mn-n+4$.

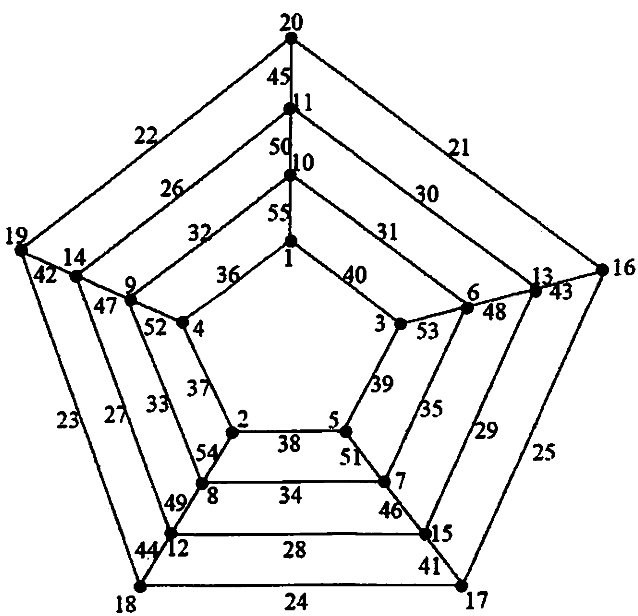

Figure 1: $C_5 x P_4$ is C_4-supermagic with supermagic sum $s(f)=199$.

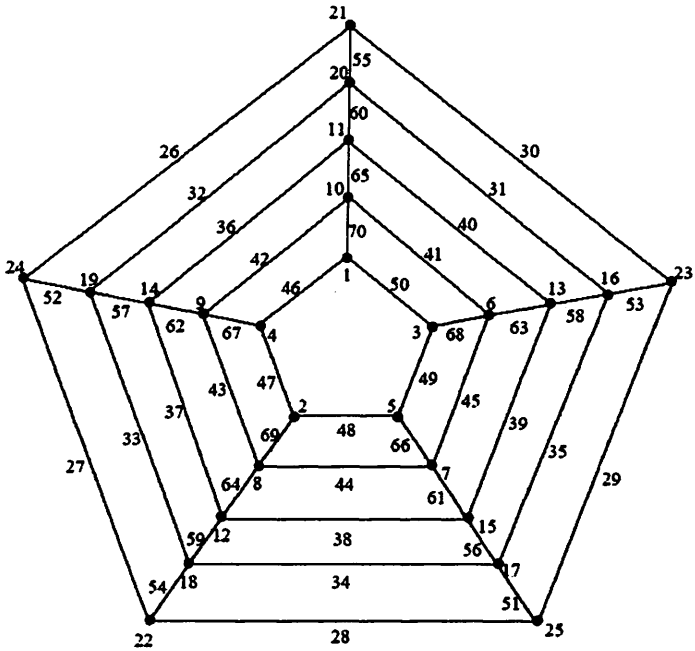

Figure 2: C_5xP_5 is C_4-supermagic with supermagic sum $s(f)$=249.

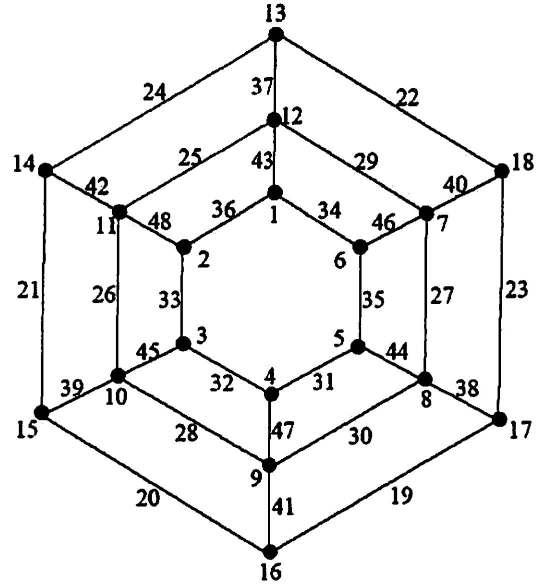

Figure 3: C_6xP_3 is C_4-supermagic with supermagic sum $s(f)$=181.

Theorem 2: The ladder graph $P_2 \times P_n$ is C_4-supermagic for all n.

Proof:

Let V be the vertex set and E be the edge set of the ladder graph $P_2 \times P_n$. Then $V = \{v_{1,j}, v_{2,j} : 1 \le j \le n\}$ and $E = \{v_{1,j}v_{1,j+1}, v_{2,j}v_{2,j+1} : 1 \le j \le n-1\} \cup \{v_{1,j}v_{2,j} : 1 \le j \le n\}$. Let H_j be the 4-cycle $H_j = \{v_{1,j}v_{1,j+1}v_{2,j+1}v_{2,j}\}$ $1 \le j \le n$-1. Define a total labeling $f : V \cup E \to [1,5n-2]$ as follows.

$f(v_{1,j}) = j$ and $f(v_{2,j}) = 2n - j + 1$ for $1 \le j \le n$.

