Asymptotic upper bounds for $K_{1,m,k}$: complete graph Ramsey Numbers*

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Abstract

It is shown that $r(K_{1,m,k}, K_n) \leq (k-1+o(1))\left(\frac{n}{\log n}\right)^{m+1}$ for any two fixed integers $k \geq m \geq 2$ and $n \to \infty$. It is obtained by the analytic method and using the function $f_m(x) = \int_0^1 \frac{(1-t)^{1/m}dt}{m+(x-m)t}, x \geq 0, m \geq 1$ on the base of the upper bounds for $r(K_{m,k}, K_n)$ which were given by Y. Li and W. Zang. Also, $(c-o(1))\left(\frac{n}{\log n}\right)^{7/3} \leq r(W_4, K_n) \leq (1+o(1))\left(\frac{n}{\log n}\right)^3$ (as $n \to \infty$). Moreover, we give $r(K_l + K_{m,k}, K_n) \leq (k-1+o(1))\left(\frac{n}{\log n}\right)^{l+m}$ for any two fixed integers $k \geq m \geq 2$(as $n \to \infty$).

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1 Introduction

Let H be a graph without isolates. The Ramsey number $r(H, K_n)$ is the smallest integer N such that for any graph G with N vertices, either G contains H as a subgraph or $\overline{G}$ (the complement of G) contains K_n as a subgraph. As usual, let $K_{m,k}$ stand for the $m \times k$ complete bipartite graph. The *join* of graphs K and H, Denoted by $K+H$, is the graph obtained by starting with vertex disjoint copies of K and H and adding uv to the edge

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set for every $u \in V(K)$ and $v \in V(H)$. In[1] Caro et al. proved that for $k \geq 2$ and $n \to \infty$,

$$r(K_{2,k}, K_n) \leq (k - 1 + o(1)) \left(\frac{n}{\log n}\right)^2.$$

In [6] Li and Zang expanded the above results into the following Lemma 1.

Lemma 1 *[6] For any fixed integers $k \geq m \geq 2$, as $n \to \infty$,*

$$r(K_{m,k}, K_n) \leq (k - 1 + o(1)) \left(\frac{n}{\log n}\right)^m. \tag{1}$$

Up to now it is the best asymptotic upper bounds for $K_{m,k}$: complete graph Ramsey numbers. Naturally, we want to know what the upper bounds for $K_{s,m,k}$: complete graph Ramsey numbers are, where $K_{s,m,k}$ denotes the $s \times m \times t$ complete 3-partite graphs, and how can we get it. However, it depends on the appearance and application of some new techniques. In this paper, we only consider the case when $s = 1$ and give the asymptotic upper bounds for $r(K_{1,m,k}, K_n)$ by the essential identical proof techniques to those of Li and Zang in [6]. By the definition of *join*, we know that $K_{1,m,k} \cong K_1 + K_{m,k}$. So we can follow up with the asymptotic upper bounds for $r(K_l + K_{m,k}, K_n)$.

On the other side, we need to know the asymptotic lower bounds for $K_{1,m,k}$: complete graph Ramsey numbers. Chvátal has proved the following Lemma 2.

Lemma 2 *[2] Let $m, n \geq 2$, then for every tree T_m of order m we have*

$$r(T_m, K_n) = 1 + (m - 1)(n - 1). \tag{2}$$

It gives us the simple but fundamental result that, for any n, all trees are K_n-*good.* Whether there is the possibility that $r(K_{1,m,k}, K_n)$ is a linear function of n? This issue has long been answered by Erdös et al. [3]. That is, for an arbitrary graph F, as $n \to \infty$,

$$r(F, K_n) \geq (c - o(1)) \left(\frac{n}{\log n}\right)^{(e(F)-1)/(m-2)}.$$

This result shows that for any connected graph G, $r(G, K_n)$ is a linear function of n, if and only if G is a tree. In Section 3 we will introduce a generalization[6] of above lower bounds and give the asymptotic relationship between $r(K_{1,m,k}, K_n)$ and $r(K_{m,k}, K_n)$. So the issue facing us at the present time is how to narrow the gap between the upper bounds and lower bounds and reach the asymptotic order of $r(K_{1,m,k}, K_n)$. So far, we know that $r(K_3, K_n)$ has order of magnitude $n^2/\log n$ in [4], and know very little about the results of other graphs: complete graph Ramsey numbers.

