Multicolored Spanning Subgraphs in G-Colorings of Complete Graphs

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Abstract

Let $G = \{g_1, \ldots, g_n\}$ be a finite abelian group. Consider the complete graph with the vertex set $\{g_1, \ldots, g_n\}$. The *G-coloring* of K_n is a proper edge coloring in which the color of edge $\{g_i, g_j\}$ is $g_i + g_j$, $1 \leq i < j \leq n$. We prove that in the G-coloring of the complete graph K_n, there exists a multicolored Hamilton path if G is not an elementary abelian 2-group. Furthermore, we show that if n is odd, then the G-coloring of K_n can be decomposed into multicolored 2-factors and there are exactly $\frac{l_r}{2}$ multicolored r-uniform 2-factors in this decomposition where l_r is the number of elements of order r in G, $3 \leq r \leq n$. This provides a generalization of a recent result due to Constantine which states: For any prime number $p > 2$, there exists a proper edge coloring of K_p which is decomposable into multicolored Hamilton cycles.

Mathematics Subject Classification: 05C05, 05C15, 05C70.

Keywords: Complete graph, multicolored tree, harmonious.

1. Introduction

Let G be a graph. A *spanning subgraph* of a graph G is a subgraph H with $V(H) = V(G)$. A *Hamilton cycle*, *Hamilton path* in G is a spanning cycle, spanning path in G, respectively. A subgraph in an edge colored graph is called *multicolored* if all its edges receive distinct colors. A *proper k-edge coloring* of a graph G is a mapping from $E(G)$ into a set of colors $\{1, \ldots, k\}$ such that incident edges of G receive distinct colors. The *edge chromatic number* of a graph G is the minimum number k for which G has a proper k-edge coloring. A *k-regular* graph is a graph all of whose degrees are k. A *k-factor* of a graph G is a spanning k-regular subgraph of G. An *r-uniform 2-factor* is a 2-factor all of whose cycles have length r. Throughout this paper K_m and $K_{m,n}$ denote the complete graph of order m and the complete bipartite graph with partite sets of sizes m and n, respectively. It is well-known that the edge chromatic number of K_m is m, if m is odd and $m-1$, if m is even. [15, p.15]

Let the edges of K_{2m} be properly colored by $2m-1$ colors. We say that the complete graph K_{2m} admits a *multicolored tree decomposition* (MTD) if all edges can be decomposed into m isomorphic multicolored spanning trees.

The following are two interesting conjectures on the decomposition of edge colored complete graphs into multicolored spanning trees.

Brualdi-Hollingsworth Conjecture [5]. *If $m > 2$, then in any proper edge coloring of K_{2m} with $2m-1$ colors, all edges can be partitioned into m multicolored spanning trees.*

Constantine's Conjecture [7]. *If $m > 2$, then any proper edge coloring of K_{2m} with $2m-1$ colors admits an MTD.*

It was proved in [3] that a complete graph on $2m$ ($m \neq 2$) vertices, K_{2m}, can be properly edge colored with $2m - 1$ colors in such a way that the edges of K_{2m} can be decomposed into m multicolored isomorphic spanning trees. In [1], the existence of multicolored spanning trees in an edge colored complete graphs (coloring is not necessary proper) are studied.

Let G be a finite abelian group of order n. In this paper using the group G, we properly color all edges of the complete graph K_n with n colors (the elements of G) and prove that if G is not an elementary abelian 2-group, then K_n contains a multicolored Hamilton path. For odd n, we prove that all edges of K_n can be decomposed into multicolored 2-factors and there are exactly $\frac{l_r}{2}$ multicolored r-uniform 2-factors in this decomposition where l_r is the number of elements of order r in G, $3 \leq r \leq n$.

