Solutions of Some Diophantine Equations Using Generalized Fibonacci and Lucas Sequences

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Abstract

In this study, we deal with some Diophantine equations. By using the generalized Fibonacci and Lucas sequences, we obtain all integer solutions of some Diophantine equations such as $x^2-kxy-y^2=\mp 1$, $x^2-kxy+y^2=1$, $x^2-kxy-y^2=\mp(k^2+4)$, $x^2-(k^2+4)xy+(k^2+4)y^2=\mp k^2$, $x^2-kxy+y^2=-(k^2-4)$, and $x^2-(k^2-4)xy-(k^2-4)y^2=k^2$. Some of the results are known but we think that our proofs are new and different from the others.

Keywords: Fibonacci number; Lucas number; Binet formula; Diophantine equation.

MSC: 11B37, 11B39,11B50,11B99

1. Introduction

For our purpose, we introduce two kinds of generalized Fibonacci and Lucas sequences $\{U_n\},\{V_n\}$ and $\{u_n\},\{v_n\}$. The generalized Fibonacci sequence $\{U_n\}$ with parameter k, is defined by $U_0=0, U_1=1$ and $U_n=kU_{n-1}+U_{n-2}$ for $n\geq 2$ and the generalized Lucas sequence $\{V_n\}$ is defined similarly by $V_0=2, V_1=k$, and $V_n=kV_{n-1}+V_{n-2}$ for $n\geq 2$ where $k\geq 1$, is an integer. Also, $U_{-n}=(-1)^{n+1}U_n$ and $V_{-n}=(-1)^nV_n$ for all $n\in\mathbb{N}$. Moreover, the generalized Fibonacci sequence $\{u_n\}$ with parameter k, is defined by $u_0=0, u_1=1$ and $u_n=ku_{n-1}-u_{n-2}$ for $n\geq 2$ and the generalized Lucas sequence $\{v_n\}$ is defined similarly by

$v_0 = 2, v_1 = k$, and $v_n = kv_{n-1} - v_{n-2}$ for $n \geq 2$ where $k \geq 3$, is an integer. Also $u_{-n} = -u_n$ and $v_{-n} = v_n$ for all $n \in \mathbb{N}$. It can be shown that $V_n = U_{n-1} + U_{n+1}$ and $v_n = u_{n+1} - u_{n-1}$ for all $n \in \mathbb{Z}$. For more information about generalized Fibonacci and Lucas sequences one can consult [12], [13], [14].

2. Main Theorems

The characteristic equation of the recurrence relation of the sequence $\{U_n\}$ is $x^2 - kx - 1 = 0$ and the roots of this equation are $\alpha = \left(k + \sqrt{k^2+4}\right)/2$ and $\overline{\alpha} = \beta = \left(k - \sqrt{k^2+4}\right)/2$. It is clear that $\alpha\beta = -1$, $\alpha^2 = k\alpha + 1$, and $\alpha + \beta = k$. Now for $a,b,c,d \in \mathbb{Z}$, we define two binary operations in $\mathbb{Z} \times \mathbb{Z}$ by

$$(a,b)(c,d) = (kac + ad + bc, ac + bd) \text{ and } (a,b) + (c,d) = (a+c, b+d). \quad (2.1)$$

It can be easily seen that

$$(a,b)(c,d) = (kac + ad + bc, ac + bd) = (kca + cb + da, ca + db) = (c,d)(a,b).$$

Furthermore the identity element is $(0,0)$ for the $+$ operation, and $(0,1)$ for the multiplication operation as defined in (2.1). Then it is seen that $\mathbb{Z} \times \mathbb{Z}$ is a commutative ring with unit element $(0,1)$.

Let $\mathbb{Z}[\alpha] = \{a\alpha + b : a,b \in \mathbb{Z}\}$. Then it can be seen that $\mathbb{Z}[\alpha]$ is a subring of the algebraic integer ring of the real quadratic field $\mathbb{Q}(\sqrt{k^2+4})$ and $\mathbb{Z}[\alpha]$ is equal to the algebraic integer ring of the real quadratic field $\mathbb{Q}(\sqrt{k^2+4})$ when k^2+4 is square free. We define a function $\phi : \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z}[\alpha]$, given by

$$\phi((a,b)) = a\alpha + b.$$

In view of the fact that $(a\alpha + b)(c\alpha + d) = \alpha^2 ac + \alpha(ad + bc) + bd = (k\alpha + 1)ac + \alpha(ad + bc) + bd = (kac + ad + bc)\alpha + ac + bd$ and $(a\alpha + b) + (c\alpha + d) = (a + c)\alpha + b + d$ for $a,b,c,d \in \mathbb{Z}$, it is easy to see that ϕ is a ring homomorphism. Moreover, since ϕ is bijective, ϕ is an isomorphism.

If $\alpha x + y$ is a unit in $\mathbb{Z}[\alpha]$, then it can be shown that

$$-x^2 + kxy + y^2 = (\alpha x + y)(\overline{\alpha} x + y) = \pm 1.$$

Theorem 2.1. *The set of the units of $\mathbb{Z}[\alpha]$ is $\{\pm\alpha^n \mid n \in \mathbb{Z}\}$.*

Proof. To prove the theorem, it is sufficient to show that every unit $\omega \geq 1$ is of the form α^n for some $n \geq 0$. Firstly, we show that there is no unit ω such that

$1 < \omega < \alpha$. Assume that $\omega = \alpha x + y$ is a unit in $\mathbb{Z}[\alpha]$ such that $1 < \omega < \alpha$. Thus since $\alpha x + y$ is a unit, $-x^2 + kxy + y^2 = (\alpha x + y)(\overline{\alpha} x + y) = \pm 1$ and therefore $|\alpha x + y|\,|\overline{\alpha} x + y| = 1$. Since $|(\alpha x + y)(\overline{\alpha} x + y)| = 1$ and $1 < \alpha x + y$, it is seen that $|\overline{\alpha} x + y| < 1$. Therefore $-1 < \overline{\alpha} x + y < 1$. Then it follows that $0 < x(\alpha + \overline{\alpha}) + 2y$, i.e., $0 < kx + 2y$.