$f(v_{1,j}v_{2,j}) = 2n + j$ for $1 \le j \le n$-1

$f(v_{1,j}v_{1,j+1}) = 4n - j$ and $f(v_{2,j}v_{2,j+1}) = 5n - j - 1$ for $1 \le j \le n$-1.

Then for $1 \le j \le n$-1,

$$f(H_j) = f(v_{1,j}) + f(v_{1,j+1}) + f(v_{2,j+1}) + f(v_{2,j}) + f(v_{1,j}v_{2,j}) + f(v_{1,j+1}v_{2,j+1}) + f(v_{1,j}v_{1,j+1}) + f(v_{2,j}v_{2,j+1}) = 17n + 2.$$

Since each H_j is isomorphic to C_4, $P_2 \times P_n$ is C_4-supermagic.

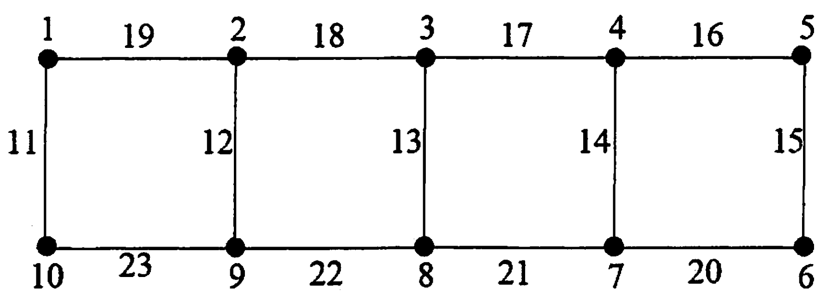

Figure 4: $P_2 \times P_5$ is C_4-supermagic with supermagic sum $s(f)$=87.

Theorem 3: The grid graph $P_3 \times P_n$ is C_4-supermagic for all n.

Proof:

Let V be the vertex set and E be the edge set of the grid graph $P_3 \times P_n$. Then $V = \{v_{i,j} : 1 \le i \le 3, 1 \le j \le n\}$ and $E = \{v_{i,j}v_{i,j+1}, : 1 \le i \le 3, 1 \le j \le n-1\} \cup \{v_{i,j}v_{i+1,j} : 1 \le i \le 2, 1 \le j \le n\}$. Let $H_{i,j}$ be the 4-cycle $H_{i,j} = \{v_{i,j}v_{i,j+1}v_{i+1,j+1}v_{i+1,j}\}$ for $1 \le i \le 2$ and $1 \le j \le n$-1. Define a total labeling $f : V \cup E \to [1,8n-3]$ as follows.

For $j = 1, 2, \ldots, n$

$$f(v_{i,j}) = \begin{cases} \left(\frac{i-1}{2}\right)n + j, & i = 1,3 \\ 2n + j, & i = 2. \end{cases} \quad \text{and} \quad f(v_{i,j}v_{i+1,j}) = \begin{cases} 8n - 2j - 1, & i = 1 \\ 8n - 2j - 2, & i = 2 \end{cases}$$

For $j = 1, 2, \ldots, n$-1

$$f(v_{i,j}v_{i,j+1}) = \begin{cases} 6n + i - ni + j - 3, & i = 1,3 \\ 5n - j - 1, & i = 2 \end{cases}$$

Then for $1 \leq i \leq 2$ and $1 \leq j \leq n$-1,

$$f(H_{i,j}) = f(v_{i,j}) + f(v_{i,j+1}) + f(v_{i+1,j+1}) + f(v_{i+1,j}) + f(v_{i,j}v_{i+1,j}) + f(v_{i,j+1}v_{i+1,j+1}) + f(v_{i,j}v_{i,j+1}) + f(v_{i+1,j}v_{i+1,j+1}) = 30n\text{-}5.$$

Since each $H_{i,j}$ is isomorphic to C_4, $P_3 \times P_n$ is C_4-supermagic.

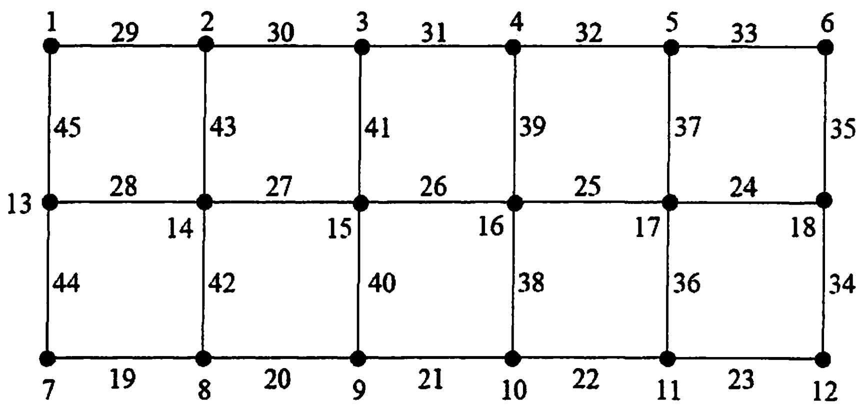

Figure 5: $P_3 x P_6$ is C_4-supermagic with supermagic sum $s(f)$=175.

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