2. Harmonious and Semi-Harmonious Groups

Let $G = \{g_1, \ldots, g_n\}$ be an abelian group of order n. Consider the complete graph K_n with the vertex set $\{g_1, \ldots, g_n\}$. Color the edge $\{g_i, g_j\}$ by $g_i + g_j$, for $1 \leq i, j \leq n, i \neq j$. Obviously, we obtain a proper edge coloring. We call this edge coloring of K_n the *G-coloring* of K_n. A group G is called *harmonious* if the elements of G can be listed as $g_1, \ldots, g_n$ so that $G = \{g_1 + g_2, g_2 + g_3, \ldots, g_{n-1} + g_n, g_n + g_1\}$. The sequence $g_1, \ldots, g_n$ is called a *harmonious* sequence. We say that G is *semi-harmonious* if the elements of G can be listed as $g_1, \ldots, g_n$ so that $g_1+g_2, g_2+g_3, \ldots, g_{n-1}+g_n$ are distinct. The sequence $g_1, \ldots, g_n$ is called a *semi-harmonious* sequence.

Let G be an abelian group of order n. Then in the G-coloring of K_n, a harmonious and a semi-harmonious sequence of G are corresponding to a multicolored Hamilton cycle and a multicolored Hamilton path, respectively.

The following interesting theorems were proved in [4].

Theorem 1. *Let G be a group of odd order. Then G is a harmonious.*

Theorem 2. *If G is a finite non-trivial abelian group, then G is harmonious if and only if G has a non-cyclic or trivial Sylow 2-subgroup and G is not an elementary abelian 2-group.*

Theorem 3. *Let n be a natural number. Then Z_n is semi-harmonious.*

Proof. If n is odd, then the result follows from Theorem 1. If n is even, then we consider the following sequence: $a_{2i-1} = i, a_{2i} = \frac{n}{2} + i$, for $i = 1, \ldots, \frac{n}{2}$. Thus we have $a_{2i-1} + a_{2i} = \frac{n}{2} + 2i$ and $a_{2i} + a_{2i+1} = \frac{n}{2} + 2i + 1$ for $i = 1, 2, \ldots, \frac{n}{2} - 1$ and $a_{n-1} + a_n = \frac{n}{2}$. Clearly, these numbers are distinct and so Z_n is semi-harmonious. □

Corollary 1. *Let n be an even number. Then Z_n has two semi-harmonious sequences $1 = a_1, a_2, \ldots, a_n = 0$, such that $\frac{n}{2}+1 \notin \{a_1+a_2, a_2+a_3, \ldots, a_{n-1}+a_n\}$ and $\frac{n}{2}+1 = b_1, b_2, \ldots, b_n = \frac{n}{2}$, $\frac{n}{2}+1 \notin \{b_1+b_2, b_2+b_3, \ldots, b_{n-1}+b_n\}$.*

Proof. The first sequence is that sequence given in the proof of Theorem 3. For the second one we note that if $g_1, \ldots, g_n$ is a semi-harmonious sequence, then clearly for any $a \in G$, $g_1 + a, \ldots, g_n + a$ is a semi-harmonious sequence too. Now, define $b_i = a_i + \frac{n}{2}$, for $i = 1, \ldots, n$. The proof is complete. □

Lemma 1. *Let n be an even number. If H is a group of odd order, then $Z_n \times H$ is a semi-harmonious sequence.*

Proof. Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be those sequences given in Corollary 1. Let $|H| = m$. By Theorem 1, there exists a harmonious sequence $h_1, \ldots, h_m$ in H. Consider the following sequence:

$$(a_1, h_1), \ldots, (a_n, h_1), (b_1, h_2), \ldots, (b_n, h_2), (a_1, h_3), \ldots, (a_n, h_3),$$

$$(b_1, h_4), \ldots, (b_n, h_4), \ldots, (b_1, h_{m-1}), \ldots, (b_n, h_{m-1}), (a_1, h_m), \ldots, (a_n, h_m).$$

Since $|H|$ is odd, we have $H = \{2h \mid h \in H\}$. Therefore it is not hard to see that the above sequence is a semi-harmonious sequence. □

Theorem 4. *Let G be a finite abelian group. If G is not an elementary abelian 2-group, then G is semi-harmonious.*

Proof. If G has odd order, then G is semi-harmonious by Theorem 2. By Theorem 3 and Theorem 2 it is enough to prove the theorem for the abelian groups of the form $Z_{2^n} \times H$, where H is an abelian group of odd order. Now, Lemma 1 completes the proof. □

The following corollary is an immediate consequence of Theorem 4.