On the other hand, since $-1 < -\overline{\alpha} x - y < 1$ and $1 < \alpha x + y$, we see that $0 < (-\overline{\alpha} x - y) + (\alpha x + y) = (\alpha - \overline{\alpha})x = (\sqrt{k^2 + 4})x$. Therefore $0 < x$. Since x is an integer, we get $1 \leq x$. By the fact $1 < \alpha x + y < \alpha$, we obtain $y < \alpha - \alpha x = \alpha(1 - x)$. Since $1 \leq x$, we get $1 - x \leq 0$. Thus it follows that $y < \alpha(1 - x) \leq 0$. That is, $y < 0$. Moreover, since $-x^2 + kxy + y^2 = \mp 1$ and $1 \leq x$, we have $kxy + y^2 = \mp 1 + x^2 \geq 0$. Then it follows that $(kx + y)y = kxy + y^2 \geq 0$. Since $y < 0$, we get $kx + y \leq 0$. By the facts $0 < kx + 2y$ and $kx + y \leq 0$, we obtain

$$y = (kx + 2y) - (kx + y) > 0.$$

But this contradicts with the fact that $y < 0$.

Now let $\omega > 1$ be a unit and $\omega \neq \alpha$. If $\omega \neq \alpha^n$ for every integer $n \geq 2$, then it follows that $\alpha^m < \omega < \alpha^{m+1}$ for some $m \in \mathbb{N}$ with $m \geq 2$. Thus $1 < \omega/\alpha^m < \alpha$ and ω/α^m is a unit in $\mathbb{Z}[\alpha]$. But this is impossible. Therefore $\omega = \alpha^n$ for some $n \geq 2$. This shows that if $\omega \geq 1$ is any unit, then $\omega = \alpha^n$ for some $n \geq 0$. ■

Since the units of the ring $\mathbb{Z}[\alpha]$ are of the form $\pm\alpha^n$ for some $n \in \mathbb{Z}$, all units of $\mathbb{Z} \times \mathbb{Z}$ are in the form $\phi^{-1}(\pm\alpha^n)$.

From the definition of the function ϕ, it is seen that $\phi((1,0)) = 1 \cdot \alpha + 0 = \alpha$ and then we get $\phi^{-1}(\alpha) = (1,0)$. Therefore we obtain

$$\phi^{-1}(\pm\alpha^n) = \pm\phi^{-1}(\alpha^n) = \pm\left(\phi^{-1}(\alpha)\right)^n = \pm(1,0)^n.$$

Thus all units of $\mathbb{Z} \times \mathbb{Z}$ are in the form $\pm(1,0)^n$ for some $n \in \mathbb{Z}$. It can be seen that $\alpha x + y$ is a unit in $\mathbb{Z}[\alpha]$ if and only if $x^2 - kxy - y^2 = \mp 1$.

Theorem 2.2. *$(1,0)^n = (U_n, U_{n-1})$ and $(1,0)^{-n} = \left((-1)^{n+1}U_n, (-1)^n U_{n+1}\right)$ for all $n \in \mathbb{N}$.*

Proof. To prove this theorem we use mathematical induction. For $n = 1$ it is clear that $(1,0) = (U_1, U_0)$. Let $(1,0)^m = (U_m, U_{m-1})$. Then

$$(1,0)^{m+1} = (1,0)^m(1,0) = (U_m, U_{m-1})(1,0) = (kU_m + U_{m-1}, U_m) = (U_{m+1}, U_m)$$

and therefore we obtain $(1,0)^n = (U_n, U_{n-1})$ for all $n \in \mathbb{N}$.

As we did above, we use mathematical induction again to prove

$$(1,0)^{-n} = \left((-1)^{n+1}U_n, (-1)^n U_{n+1}\right).$$

For $n=1$, $(1,0)^{-1}=(1,-k)=(U_{-1},U_{-2})$. Now let

$$(1,0)^{-m}=\left((-1)^{m+1}U_m,(-1)^m U_{m+1}\right).$$

Then it follows that

$$\begin{aligned}(1,0)^{-(m+1)} &= (1,0)^{-m}(1,0)^{-1}=\left((-1)^{m+1}U_m,(-1)^m U_{m+1}\right)(1,-k)\\ &= ((-1)^m U_{m+1},(-1)^{m+1}(kU_{m+1}+U_m))\\ &= ((-1)^m U_{m+1},(-1)^{m+1}U_{m+2}).\end{aligned}$$

As a result, we obtain

$$(1,0)^{-n}=\left((-1)^{n+1}U_n,(-1)^n U_{n+1}\right)$$

for all $n\in\mathbb{N}$. ■

According to Theorem 2.2, it is obvious that

$$(1,0)^{-n}=\left((-1)^{n+1}U_n,(-1)^n U_{n+1}\right)=(U_{-n},U_{-n-1})$$

for all $n\in\mathbb{N}$. Therefore $(1,0)^n=(U_n,U_{n-1})$ for all $n\in\mathbb{Z}$.

Theorem 2.3. *$U_n^2-kU_nU_{n-1}-U_{n-1}^2=(-1)^{n+1}$ for all $n\in\mathbb{Z}$.*

Proof. If $n=0$, then the assertation is correct. Now let $n>0$ and $x=(1,0)$. By using $x^n=(1,0)^n=(U_n,U_{n-1})$ and $x^{-n}=(1,0)^{-n}=((-1)^{n+1}U_n,(-1)^n U_{n+1})$, we get immediately

$$\begin{aligned}(0,1) &= x^n x^{-n}=(U_n,U_{n-1})((-1)^{n+1}U_n,(-1)^n U_{n+1})\\ &= \left((-1)^n\left(-kU_n^2+U_nU_{n+1}-U_nU_{n-1}\right),(-1)^n\left(-U_n^2+U_{n-1}U_{n+1}\right)\right).\end{aligned}$$

It follows that $U_n^2-U_{n-1}U_{n+1}=(-1)^{n+1}$. Furthermore, if we use the fact that $U_{n+1}=kU_n+U_{n-1}$, then we see that

$$U_n^2-kU_nU_{n-1}-U_{n-1}^2=(-1)^{n+1}. \tag{2.2}$$

If $n<0$, then by using the definitions of U_n and U_{n-1} it can be shown that $U_n^2-kU_nU_{n-1}-U_{n-1}^2=(-1)^{n+1}$. ■

Lemma 1. *$\alpha^n=\alpha U_n+U_{n-1}$ and $\beta^n=\beta U_n+U_{n-1}$ for all $n\in\mathbb{Z}$. Furthermore $U_n=(\alpha^n-\beta^n)/\sqrt{k^2+4}$ and $V_n=\alpha^n+\beta^n$.*