Now, let us introduce the proof techniques used in this paper. Clearly, to obtain an upper bound of $r(H, K_n)$, one may turn to establish a lower bound of the independence number(the maximal size of some independent set) of any H-free graph with a fixed number of vertices. A classical theorem of Turán asserts that the independence number $\alpha(G)$ of any graph G with N vertices and average degree d satisfies $\alpha(G) \geq N/(1+d)$. In case G is triangle-free, Shearer[8] verified that $\alpha(G) \geq Nf(d)$, where $f(x) = \frac{x\log x - x + 1}{(x-1)^2}$ which is asymptotically equal to $\frac{\log x}{x}$ as $x \to \infty$. Li, et al[5] generalized Shearer's inequality in terms of the upper bound of the average degree of any neighborhood induced subgraph(see Lemma 3). So we can get the inequality on n and $N = r(H, K_n) - 1$, then resolve the upper bounds of $N + 1 = r(H, K_n)$. Throughout the remainder of this paper, we shall let G_v stand for the subgraph of a graph G induced by the neighborhood of v. Now we enter the proof of the upper bounds for $r(K_{1,m,k}, K_n)$.

2 Upper bounds for $r(K_{1,m,k}, K_n)$

We will use the function $f_m(x)$([5],[7]), defined as follows

$$f_m(x) = \int_0^1 \frac{(1-t)^{1/m}dt}{m + (x-m)t}, \quad x \geq 0, \quad m \geq 1,$$

which plays a central role. We introduce some basic properties of $f_m(x)$ which are needed in this paper. Clearly, $f_m(x)$ satisfies the differential equation

$$x(x-m)f'_m(x) + (x+1)f_m(x) = 1.$$

Moreover, $f_m(x)$ is completely monotonic on $(0, \infty)$, that is, $(-1)^k f_m^{(k)}(x) \geq 0$ for all $k \geq 0$ and $x \geq 0$. In particular, f_m is positive, decreasing and convex .

Since $(1-t)^{1/m} \geq (1-t)$ for $0 \leq t \leq 1$ and $m \geq 1$, a simple calculation gives

$$\begin{aligned} f_m(x) &\geq \int_0^1 \frac{(1-t)dt}{m + (x-m)t} = \frac{x\log(x/m) - (x-m)}{(x-m)^2} \\ &> \frac{\log(x/m) - 1}{x}, \quad x > m, \end{aligned} \tag{3}$$

where $\log x$ is the natural logarithmic function. [With $(x/m) = 1 + u$, the last inequality is equivalent to $(1+2u)\log(1+u) > u$, which holds for all $u > 0$ since $\log(1+u) > u/(1+u)$.]

Also, see Ref.[5],

$$f_m(x) \geq 1/(1+x), \quad \text{if } x \geq m. \tag{4}$$

Lemma 3 *[5] Let G be a graph with N vertices and average degree d (the average degree of graph G of order n is defined to be $2e(G)/n$). If for any vertex v of G, the average degree of G_v is at most a, then $\alpha(G) \geq N f_{a+1}(d)$.*

By combining the upper bounds for $r(K_{m,k}, K_n)$ in (1) and Lemma 3, we shall manage to get the following upper bounds for $r(K_{1,m,k}, K_n)$.