Corollary 2. *Let G be a finite abelian group of order n which is not an elementary abelian 2-group. Then the G-coloring of the complete graph K_n contains a multicolored Hamilton path.*

Let G be a finite group of order n. A *transversal* of the Cayley group table of G is a set of n cells which are in distinct rows and columns such that these cells contain n different elements. The following theorem was proved in [11].

Theorem 5. *Let G be a finite solvable group of order n such that the Sylow 2-subgroups of G are trivial or non-cyclic. Then the Cayley group table of G has a transversal.*

Theorem 6. *Let G be a finite abelian group of odd order n. Then the G-coloring of K_n can be decomposed into multicolored 2-factors. Furthermore, if l_r is the number of elements of order r in G, $3 \leq r \leq n$, then in this decomposition, there are exactly $\frac{l_r}{2}$ multicolored r-uniform 2-factors.*

Proof. Let C be the Cayley group table of G. Since n is odd, the main diagonal of C is a transversal. Let e be the identity of G. We claim that C has n mutually disjoint transversals C_g, $g \in G$ in which C_e is the main diagonal of G. For any $g \in G$, in the ith row of G, $1 \leq i \leq n$, choose the element $g + 2g_i$. Clearly, it is a transversal since $G = \{g + 2g_i | g_i \in G\}$.

Let this transversal be C_g. Since $g + 2g_i = g' + 2g_i$ implies that $g = g'$, we conclude that $\{C_g | g \in G\}$ are n disjoint transversals which cover all cells of C. On the other hand, since C is symmetric, thus if a transversal contains the ijth cell of C, it cannot contain the jith cell of C. Thus every transversal except C_e is corresponding to a multicolored 2-factor. Let $g \in G$. The transversal C_g should have $g + 2g_i$ from the ith row, where $1 \leq i \leq n$. But this element is in the $(g_i, g + g_i)$th cell of C. Now, consider the $(g + g_i)$th row of C. The transversal corresponding to this row should have $3g + 2g_i$. But this entry is in the $(g + g_i, 2g + g_i)$th cell of C. By continuing this procedure we reach to a multicolored cycle with the vertices,

$$g_i, g + g_i, 2g + g_i, \ldots, o(g)g_i + g_i = g_i,$$

$1 \leq i \leq n$, which has length $o(g)$. Therefore the transversal corresponding to g is actually a multicolored $o(g)$-uniform 2-factor. Also, if we repeat the procedure for $-g$ instead of g, we obtain a multicolored cycle with the vertices,

$$g_i, -g + g_i, -2g + g_i, \ldots, -o(g)g + g_i = g_i,$$

that is the multicolored cycle corresponding to C_{-g}. Indeed the transpose of C_g in the Cayley group table of G is C_{-g}. This completes the proof. □

Constantine proved that for any prime number $p > 2$, there exists a proper edge coloring of K_p that is decomposable into multicolored Hamilton cycles, see [8]. Also recently it was proved that for any odd n, there exists a proper edge coloring of K_n which is decomposable into multicolored Hamilton cycles, see [10]. Now, as a corollary of Theorem 6, we obtain Constantine's result.

Corollary 3. *Let $p \geq 3$ be a prime number. Then the Z_p-coloring of K_p can be decomposed into multicolored Hamilton cycles.*

3. Stong Colorings of Complete Graphs

Let G be a finite group of order n with a generating set Ω such that for each $a \in \Omega$, $o(a)$ is even. Stong used the group G to introduce a proper edge coloring for Cayley(G, Ω) as follows(see [13]).