Proof. Since $\phi((1,0)) = \alpha$, it is obvious that $\phi((1,0)^n) = (\phi((1,0)))^n = \alpha^n$. Also from the definition of ϕ we see that

$$\phi((1,0)^n) = \phi((U_n, U_{n-1})) = \alpha U_n + U_{n-1}.$$

Therefore it follows that

$$\alpha^n = \alpha U_n + U_{n-1}. \tag{2.3}$$

Since $\alpha\beta = -1$, we get

$$\begin{aligned} \beta^n &= \left(-\alpha^{-1}\right)^n = (-1)^n \alpha^{-n} = (-1)^n (\alpha U_{-n} + U_{-n-1}) \\ &= (-1)^n \left(\alpha(-1)^{n+1} U_n + (-1)^n U_{n+1}\right) = -\alpha U_n + U_{n+1} \end{aligned}$$

so that,

$$\beta^n = -\alpha U_n + U_{n+1} \tag{2.4}$$

and if we use $\alpha = k - \beta$, then we see that

$$\beta^n = \beta U_n + U_{n-1}. \tag{2.5}$$

From (2.3), (2.4) and (2.5), we obtain $V_n = \alpha^n + \beta^n$ and $U_n = (\alpha^n - \beta^n) / \sqrt{k^2+4}$. ■

Since $V_n = \alpha^n + \beta^n$ and $U_n = (\alpha^n - \beta^n) / \sqrt{k^2+4}$, it follows that $V_n = U_{n-1} + U_{n+1}$ for all $n \in \mathbb{Z}$.

Corollary 1. *$V_n^2 - (k^2+4)U_n^2 = 4(-1)^n$ for all $n \in \mathbb{Z}$.*

Proof. From Theorem 2.3, we have

$$U_{n+1}^2 - kU_{n+1}U_n - U_n^2 = (-1)^n \tag{2.6}$$

and multiplying both sides of (2.6) by 4, we get $(2U_{n+1} - kU_n)^2 - (k^2+4)U_n^2 = 4(-1)^n$. Since $2U_{n+1} - kU_n = V_n$ we obtain

$$V_n^2 - (k^2+4)U_n^2 = 4(-1)^n. \tag{2.7}$$

■

Lemma 2. *Let $x = (1,0)$. Then $(2,-k)x^n = (V_n, V_{n-1})$ for all $n \in \mathbb{Z}$.*

Proof. Since $x^{n+1} + x^{n-1} = (U_{n+1}, U_n) + (U_{n-1}, U_{n-2}) = (V_n, V_{n-1})$ and $x^{n+1} + x^{n-1} = x^n(x + x^{-1}) = x^n((1,0) + (1,-k)) = (2,-k)x^n$, the proof follows. ■

Theorem 2.4. *$V_n^2 - kV_nV_{n-1} - V_{n-1}^2 = (-1)^n(k^2+4)$ for all $n \in \mathbb{Z}$.*

Proof. Since $(2,-k)x^n = (V_n, V_{n-1})$ and

$$(2,-k)x^{-n} = (V_{-n}, V_{-n-1}) = ((-1)^n V_n, (-1)^{n+1} V_{n+1}),$$

it follows that

$$\begin{aligned}(0,k^2+4) &= (2,-k)x^n(2,-k)x^{-n} = (V_n,V_{n-1})(V_{-n},V_{-n-1}) \\ &= (V_n,V_{n-1})((-1)^n V_n,(-1)^{n+1}V_{n+1}) = (0,(-1)^n(V_n^2 - V_{n+1}V_{n-1})).\end{aligned}$$

That is, we get $k^2+4 = (-1)^n(V_n^2 - V_{n+1}V_{n-1})$. Taking $V_{n+1} = kV_n + V_{n-1}$, we see that $V_n^2 - kV_nV_{n-1} - V_{n-1}^2 = (-1)^n(k^2+4)$. Therefore

$$V_n^2 - kV_nV_{n-1} - V_{n-1}^2 = (-1)^n(k^2+4) \tag{2.8}$$

for all $n \in \mathbb{Z}$. ■

Theorem 2.5. *All integer solutions of the equation $x^2 - kxy - y^2 = \mp 1$ are given by $(x,y) = \mp(U_n, U_{n-1})$ with $n \in \mathbb{Z}$.*

Proof. It is seen from (2.2) that $(x,y) = \mp(U_n,U_{n-1})$ verifies the equation $x^2 - kxy - y^2 = \mp 1$. Now assume that $x^2 - kxy - y^2 = \mp 1$. Then $\alpha x + y$ is a unit in $\mathbb{Z}[\alpha]$. Therefore there exists some $n \in \mathbb{Z}$ such that $\alpha x + y = \mp \alpha^n$. Thus $(x,y) = \phi^{-1}(\alpha x + y) = \phi^{-1}(\mp \alpha^n) = \mp(\phi^{-1}(\alpha))^n = \mp(1,0)^n = \mp(U_n,U_{n-1})$. ■

Corollary 2. *All integer solutions of the equation $x^2 - xy - y^2 = \mp 1$ are given by $(x,y) = \mp(F_n, F_{n-1})$ with $n \in \mathbb{Z}$.*

Corollary 3. *All nonnegative integer solutions of the equation $x^2 - kxy - y^2 = \mp 1$ are given by $(x,y) = (U_n, U_{n-1})$ with $n \geq 0$.*

Proof. Assume that x and y are nonnegative integers and $x^2 - kxy - y^2 = \mp 1$. Then $\alpha x + y \geq 1$ and since $\alpha x + y$ is a unit, it follows that $\alpha x + y = \alpha^n$ for some $n \geq 0$. Then $\alpha x + y = \alpha^n = \alpha U_n + U_{n-1}$ and therefore the proof follows. ■

Theorem 2.6. *All nonnegative solutions of the equation $u^2 - (k^2+4)v^2 = \mp 4$ are given by $(u,v) = (V_n, U_n)$ with $n \geq 0$.*

Proof. If $(u,v) = (V_n,U_n)$, then from Corollary 1, it follows that $u^2 - (k^2+4)v^2 = \mp 4$. Assume that x and y are nonnegative integers and $u^2 - (k^2+4)v^2 = \mp 4$. Then $u^2 - k^2v^2$ is an even integer and therefore u and kv have the same parity. Thus if we take $x = (u+kv)/2$ and $y = v$, then we get

$$x^2 - kxy - y^2 = \frac{(u+kv)^2 - 2kv(u+kv) - 4v^2}{4} = \frac{u^2 - (k^2+4)v^2}{4} = \mp 1.$$