Theorem 1 *For any two fixed integers $k \geq m \geq 2$, as $n \to \infty$,*

$$r(K_{1,m,k}, K_n) \leq (k-1+o(1)) \left(\frac{n}{\log n}\right)^{m+1}. \tag{5}$$

Proof. Let G be a graph of order $N = r(K_{1,m,k}, K_n) - 1$ such that G contains no $K_{1,m,k}$ and $\alpha(G) \leq n-1$. Then for each vertex v of G, we have that the degree of v is at most $r(K_{m,k}, K_n) - 1$, and the maximum degree and therefore the average degree of G_v is at most $r(K_{m-1,k}, K_n) - 1$. Thus, it follows from Lemma 3 and the properties of $f_m(x)$ that

$$n > \alpha(G) \geq N f_a(r(K_{m,k}, K_n) - 1) \geq N f_a(r(K_{m,k}, K_n)), \tag{6}$$

where $a = r(K_{m-1,k}, K_n) < r(K_{m,k}, K_n)$. By means of replacing $r(K_{m,k}, K_n)$ in the last inequality by its upper bounds in (1) and applying the inequality (3) we can get that as $n \to \infty$,

$$n \geq N f_a((k-1+o(1)) \left(\frac{n}{\log n}\right)^m) \geq N \frac{\log \frac{(k-1+o(1))(n/\log n)^m}{a} - 1}{(k-1+o(1))(n/\log n)^m}. \tag{7}$$

For $m = 2$, note that by Chvátal's theorem(Lemma 2), $a = r(K_{1,k}, K_n) = k(n-1)+1$, So (7) becomes $n \geq N \frac{(1-o(1))\log n}{(k-1+o(1))(n/\log n)^2}$, which yields $N \leq (k-1+o(1)) \left(\frac{n}{\log n}\right)^{2+1}$.

For $m \geq 3$, since $a = r(K_{m-1,k}, K_n) \leq (k-1+o(1))(n \backslash \log n)^{m-1}$, (7) becomes $n \geq N \frac{(1-o(1))\log n}{(k-1+o(1))(n/\log n)^m}$, which yields $N \leq (k-1+o(1)) \left(\frac{n}{\log n}\right)^{m+1}$. Therefore, as $n \to \infty$,

$$r(K_{1,m,k}, K_n) = N + 1 \leq (k-1+o(1)) \left(\frac{n}{\log n}\right)^{m+1}.$$

This completes the proof of Theorem 1.

3 The asymptotic bounds for some related Ramsey numbers

In this section we will introduce the lower bounds for $r(K_{m,k}, K_n)$ and $r(K_{1,m,k}, K_n)$, and give the asymptotic relationship between the two. In addition, we also introduce the upper bounds and lower bounds for wheel(W_4): complete graph Ramsey number.

Lemma 4 *[6] For any fixed integer $m \geq 3$, constants $\delta > 0$ and $\alpha \geq 0$, if F is a graph on m vertices and G is a graph on n vertices with $e(G) \geq (\delta - o(1))n^2/(\log n)^\alpha$ as $n \to \infty$, then there exists a constant $c = c(m, \delta) > 0$ such that*

$$r(F, G) \geq (c - o(1)) \left(\frac{n}{(\log n)^{\alpha+1}} \right)^{(e(F)-1)/(m-2)}.$$

By Lemma 4 we can get the following Corollary 1 and Corollary 2.

Corollary 1 *For any two fixed integers $k \geq m \geq 2$, there is a constant $c = c(m, k) > 0$ such that as $n \to \infty$,*

$$r(K_{m,k}, K_n) \geq (c - o(1)) \left(\frac{n}{\log n} \right)^{\frac{mk-1}{m+k-2}}.$$

Corollary 2 *For any two fixed integers $k \geq m \geq 2$, there is a constant $c = c(1, m, k) > 0$ such that as $n \to \infty$,*

$$r(K_{1,m,k}, K_n) \geq (c - o(1)) \left(\frac{n}{\log n} \right)^{1+\frac{mk}{m+k-1}}.$$

So we can see the gap between the upper bounds and the lower bounds for $r(K_{1,m,k}, K_n)$. I would like to know what the the asymptotic order of $r(K_{1,m,k}, K_n)$ is, does the index of $\frac{n}{\log n}$ have relevance to k? These are all the directions of our research.