Let $g \in G$ and $a \in \Omega$. Assume that $o(a) = 2k$, where $k > 1$ is natural number. Consider the cycle C_{2k} on the vertices $g, ga, ga^2, \ldots, ga^{2k-1}$. Now, color the edges $\{ga^i, ga^{i+1}\}$ by a for $i = 0, 2, \ldots, 2k-2$ and by a^{-1} for $i = 1, 3, \ldots, 2k-1$. Let $S = G \setminus \{g, ga, ga^2, \ldots, ga^{2k-1}\}$. If $S \neq \emptyset$, then choose $g' \in S$ and repeat the argument for g' instead of g. We continue this procedure until all elements of G are covered. Using this we obtain a $(2k)$-uniform 2-factor such that all edges of every cycle in this 2-factor alternatively colored by a and a^{-1}. Now, consider $b \in \Omega \setminus \{a, a^{-1}\}$ and repeat the procedure for b instead of a. Therefore we obtain another 2-factor for G containing even cycles of length $o(b)$ such that all edges of each even cycle alternatively colored by color b and b^{-1}. Continuing this algorithm we find a proper edge coloring of G with colors in Ω. If $k = 1$, then we find a 1-factor $\{\{g, ga\} | g \in G\}$, all of whose edges have color a.

Note that this kind of coloring is not unique. If we start by ga instead of a, then we obtain a coloring in which the color of the edge $\{ga, ga^2\}$ is a. Indeed if Ω has l_i elements of order i, $1 \leq i \leq n$, then the number of these colorings is $2^{\sum_{i=3}^{n} \frac{n l_i}{2i}}$.

We call any coloring obtained by this method a *Stong G-coloring*, or shortly an *SG-coloring* for G. Note that if G is a group of order 2^n and $\Omega = G \setminus \{e\}$, then $K_{2^n} = Cayley(G, \Omega)$. Now, we have two conjectures.

Conjecture 1. *Let $G = Z_{2^n}$ and $n \geq 3$ be a natural number. If the edge coloring of K_{2^n} is an SG-coloring, then K_{2^n} admits an MTD.*

Conjecture 2. *Let G be a group of order 2^n, $n \geq 3$. If the edge coloring of K_{2^n} is an SG-coloring, then K_{2^n} admits an MTD.*

We show that these conjectures are equivalent. Before stating the proof we need a result from group theory.

Lemma 2. *Let G be a finite group of prime power order such that all of whose maximal subgroups are cyclic. Then G is either cyclic or isomorphic to the quaternion group of order* 8.

Proof. By Theorem 4.1 and Theorem 4.2 Part (iii) of [14], we are done. □

Theorem 7. *Conjecture* 1 *and Conjecture* 2 *are equivalent.*

Proof. Let G be a group of order 2^n. It is enough to show that if Conjecture 1 is true and the edge coloring of K_{2^n} is an SG-coloring, then K_{2^n} admits an MTD. The proof is by applying induction on n. For $n = 3$, $|G| = 8$. By [6], for any proper 7-edge coloring, K_8 admits an MTD and so that the assertion holds. Now, suppose that G is a group of order 2^n, $n > 3$. If G is a cyclic group, then there is nothing to prove. Thus by Lemma 2 we may assume that G has a non-cyclic subgroup H such that $|H| = 2^{n-1}$. If we consider the induced subgraph on H, then we obtain an SH-coloring for $K_{2^{n-1}}$. We note that the colors used in the edge coloring of H are the elements of H. Thus by induction hypothesis, H admits an MTD with 2^{n-2} spanning trees.

Let $G = H \bigcup aH$, where H and aH are distinct left cosets of H in G. Similarly, the induced subgraph on aH, is isomorphic to $K_{2^{n-1}}$, which is edge colored by the colors in H, and so it admits an MTD with 2^{n-2} spanning trees. Now, we claim that all edges of the complete bipartite graph $K_{2^{n-1},2^{n-1}}$ with part sets (H, aH) can be decomposed into multicolored perfect matchings. Since H is a solvable group, by Theorem 5, the Cayley group table of H is decomposable into disjoint transversals. Thus the table $aH \times H$ and so the table of $Ha^{-1} \times H$ is decomposable into disjoint transverals. But the color of every edge with an end point in H and other end point in aH is one of the elements appeared in the table $Ha^{-1} \times H$.