This shows that $x = (u+kv)/2 = U_{n+1}$ and $v = y = U_n$ with $n \geq 0$. Thus $u = 2x - kv = 2U_{n+1} - kU_n = V_n$. That is, $(u,v) = (V_n, U_n)$. ■

Theorem 2.7. *Let k^2+4 be a square free integer. Then all integer solutions of the equation $x^2 - kxy - y^2 = \mp(k^2+4)$ are given by $(x,y) = \mp(V_n, V_{n-1})$ with $n \in \mathbb{Z}$.*

Proof. If $(x,y) = \mp(V_n, V_{n-1})$, then by (2.8) it follows that $x^2 - kxy - y^2 = \mp(k^2+4)$. Now assume that $x^2 - kxy - y^2 = \mp(k^2+4)$. Then $(2x-ky)^2 - (k^2+4)y^2 = \mp 4(k^2+4)$. Since (k^2+4) is square free, it is seen that $(k^2+4)|2x-ky$. Similarly we see that $(k^2+4)|2y+kx$. Therefore if we take

$$u = (2y+kx)/(k^2+4) \text{ and } v = (2x-ky)/(k^2+4), \tag{2.9}$$

we get $u^2 - kuv - v^2 = \mp 1$. Then, by Theorem 2.5, we obtain

$$(u,v) = \mp(U_n, U_{n-1}) \tag{2.10}$$

From (2.9) and (2.10), it is seen that $(x,y) = \mp(V_n, V_{n-1})$. ■

Corollary 4. *All integer solutions of the equation $x^2 - xy - y^2 = \mp 5$ are given by $(x,y) = \mp(L_n, L_{n-1})$ with $n \in \mathbb{Z}$.*

Corollary 5. *Let k^2+4 be a square free integer. Then all nonnegative integer solutions of the equation $x^2 - kxy - y^2 = \mp(k^2+4)$ are given by $(x,y) = (V_n, V_{n-1})$ with $n \in \mathbb{N}$.*

Using Theorem 2.3 and the identities $U_n = (2U_{n+1} - V_n)/k$ and $U_{n+1} = (V_{n+1} - 2U_n)/k$, respectively, we can give the following corollary.

Corollary 6.

$$V_n^2 - (k^2+4)V_nU_{n+1} + (k^2+4)U_{n+1}^2 = (-1)^{n+1}k^2$$

and

$$V_{n+1}^2 - (k^2+4)V_{n+1}U_n + (k^2+4)U_n^2 = (-1)^n k^2$$

for all $n \in \mathbb{Z}$.

Proof. For the proof of the first identity, we will use the identity (2.2) given in Theorem 2.3 and the identity $U_n = (2U_{n+1} - V_n)/k$. Since

$$U_n^2 - kU_nU_{n-1} - U_{n-1}^2 = (-1)^{n+1},$$

we get

$$((2U_{n+1}-V_n)/k)^2-k((2U_{n+1}-V_n)/k)U_{n-1}-U_{n-1}^2=(-1)^{n+1}.$$

Thus it follows that

$$\begin{aligned}(-1)^{n+1}k^2 &= 4U_{n+1}^2-4U_{n+1}V_n+V_n^2-2k^2U_{n+1}U_{n-1}+k^2V_nU_{n-1}-k^2U_{n-1}^2\\ &= 4U_{n+1}^2-4U_{n+1}V_n+V_n^2-k^2U_{n-1}(2U_{n+1}-V_n+U_{n-1}). \quad (2.11)\end{aligned}$$

Using the fact $U_{n-1}=V_n-U_{n+1}$ in (2.11), we obtain

$$\begin{aligned}(-1)^{n+1}k^2 &= 4U_{n+1}^2-4U_{n+1}V_n+V_n^2-k^2(V_n-U_{n+1})(2U_{n+1}-V_n+V_n-U_{n+1})\\ &= 4U_{n+1}^2-4U_{n+1}V_n+V_n^2-k^2(V_n-U_{n+1})U_{n+1}\\ &= 4U_{n+1}^2-4U_{n+1}V_n+V_n^2-k^2V_nU_{n+1}+k^2U_{n+1}^2.\end{aligned}$$

Then it is seen that

$$V_n^2-(k^2+4)V_nU_{n+1}+(k^2+4)U_{n+1}^2=(-1)^{n+1}k^2.$$

For the proof of the second identity, we will consider the equation (2.6). Using the fact $U_{n+1}=(V_{n+1}-2U_n)/k$ in (2.6), we get

$$(-1)^n=((V_{n+1}-2U_n)/k)^2-k((V_{n+1}-2U_n)/k)U_n-U_n^2.$$

Thus it follows that

$$(-1)^nk^2=V_{n+1}^2-4V_{n+1}U_n+4U_n^2-k^2V_{n+1}U_n+2k^2U_n^2-k^2U_n^2.$$

That is,

$$V_{n+1}^2-(k^2+4)V_{n+1}U_n+(k^2+4)U_n^2=(-1)^nk^2.$$

■

Corollary 7. *All integer solutions of the equation* $x^2-(k^2+4)xy+(k^2+4)y^2=\mp k^2$ *are given by* $(x,y)=\mp(V_{n+1},U_n)$ *with* $n\in\mathbb{Z}$.

Proof. If $(x,y)=\mp(V_{n+1},U_n)$, then

$$x^2-(k^2+4)xy+(k^2+4)y^2=V_{n+1}^2-(k^2+4)V_{n+1}U_n+(k^2+4)U_n^2=(-1)^nk^2=\mp k^2.$$

Now assume that $x^2-(k^2+4)xy+(k^2+4)y^2=\mp k^2$ for some integer x and y. Then it follows that $k^2|(x-2y)^2$, i.e., $k|(x-2y)$. Let $u=(x-2y)/k$ and $v=y$. Then it can be seen that $u^2-kuv-v^2=\mp 1$. Therefore by Theorem 2.5, we get $u=\mp U_{n+1}$ and $v=\mp U_n$. Thus the proof follows. ■

Let m be an odd integer and $k = L_m$. Then by using $L_m^2 - 5F_m^2 = -4$, we see that

$$\begin{aligned}\alpha &= \left(k+\sqrt{k^2+4}\right)/2 = (L_m+\sqrt{5}F_m)/2 = \left(\left(1+\sqrt{5}\right)/2\right)^m \\ \beta &= \left(k-\sqrt{k^2+4}\right)/2 = (L_m-\sqrt{5}F_m)/2 = \left(\left(1-\sqrt{5}\right)/2\right)^m.\end{aligned}$$

Thus it follows that $U_n = F_{mn}/F_m$ and $V_n = L_{mn}$. Now we can give the following corollaries easily. The proofs of the corollaries follow from Theorem 2.3, Theorem 2.5, and Theorem 2.4, respectively.