Theorem 2 *For any two fixed integers $k > m^2 - 3m + 3$, $m \geq 2$, as $n \to \infty$, we have*

$$r(K_{1,m,k}, K_n) \leq (1 + o(1)) \frac{m + k - 2}{k - (m^2 - 3m + 3)} r(K_{m,k}, K_n) \frac{n}{\log n}. \quad (8)$$

Outline of proof. The identical analysis with Theorem 1 shows that $n \geq N^{\frac{\log \frac{r(K_{m,k},K_n)}{r(K_{m-1,k},K_n)} - 1}{r(K_{m,k},K_n)}}$ from (6) and (3). For the numerator on the right side of the inequality we replace $r(K_{m,k}, K_n)$ by its lower bounds in Corollary

1 and replace $r(K_{m-1,k}, K_n)$ by its upper bounds in Lemma 1. Simple asymptotic calculations show that (8) holds.

The role of Theorem 2 is that it inspired us to use mathematical induction to derive the asymptotic upper bounds for $r(K_l + K_{m,k}, K_n)$ on the base of that of $r(K_{m,k}, K_n)$ and $r(K_1 + K_{m,k}, K_n)$ (See Section 4).

Let C_m stand for the cycle of length m and W_m for the wheel with m spokes. That is, $W_m \cong K_1 + C_m$. In addition, we noted that $W_4 \cong K_1 + K_{2,2} \cong K_{1,2,2}$. As an application of Theorem 1 and Corollary 2, we can get the following Corollary 3.

Corollary 3 *As $n \to \infty$, there exists a constant $c = c(1,2,2) > 0$ such that*

$$(c - o(1))\left(\frac{n}{\log n}\right)^{7/3} \le r(W_4, K_n) \le (1 + o(1))\left(\frac{n}{\log n}\right)^3.$$

In [9] the writers have given that $r(W_4, K_n) \le (1 + o(1))c(4)(\frac{n}{\log n})^3$, where $c(4) = 3240\sqrt[3]{3}$. So we know that Corollary 3 improves the upper bounds for $r(W_4, K_n)$.

4 Upper bounds for $r(K_l + K_{m,k}, K_n)$

In this section we give the upper bounds for $r(K_l + K_{m,k}, K_n)$, as an application of which we give the upper bounds for $r(K_l - 2e, K_n)$, where $K_l - 2e$ is formed from K_l by deleting its two independent edges. So we know that $K_l - 2e \cong K_{l-4} + K_{2,2}$.

Theorem 3 *Let m, k and l be any three fixed integers and $k \ge m \ge 2$, $l \ge 0$. Then, as $n \to \infty$,*

$$r(K_l + K_{m,k}, K_n) \le (k - 1 + o(1))\left(\frac{n}{\log n}\right)^{l+m}. \tag{9}$$

Proof. We apply induction on l. For $l = 0$, (1) implies that (9) holds. For $l = 1$, (5) implies that (9) holds. Our statement follows.

Suppose the statement holds for $1, 2, ..., l$. We proceed to the induction step. Let $r(l, m, k; n)$ denote $r(K_l + K_{m,k}, K_n)$ and let G be a graph of order $N = r(l+1, m, k; n) - 1$ such that G contains no $K_{l+1} + K_{m,k}$ and that $\alpha(G) \le n - 1$. Then for each vertex v of G, we have that the degree of v is at most $r(l, m, k; n) - 1$, and the maximum degree and therefore the average degree of G_v is at most $r(l-1, m, k; n) - 1$. Thus, by Lemma 3, we have

$$n > \alpha(G) \ge N f_a(r(l, m, k; n) - 1) \ge N f_a(r(l, m, k; n)), \tag{10}$$

where $a = r(l-1, m, k; n) < r(l, m, k; n)$. Now let ϵ be an arbitrary number with $0 < \epsilon < 1$. Then, by (3) we know that there exists an $M > 0$ such that $f_a(x) > (1-\epsilon)\log(x/a)/x$ whenever $x/a > M$. We decompose the set of natural numbers into n' and n'' such that