Therefore we can decompose all edges of $K_{2^{n-1},2^{n-1}}$ into 2^{n-1} multicolored perfect matchings with colors in aH. Attach each of multicolored perfect matchings to exactly one of the spanning trees of H or aH. Each edge in the complete graph $K_{2^{n-1},2^{n-1}}$ has a color of the set aH. Therefore K_{2^n} can be decomposed into 2^{n-1} multicolored spanning trees and the proof is complete. □

There are six non-isomorphic edge colorings of K_8, see [5]. We prove that K_8 with edge coloring (a) has no multicolored Hamilton path (see Theorem 9). In the following we give a decomposition into multicolored Hamilton paths for the edge colorings (b), (c) and (e) in [5]. By a computer search by E. Ghorbani we noted here that by the two colorings (d) and (f), K_8 cannot decompose into multicolored Hamilton paths.

	P_1	P_2	P_3	P_4
1 :	23	45	67	01
2 :	46	02	57	13
3 :	03	47	12	56
4 :	15	26	04	37
5 :	14	36	05	27
6 :	25	17	34	06
7 :	07	35	16	24

(b)

	P_1	P_2	P_3	P_4
1 :	67	23	01	45
2 :	46	02	13	57
3 :	12	56	47	03
4 :	04	15	36	27
5 :	05	37	24	16
6 :	17	06	25	34
7 :	35	14	07	26

(c)

	P_1	P_2	P_3	P_4
1 :	23	01	45	67
2 :	57	46	02	13
3 :	14	27	03	56
4 :	04	16	25	37
5 :	26	34	17	05
6 :	06	35	47	12
7 :	15	07	36	24

(e)

	P_1	P_2	P_3	P_4
1 :	01	23	45	67
2 :	13	02	57	46
3 :	47	56	03	12
4 :	04	37	26	15
5 :	27	05	14	36
6 :	35	17	06	24
7 :	16	34	25	07

(a)

	P_1	P_2	P_3	P_4
1 :	01	23	45	67
2 :	46	02	57	13
3 :	47	56	03	12
4 :	27	35	16	04
5 :	05	17	26	34
6 :	37	14	25	06
7 :	07	24	36	15

(d)

	P_1	P_2	P_3	P_4
1 :	01	23	45	67
2 :	36	02	57	14
3 :	15	46	03	27
4 :	37	25	16	04
5 :	34	17	26	05
6 :	12	47	35	06
7 :	07	24	56	13

(f)

Consider the following SZ_8-coloring for K_8. Then K_8 admits an MTD. To see this consider the complete graph K_8 with the vertex set $\{0, \dots, 7\}$. The following table gives a proper edge coloring of K_8 with colors $1, \dots, 7$ and its decomposition into multicolored Hamilton paths. The ith row of this table contains all edges with color i, for $i = 1, \dots, 7$. Each column denotes all edges of a multicolored Hamilton path.

	P_1	P_2	P_3	$P4$
1 :	07	56	12	34
2 :	24	57	13	06
3 :	36	14	05	27
4 :	15	37	04	26
5 :	03	16	25	47
6 :	17	02	46	35
7 :	45	23	67	01

Consider the following SQ_8-coloring for K_8. Then K_8 admits an MTD. To see this consider the complete graph K_8 with the vertex set $\{\pm 1, \pm i, \pm j, \pm k\}$. The following table gives a proper edge coloring of K_8 with colors $\{-1, \pm i, \pm j, \pm k\}$ and its decomposition into multicolored Hamilton paths. The xth row of this table contains all edges with color x. Each column denotes all edges of a multicolored Hamilton path.