Corollary 8. *Let m be an odd integer. Then*

$$F_{mn}^2 - L_m F_{mn} F_{m(n-1)} - F_{m(n-1)}^2 = (-1)^{n+1} F_m^2$$

for all $n \in \mathbb{Z}$.

Corollary 9. *Let m be an odd integer. Then all integer solutions of the equation $x^2 - L_m xy - y^2 = \mp 1$ are given by*

$$(x,y) = \mp(F_{mn}/F_m, F_{m(n-1)}/F_m)$$

with $n \in \mathbb{Z}$.

Corollary 10. *Let m be an odd integer. Then*

$$L_{mn}^2 - L_m L_{mn} L_{m(n-1)} - L_{m(n-1)}^2 = (-1)^n 5F_m^2$$

for all $n \in \mathbb{Z}$.

Now we turn to the sequences $\{u_n\}$ and $\{v_n\}$. The characteristic equation of the recurrence relation of the sequence $\{u_n\}$ is $x^2 - kx + 1 = 0$ and the roots of this equation are $\alpha = \left(k+\sqrt{k^2-4}\right)/2$ and $\overline{\alpha} = \beta = \left(k-\sqrt{k^2-4}\right)/2$. It is clear that $\alpha\beta = 1$, $\alpha^2 = k\alpha - 1$, and $\alpha + \beta = k$.

Now for $a,b,c,d \in \mathbb{Z}$, we define two binary operations in $\mathbb{Z}\times\mathbb{Z}$ by

$$(a,b)(c,d) = (kac+ad+bc, -ac+bd) \text{ and } (a,b)+(c,d) = (a+c, b+d).$$

It can be easily seen that

$$(a,b)(c,d) = (kac+ad+bc, -ac+bd) = (kca+cb+da, -ca+db) = (c,d)(a,b).$$

Furthermore the identity element is $(0,0)$ for the $+$ operation, and $(0,1)$ for the multiplication operation. Then it is seen that $\mathbb{Z}\times\mathbb{Z}$ is a commutative ring with unit element $(0,1)$.

Let $\mathbb{Z}[\alpha] = \{a\alpha+b : a,b\in\mathbb{Z}\}$. Then it can be seen that $\mathbb{Z}[\alpha]$ is a subring of the algebraic integer ring of the real quadratic field $\mathbb{Q}(\sqrt{k^2-4})$ and $\mathbb{Z}[\alpha]$ is equal to the algebraic integer ring of the real quadratic field $\mathbb{Q}(\sqrt{k^2-4})$ when k^2-4 is square free. We define a function $\phi:\mathbb{Z}\times\mathbb{Z}\longrightarrow\mathbb{Z}[\alpha]$, given by

$$\phi((a,b)) = a\alpha+b.$$

In view of the fact that $(a\alpha+b)(c\alpha+d) = \alpha^2ac+\alpha(ad+bc)+bd = (k\alpha-1)ac+\alpha(ad+bc)+bd = (kac+ad+bc)\alpha-ac+bd$ and $(a\alpha+b)+(c\alpha+d) = (a+c)\alpha+b+d$ for $a,b,c,d\in\mathbb{Z}$, it is easy to see that ϕ is a ring homomorphism. Moreover, since ϕ is bijective, ϕ is an isomorphism.

It can be shown that $x+\alpha y$ is a unit in $\mathbb{Z}[\alpha]$ if and only if $x^2+kxy+y^2=\mp1$. Now we give a theorem, which is important for solutions of some Diophantine equations.

Theorem 2.8. *Let $k>3$. Then the set of the units of $\mathbb{Z}[\alpha]$ is $\{\mp\alpha^n : n\in\mathbb{Z}\}$.*

Proof. The proof of the theorem is similar to that of Theorem 2.1. It can be shown easily that if $\alpha x+y$ is a unit and $1<\alpha x+y<\alpha$, then $x>0$ and $y<0$. Firstly, we show that there is no unit $\alpha x+y$ such that $1<\alpha x+y<\alpha$. On the contrary, assume that there exist some units $\alpha x+y$ such that $1<\alpha x+y<\alpha$. We consider the smallest positive integer x such that $1<\alpha x+y<\alpha$ and $\alpha x+y$ is a unit. Since $1<\alpha x+y<\alpha$, $\overline{\alpha}<x+\overline{\alpha}y<1$ and $\overline{\alpha}(\alpha x+y) = x+\overline{\alpha}y$ is also a unit. This shows that $1/(x+\overline{\alpha}y)$ is a unit in $\mathbb{Z}[\alpha]$ and $1<1/(x+\overline{\alpha}y)<1/\overline{\alpha}=\alpha$. Let $\varepsilon = (x+\overline{\alpha}y)(x+\alpha y)$. Then we have $\varepsilon=\mp1$ and $1<(x+\alpha y)/\varepsilon<\alpha$. Assume that $\varepsilon=1$. Then $1<x+\alpha y<\alpha$, which implies that $y>0$. But this is a contradiction, since $y<0$. Therefore $\varepsilon=-1$ and this shows that $1<-x+\alpha(-y)<\alpha$. Thus it follows that $x\le -y$. Then we see that $-x+\alpha x\le -x-\alpha y<\alpha$, which implies that $x(\alpha-1)\le\alpha$. Since $k>3$, it follows that $\alpha = \left(k+\sqrt{k^2-4}\right)/2>(3+\sqrt{5})/2>2$ and therefore $x\le\alpha/(\alpha-1) = 1+1/(\alpha-1)<2$. Thus $x=1$. Since $-1=\varepsilon = (x+\overline{\alpha}y)(x+\alpha y) = x^2+kxy+y^2$, it is seen that $-1 = 1+ky+y^2$ and therefore $y(k+y) = -2$. This shows that $y|2$ and thus $y=-2$ or $y=-1$, which implies that $k=3$. This contradicts with the hypothesis. The proof of the theorem then follows. ■

Note that theorem is not correct when $k=3$. In this case, $\alpha = (3+\sqrt{5})/2 = \left[(1+\sqrt{5})/2\right]^2 = 1+(1+\sqrt{5})/2$ and therefore $(1+\sqrt{5})/2\in\mathbb{Z}[\alpha]$. Moreover $(1+\sqrt{5})/2\neq\left[(1+\sqrt{5})/2\right]^{2n}=\alpha^n$ for every $n\in\mathbb{Z}$.