$$\frac{r(l, m, k; n')}{r(l-1, m, k; n')} > (n')^{1-\epsilon},$$

$$\frac{r(l, m, k; n'')}{r(l-1, m, k; n'')} \leq (n'')^{1-\epsilon}.$$

Thus $\log \frac{r(l,m,k;n')}{r(l-1,m,k;n')} > (1-\epsilon)\log n'$. Without loss of generality, we may suppose that all $n', (n')^{(1-\epsilon)} > M$. So from (10) it follows that

$$n' > N f_a(r(l, m, k; n')) \geq (1-\epsilon) N \frac{\log \frac{r(l,m,k;n')}{a}}{r(l, m, k; n')} \geq \frac{(1-\epsilon)^2 N \log n'}{r(l, m, k; n')}.$$

where $a = r(l-1, m, k; n')$. Hence $N \leq \frac{n'}{(1-\epsilon)^2} \frac{r(l,m,k;n')}{\log n'}$, and the desired inequality for $r(l+1, m, k; n')$ follows from the inductive hypothesis on $r(l, m, k; n')$. Recall the inequality (4), $f_a(x) \geq 1/(1+x)$, if $x \geq a$. We get

$$n'' > N f_a(r(l, m, k; n'')) \geq N/(1 + r(l, m, k; n'')),$$

where $a = r(l-1, m, k; n'') < r(l, m, k; n'')$. Hence,

$$N \leq n''(1 + r(l, m, k; n'')) \leq n''(1 + (n'')^{1-\epsilon} r(l-1, m, k; n'')).$$

The desired inequality for $r(l+1, m; n'')$ follows from the inductive hypothesis on $r(l-1, m, k; n'')$ since $(n'')^{2-\epsilon} < (n''/\log n'')^2$ for sufficiently large n''.

This completes the proof of Theorem 3.

By Lemma 4 and Theorem 3 we can get the following Corollary 4.

Corollary 4 *Let l be any fixed integer and $l \geq 4$, then, as $n \to \infty$, there exists a constant $c = c(l-4, 2, 2) > 0$ such that*

$$(c - o(1)) \left(\frac{n}{\log n}\right)^{\frac{(l-3)(l+2)}{2(l-2)}} \leq r(K_l - 2e, K_n) \leq (1 + o(1)) \left(\frac{n}{\log n}\right)^{l-2}.$$

References

[1] Y. Caro, Y. Li, C.C. Rousseau, Y. Zhang, Asymptotic bounds for some bipatite graph: complete graph Ramsey numbers[J]. *Discrete Mathematics,* 2000, **220**: 51-56.

[2] V. Chvátal, Tree-complete graph Ramsey numbers[J]. *Journal of Graph Theory,* 1977, **1**: 93.

[3] P. Erdös, R.J. Faudree, C.C. Rousseau, R.H. Schelp, A Ramsey problem of Harary on graphs with prescribed size[J]. *Discrete Mathematics,* 1987, **67**: 227-233.

[4] J.H. Kim, The Ramsey number $r(3,t)$ has order of magnitude $t^2/\log t$[J]. *Random Structures and Algorithms,* 1995, **7**: 173-207.

[5] Y. Li, C.C. Rousseau, W. Zang, Asymptotic upper bounds for Ramsey functions[J]. *Graphs and Combinatorics* 2001, **17**: 123-128.

[6] Y. Li, W. Zang, Ramsey numbers involving large dense graphs and bipartite Turán numbers[J]. *J.Combin.Theory Ser.B* 2003, **87**: 280-288.

[7] Y. Li, C.C. Rousseau, On book-complete graph Ramsey numbers[J]. *J.Combin.Theory Ser.B* 1996, **68**: 36-44.

[8] J.B. Shearer, A note on the independence number of triangle-free graphs[J]. *Discrete Mathematics* 1983, **46**: 83-87.

[9] H. Song, L. Bai, S. Liu, Asymptotic upper bounds for wheel: complete graph Ramsey numbers[J]. *Acta Mathematica Scientia, Ser.A* 2006, **26**: 741-746.

[10] J. Spencer, Asymptotic lower bounds for Ramsey functions[J]. *Discrete Mathematics* 1977, **20**: 69-76.