	P_1	P_2	P_3	P_4
$-1:$	$\{i,-i\}$	$\{j,-j\}$	$\{-k,k\}$	$\{-1,1\}$
$i:$	$\{-j,k\}$	$\{-1,-i\}$	$\{1,i\}$	$\{j,-k\}$
$-i:$	$\{-j,-k\}$	$\{j,k\}$	$\{1,-i\}$	$\{-1,i\}$
$j:$	$\{-i,-k\}$	$\{i,k\}$	$\{-1,-j\}$	$\{1,j\}$
$-j:$	$\{-1,j\}$	$\{1,-j\}$	$\{i,-k\}$	$\{-i,k\}$
$k:$	$\{1,k\}$	$\{-1,-k\}$	$\{-i,j\}$	$\{i,-j\}$
$-k:$	$\{i,j\}$	$\{1,-k\}$	$\{-1,k\}$	$\{-i,-j\}$

Consider the following SD_4-coloring for K_8, where $D_4 =< a,b \mid a^4 = b^2 = 1, bab = a^3 >$. Then K_8 admits an MTD. To see this consider the complete graph K_8 with the vertex set $\{e,a,a^2,a^3,b,ab,a^2b,a^3b\}$. The following table gives a proper edge coloring of K_8 with colors $\{a,a^2,a^3,b,ab,a^2b,a^3b\}$ and its decomposition into multicolored Hamilton paths. The ith row of this table contains all edges with color i. Each column denotes all edges of a multicolored Hamilton path.

	P_1	P_2	P_3	P_4
$a:$	$\{a^2b,a^3b\}$	$\{b,ab\}$	$\{a^2,a^3\}$	$\{e,a\}$
$a^2:$	$\{a,a^3\}$	$\{e,a^2\}$	$\{b,a^2b\}$	$\{ab,a^3b\}$
$a^3:$	$\{e,a^3\}$	$\{a^2b,ab\}$	$\{a,a^2\}$	$\{b,a^3b\}$
$b:$	$\{e,b\}$	$\{a^2,a^2b\}$	$\{a,a^3b\}$	$\{a^3,ab\}$
$ab:$	$\{a^2,a^3b\}$	$\{a,b\}$	$\{e,ab\}$	$\{a^3,a^2b\}$
$a^2b:$	$\{a,ab\}$	$\{a^3,a^3b\}$	$\{e,a^2b\}$	$\{a^2,b\}$
$a^3b:$	$\{a^2,ab\}$	$\{e,a^3b\}$	$\{a^3,b\}$	$\{a,a^2b\}$

It can be checked that in every $S(Z_4 \times Z_2)$-coloring and $S(Z_2)^3$-coloring of K_8 the union of all edges with two given colors is a 4-uniform 2-factor. So by [9], K_8 with these Stong colorings admits an MTD but no multicolored Hamilton path, see Theorem 9 in next section.

4. The Uniqueness of Decomposition of K_{2^n} into Commuting Perfect Matchings

In [2] it has been proved that two perfect matchings of K_n commute if and only if their union is a 4-uniform 2-factor. Also K_{2^n} is decomposable into commuting perfect matchings if and only if n is a 2-power. We recall that two 1-factorizations of a graph are isomorphic if there exists an automorphism such that it induces a permutation between the sets of 1-factors of two 1-factorizations. The following result was proved in [12]. In the following we give another proof.

Theorem 8. *Let n be a natural number. Then every two edge decompositions of K_{2^n} into commuting perfect matchings are isomorphic.*

Proof. Let c_1 and c_2 be two edge decompositions of K_{2^n} into commuting perfect matchings. Each of c_1 and c_2 induces a proper edge coloring. By induction on n we prove the theorem. For $n = 2$ the assertion is obvious. Thus assume that $n \geq 3$. By Lemma 2 of [9], c_1 and c_2 contain cliques H_1 and H_2, respectively such that $H_1 \simeq H_2 \simeq K_{2^{n-1}}$, and the number of colors used in edge colorings of H_1 and H_2 is $2^{n-1} - 1$. Without loss of generality assume that the set of all colors used in the edge colorings of H_1 and H_2 is $L = \{1, \ldots, 2^{n-1} - 1\}$. Set $H'_i = K_{2^n} \setminus V(H_i)$, for $i = 1, 2$. Since for each $v \in V(K_{2^n})$, we have $d(v) = 2^n - 1$, thus the set of colors used in the edges between H_i and H'_i is $\{1, \ldots, 2^n - 1\} \setminus L$, for $i = 1, 2$. Hence the set of colors in edge colorings of H'_i is L, for $i = 1, 2$.