Theorem 2.9. $(1,0)^n = (u_n, -u_{n-1})$ *and* $(1,0)^{-n} = (-u_n, u_{n+1})$ *for all* $n \in \mathbb{N}$.

Proof. Since the proof of this theorem is similar to that of Theorem 2.2, we omit it. ■

Since $(1,0)^{-n} = (-u_n, u_{n+1}) = (u_{-n}, -u_{-(n+1)})$ and $(1,0)^n = (u_n, -u_{n-1})$ for all $n \in \mathbb{N}$, it follows that $(1,0)^n = (u_n, -u_{n-1})$ for all $n \in \mathbb{Z}$.

Theorem 2.10. $u_n^2 - ku_nu_{n-1} + u_{n-1}^2 = 1$ *for all* $n \in \mathbb{Z}$.

Proof. Theorem is correct for $n = 0$. Assume that $n \in \mathbb{N}$. Then

$$(0,1) = (1,0)^n(1,0)^{-n} = (u_n, -u_{n-1})(-u_n, u_{n+1}) = (0, u_n^2 - u_{n-1}u_{n+1})$$

and therefore $1 = u_n^2 - u_{n-1}u_{n+1}$. Since $u_{n+1} = ku_n - u_{n-1}$, it follows that $u_n^2 - ku_nu_{n-1} + u_{n-1}^2 = 1$. If $n \in \mathbb{Z}$ and $n < 0$, then using the definition of u_n and u_{n-1}, it can be shown that $u_n^2 - ku_nu_{n-1} + u_{n-1}^2 = 1$. ■

Lemma 3. $\alpha^n = \alpha u_n - u_{n-1}$ *and* $\beta^n = \beta u_n - u_{n-1}$ *for all* $n \in \mathbb{Z}$. *Furthermore* $u_n = (\alpha^n - \beta^n)/\sqrt{k^2-4}$ *and* $v_n = \alpha^n + \beta^n$.

Proof. From the definition of ϕ, we see that $\alpha u_n - u_{n-1} = \phi((u_n, -u_{n-1})) = \phi((1,0)^n) = (\phi((1,0)))^n = \alpha^n$ and $\beta^n = \alpha^{-n} = \alpha u_{-n} - u_{-n-1} = -\alpha u_n + u_{n+1} = (\beta - k)u_n + u_{n+1} = \beta u_n - u_{n-1}$. Since $\alpha^n = \alpha u_n - u_{n-1}$ and $\beta^n = \beta u_n - u_{n-1}$, it is seen that $u_n = (\alpha^n - \beta^n)/\sqrt{k^2-4}$ and $v_n = \alpha^n + \beta^n$. ■

From the above lemma it follows that $v_n = u_{n+1} - u_{n-1}$. By using the identity $u_n^2 - ku_nu_{n-1} + u_{n-1}^2 = 1$, we can give the following corollary.

Corollary 11. $v_n^2 - (k^2-4)u_n^2 = 4$ *for all* $n \in \mathbb{Z}$.

Lemma 4. *Let* $x = (1,0)$. *Then* $(2,-k)x^n = (v_n, -v_{n-1})$ *for all* $n \in \mathbb{Z}$.

Proof. Since $x^{n+1} - x^{n-1} = x^n(x - x^{-1}) = x^n((1,0) - (-1,k)) = (2,-k)x^n$ and $x^{n+1} - x^{n-1} = (u_{n+1}, -u_n) - (u_{n-1}, -u_{n-2}) = (v_n, -v_{n-1})$, it follows that $(2,-k)x^n = (v_n, -v_{n-1})$. ■

Theorem 2.11. $v_n^2 - kv_nv_{n-1} + v_{n-1}^2 = -(k^2-4)$ *for all* $n \in \mathbb{Z}$.

Proof. Multiplying $(2,-k)x^n = (v_n, -v_{n-1})$ and $(2,-k)x^{-n} = (v_{-n}, -v_{-n-1}) = (v_n, -v_{n+1})$, we get

$$\begin{aligned}(0,k^2-4) &= (v_n, -v_{n-1})(v_n, -v_{n+1}) \\ &= (0, -v_n^2 + v_{n-1}v_{n+1}) = (0, -v_n^2 + kv_nv_{n-1} - v_{n-1}^2).\end{aligned}$$

From this, it follows that

$$v_n^2 - kv_n v_{n-1} + v_{n-1}^2 = -(k^2-4)$$

for all $n \in \mathbb{Z}$. ■

Theorem 2.12. *Let $k > 3$. Then all integer solutions of the equation $x^2 - kxy + y^2 = 1$ are given by $(x,y) = \mp(u_n, u_{n-1})$ with $n \in \mathbb{Z}$.*

Proof. It is obvious from Theorem 2.10 that if $(x,y) = \mp(u_n, u_{n-1})$, then $x^2 - kxy + y^2 = 1$. Now assume that $x^2 - kxy + y^2 = 1$. Then $\alpha x - y$ is a unit in $\mathbb{Z}[\alpha]$. Therefore there exists some $n \in \mathbb{Z}$ such that $\alpha x - y = \mp\alpha^n$. Consequently, we obtain

$$\begin{aligned}(x,-y) &= \phi^{-1}(\alpha x - y) = \phi^{-1}(\mp\alpha^n) \\ &= \mp(\phi^{-1}(\alpha))^n = \mp(1,0)^n = \mp(u_n, -u_{n-1}).\end{aligned}$$

Then the proof follows. ■

Corollary 12. *Let $k > 3$. Then all nonnegative integer solutions of the equation $x^2 - kxy + y^2 = 1$ are given by $(x,y) = (u_n, u_{n-1})$ with $n \geq 1$.*

Corollary 13. *Let $k > 3$. Then all nonnegative integer solutions of the equation $u^2 - (k^2-4)v^2 = 4$ are given by $(u,v) = (v_n, u_n)$ with $n \geq 1$.*

Proof. It is obvious from Corollary 11 that if $(u,v) = (v_n, u_n)$, then $u^2 - (k^2 - 4)v^2 = 4$. Now assume that $u^2 - (k^2-4)v^2 = 4$. Then $u^2 - k^2v^2 = 4 + 4v^2$ and therefore $u^2 - k^2v^2$ is an even integer. Thus u and kv have the same parity. Let $x = (u+kv)/2$ and $y = v$. Then it follows that

$$x^2 - kxy + y^2 = \frac{(u+kv)^2 - 2kv(u+kv) + 4v^2}{4} = \frac{u^2 - (k^2-4)v^2}{4} = \frac{4}{4} = 1.$$

By Corollary 12, $x = u_{n+1}$ and $y = u_n$ for some $n \geq 0$. That is, $u + kv = 2u_{n+1}$ and $y = v = u_n$. From this, it is seen that $u = v_n$ and $v = u_n$. ■

Theorem 2.13. *Let $k > 3$ and $k^2 - 4$ be a square free integer. Then all integer solutions of the equation $x^2 - kxy + y^2 = -(k^2-4)$ are given by $(x,y) = \mp(v_n, v_{n-1})$ with $n \in \mathbb{Z}$.*

Proof. From Theorem 2.11, it is seen that if $(x,y) = \mp(v_n, v_{n-1})$, then $x^2 - kxy + y^2 = -(k^2-4)$. Now let $x^2 - kxy + y^2 = -(k^2-4)$. Then $(2x - ky)^2 - (k^2-4)y^2 =$

$(2y-kx)^2-(k^2-4)x^2=-4(k^2-4)$. Since (k^2-4) is square free, it follows that $(k^2-4)|(2x-ky)$ and $(k^2-4)|(2y-kx)$. Let

$$u=\frac{kx-2y}{k^2-4} \text{ and } v=\frac{2x-ky}{k^2-4}.$$

Then it is seen that

$$\begin{aligned} u^2-kuv+v^2 &= \frac{(kx-2y)^2-k(kx-2y)(2x-ky)+(2x-ky)^2}{(k^2-4)^2} \\ &= \frac{-(k^2-4)(x^2-kxy+y^2)}{(k^2-4)^2}=\frac{-(k^2-4)(-(k^2-4))}{(k^2-4)^2}=1. \end{aligned}$$

According to Theorem 2.12, $(u,v)=\mp(u_n,u_{n-1})$. A simple computation shows that $(x,y)=\mp(v_n,v_{n-1})$. ■

Corollary 14. *Let $k>3$ and k^2-4 be a square free integer. Then all nonnegative integer solutions of the equation $x^2-kxy+y^2=-(k^2-4)$ are given by $(x,y)=(v_n,v_{n-1})$ with $n\in\mathbb{Z}$.*

Using Theorem 2.10 and the identities $u_n=(2u_{n+1}-v_n)/k$ and $u_{n+1}=(v_{n+1}+2u_n)/k$, respectively, we can give the following.

Corollary 15.

$$v_n^2-(k^2-4)v_nu_{n+1}-(k^2-4)u_{n+1}^2=k^2$$

and

$$v_{n+1}^2-(k^2-4)v_{n+1}u_n-(k^2-4)u_n^2=k^2$$

for all $n\in\mathbb{Z}$.

Now we can give the following corollary. The proof of the corollary is similar to that of Corollary 7.

Corollary 16. *Let $k>3$. Then all integer solutions of the equation $x^2-(k^2-4)xy-(k^2-4)y^2=k^2$ are given by $(x,y)=\mp(v_{n+1},u_n)$ with $n\in\mathbb{Z}$.*

Let m be an even integer different from zero and $k=L_m$. Then by using $L_m^2-5F_m^2=4$, we see that

$$\begin{aligned} \alpha &= \left(k+\sqrt{k^2-4}\right)/2=(L_m+\sqrt{5}F_m)/2=\left(\left(1+\sqrt{5}\right)/2\right)^m \\ \beta &= \left(k-\sqrt{k^2-4}\right)/2=(L_m-\sqrt{5}F_m)/2=\left(\left(1-\sqrt{5}\right)/2\right)^m. \end{aligned}$$

Thus it follows that $u_n=F_{mn}/F_m$ and $v_n=L_{mn}$. Now we can give the following corollaries easily. The proofs of the corollaries follow from Theorem 2.10, Theorem 2.12, and Theorem 2.11,respectively.

Corollary 17. *Let $m > 2$ be an even integer. Then*

$$F_{mn}^2 - L_m F_{mn} F_{m(n-1)} + F_{m(n-1)}^2 = F_m^2$$

for all $n \in \mathbb{Z}$.

Corollary 18. *Let $m > 2$ be an even integer. Then all integer solutions of the equation $x^2 - L_m xy + y^2 = 1$ are given by*

$$(x,y) = \mp(F_{mn}/F_m, F_{m(n-1)}/F_m)$$

with $n \in \mathbb{Z}$.

Corollary 19. *Let $m > 2$ be an even integer. Then*

$$L_{mn}^2 - L_m L_{mn} L_{m(n-1)} + L_{m(n-1)}^2 = -5F_m^2$$

for all $n \in \mathbb{Z}$.

Theorem 2.14. *Let $k > 3$. Then the equation $x^2 - kxy + y^2 = -1$ has no integer solutions.*

Proof. Assume that $x^2 - kxy + y^2 = -1$ for some integers x and y. Then

$$(\alpha x - y)(\overline{\alpha} x - y) = -1$$

and therefore $\alpha x - y$ is a unit in $\mathbb{Z}[\alpha]$. Thus $\alpha x - y = \mp\alpha^n$ for some $n \in \mathbb{Z}$. Then it follows that $(x,y) = \mp(u_n, u_{n-1})$. This shows that

$$-1 = x^2 - kxy + y^2 = u_n^2 - ku_n u_{n-1} + u_{n-1}^2 = 1,$$

which is a contradiction. ■

We can give the following two corollaries.

Corollary 20. *Let $k > 3$. Then the equation $u^2 - (k^2 - 4)v^2 = -4$ has no integer solutions.*

Proof. Assume that $u^2 - (k^2 - 4)v^2 = -4$ for some integers u and v. Then $u^2 - k^2v^2 = 4v^2 - 4$ and therefore $u^2 - k^2v^2$ is an even integer. Thus u and v have the same parity. Let $x = (u + kv)/2$ and $y = v$. Then it follows that

$$x^2 - kxy + y^2 = \frac{(u+kv)^2 - 2kv(u+kv) + 4v^2}{4} = \frac{u^2 - (k^2-4)v^2}{4} = -1,$$

which is impossible by Theorem 2.14. ■

Corollary 21. *Let $k > 3$ and $k^2 - 4$ be a square free integer. Then $x^2 - kxy + y^2 = k^2 - 4$ has no integer solutions.*

Also the following corollary can be given.