Note that $H'_1 \simeq H'_2 \simeq K_{2^{n-1}}$. Since any two colors in K_{2^n} commute, thus any two colors in H_i and H'_i commutes, for $i = 1, 2$. Now, by induction hypothesis the 1-factorization of H_1 and the 1-factorization of H_2 are isomorphic. The same holds for H'_1 and H'_2. Thus we may assume that in two edge colorings of c_1 and c_2, the edges with colors in L are the same.

Now, let $e = uv$, $u \in V(H_1)$, $v \in V(H'_1)$ and $c_1(e) = i$, where $i \in \{1, \ldots, 2^n - 1\} \setminus L$. We know that for any j, $j \in L$, there are two edges ux and vy, such that $x \in V(H_1)$, $y \in V(H'_2)$ and $c_1(ux) = c_1(vy) = j$. Since the decomposition is commuting by a theorem of [2], the edges with colors i and j in coloring c_1 form a 4-uniform 2-factor. Thus $c_1(xy) = i$. This implies that for any i, $i \in \{1, \ldots, 2^n - 1\} \setminus L$, the edges with color i uniquely determined in the coloring of c_1. Since j is arbitrary, so all 1-factors uniquely determined and the proof is complete. □

Now, by Lemma 1 of [2], we have the following corollary.

Corollary 4. *Let G be the elementary abelian group of order 2^n. Then up to isomorphism there exists an SG-coloring. Furthermore, in a proper edge coloring of K_{2^n} the union of any two colors form a 4-uniform 2-factor if and only if the proper edge coloring is an SG-coloring.*

Conjecture 3. *Let G be a group of order 2^n and H be the elementary abelian group of order 2^n. If an SG-coloring of K_{2^n} is not isomorphic to the SH-coloring, then K_{2^n} contains a multicolored Hamilton path.*

Conjecture 4. *Let G be a group of order 2^n and H be the elementary abelian group of order 2^n. If an SG-coloring of K_{2^n} is not isomorphic to the SH-coloring, then K_{2^n} can be decomposed into multicolored Hamilton paths.*

Theorem 9. *Suppose that K_{2^n} can be properly edge colored such that the union of any two colors forms a 4-uniform 2-factor. Then K_{2^n} has no multicolored Hamilton path.*

Proof. Let c be a proper edge coloring of K_{2^n} with given property. By Corollary 4, this coloring is just an SG-coloring, where G is the elementary abelian 2-group of order 2^n. By the definition of SG-coloring one may see that if $G = \{0 = g_1, g_2, \ldots, g_{2^n}\}$, then the color of edge with end points g_i and g_j is $g_i + g_j$. Now, if we have a multicolored Hamilton path P, then each non-zero element of G has been appeared on the edges once. Clearly, for any i, the number of elements in $(Z_2)^n$ whose the ith components equal 1 is 2^{n-1}. Hence if two end points of P are g_i and g_j, then by sum of all colors on the edges of P, we obtain $g_i + g_j + 2\sum_{r \neq i,j} g_r = 0$. Thus $g_i = g_j$, a contradiction. The proof is complete. □

Acknowledgements. The first author is indebted to the School of Mathematics, Institute for Research in Fundamental Sciences (IPM) for support. Also, the first author is indebted to the Abdus Salam International Centre for Theoretical Physics (ICTP) for hospitality during his visit. The research of the first author was in part supported by a grant (No. 87050212) from IPM. Finally, the authors wish to thank professor Alireza Abdollahi for his constructive comments in the preparation of this paper.

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