Corollary 22. *Let $k > 3$ and k be an odd integer. Then there are no integer solutions of the equations $u^2 - (k^2 - 4)v^2 = -16$ and $x^4 - kx^2y^2 + y^4 = -4$.*

Proof. Assume that $u^2 - (k^2 - 4)v^2 = -16$ for some integers u and v. Then u and v have the same parity. Since $k^2 - 4 \equiv -3 \pmod 8$, it follows that $u^2 + 3v^2 \equiv 0 \pmod 8$. This shows that u and v are even integers. This implies that $(u/2)^2 - (k^2 - 4)(v/2)^2 = -4$, which is impossible by Corollary 20. Therefore there is no solutions of the equation $u^2 - (k^2 - 4)v^2 = -16$. Now assume that $x^4 - kx^2y^2 + y^4 = -4$ for some integers x and y. Then $(2x^2 - ky^2)^2 - (k^2 - 4)y^4 = -16$, which is a contradiction. ■

Corollary 23. *Let $k > 3$. Then there are no integer solutions of the equation $x^2 - (k^2 - 4)xy - (k^2 - 4)y^2 = -k^2$.*

Proof. Assume $x^2 - (k^2 - 4)xy - (k^2 - 4)y^2 = -k^2$ for some integers x and y. Then it is seen that $k^2 | (x + 2y)^2$. That is, $k | (x + 2y)$. Let $u = (x + 2y)/k$ and $v = y$. Then it follows that

$$u^2 - kuv + v^2 = \frac{x^2 - (k^2 - 4)xy - (k^2 - 4)y^2}{k^2} = \frac{-k^2}{k^2} = -1,$$

which is a contradiction by Theorem 2.14. ■

Corollary 24. *Let $k \geq 3$. Then there are no integer solutions of the equation $x^2 - (k^2 - 4)xy - (k^2 - 4)y^2 = -1$.*

Proof. Assume that $k > 3$ and $x^2 - (k^2 - 4)xy - (k^2 - 4)y^2 = -1$ for some integers x and y. Then $(kx)^2 - (k^2 - 4)kxky - (k^2 - 4)(ky)^2 = -k^2$,which is impossible by Corollary 23. Now assume that $k = 3$. Then the equation becomes $x^2 - 5xy - 5y^2 = -1$. Then $(2x - 5y)^2 - 5(3y)^2 = -4$ and therefore $|2x - 5y| = L_n$ and $3|y| = F_n$ for some odd integer n. Thus $3 | F_n$ and this implies that $4 | n$, which is a contradiction. Recall that $F_m | F_n$ if and only if $m | n$ where $m \geq 3$.(See [9]). ■

Corollary 25. *Let $k \geq 2$. Then equations $x^2 - 4(k^2 - 1)xy - 4(k^2 - 1)y^2 = -k^2$ and $x^2 - 4(k^2 - 1)xy - 4(k^2 - 1)y^2 = -1$ have no integer solutions.*

Proof. Assume that

$$x^2-4(k^2-1)xy-4(k^2-1)y^2=-k^2$$

for some integers x and y. Then we have

$$x^2-((2k)^2-4)xy-((2k)^2-4)y^2=-k^2. \tag{2.12}$$

Multiplying (2.12) by 4, we get

$$(2x)^2-((2k)^2-4)(2x)(2y)-((2k)^2-4)(2y)^2=-(2k)^2,$$

which is impossible by Corollary 23.

Now assume that

$$x^2-4(k^2-1)xy-4(k^2-1)y^2=-1 \tag{2.13}$$

for some integers x and y. Then multiplying (2.13) by k^2, we get

$$(kx)^2-4(k^2-1)(kx)(ky)-4(k^2-1)(ky)^2=-k^2,$$

which is also impossible. ■

When $k=3$, $k^2-4=5$, $u_n=F_{2n}$ and $v_n=L_{2n}$ and the equations $x^2-3xy+y^2=-1$, $x^2-3xy+y^2=5$ and $u^2-5v^2=-4$ have solutions. Also, Theorem 2.12, Theorem 2.13, Corollary 12, and Corollary 16 are correct. This follows from the following theorem.

Theorem 2.15. *All integer solutions of the equation $x^2-3xy+y^2=\mp 1$ are given by $(x,y)=\mp(F_{n+2},F_n)$ with $n\in\mathbb{Z}$ and all integer solutions of the equation $x^2-3xy+y^2=\mp 5$ are given by $(x,y)=\mp(L_{n+2},L_n)$ with $n\in\mathbb{Z}$. Also, all nonnegative integer solutions of the equation $u^2-5v^2=\mp 4$ are given by $(u,v)=(L_n,F_n)$ with $n\in\mathbb{N}$ and all integer solutions of the equation $x^2-5xy-5y^2=9$ are given by $(x,y)=\mp(L_{2n+2},F_{2n})$ with $n\in\mathbb{Z}$.*

Proof. Let $u=x-y$ and $v=y$. Then $u^2-uv-v^2=(x-y)^2-y(x-y)-y^2=x^2-3xy+y^2=\mp 1$. Therefore by Corollary 2, we get $(u,v)=\mp(F_{n+1},F_n)$. That is, $x-y=\mp F_{n+1}$ and $y=\mp F_n$. It follows that $(x,y)=\mp(F_{n+2},F_n)$. Similarly, if $x^2-3xy+y^2=\mp 5$, then $u^2-uv-v^2=(x-y)^2-y(x-y)-y^2=x^2-3xy+y^2=\mp 5$. Thus $(u,v)=\mp(L_{n+1},L_n)$ by Corollary 4. That is, $x-y=\mp L_{n+1}$ and $y=\mp L_n$. This shows that $(x,y)=\mp(L_{n+2},L_n)$. Conversely, it can be shown by induction that $F_{n+2}^2-3F_nF_{n+2}+F_n^2=(-1)^n$ and $L_{n+2}^2-3L_nL_{n+2}+L_n^2=5(-1)^{n+1}$. By Theorem 2.6, it is seen that all nonnegative integer solutions of the equation $u^2-5v^2=\mp 4$ are given by $(u,v)=(L_n,F_n)$ with $n\in\mathbb{N}$. Moreover, all integer solutions of the equation $x^2-5xy-5y^2=9$ can be found similarly. ■

When k^2+4 or k^2-4 (in this case, $k>3$) are not square free, Theorem 2.7 and Theorem 2.13 are not correct. For instance, $x^2-4xy-y^2=20$ has a solution $(x,y)=(7,1)$ but $(7,1)\neq(V_n,V_{n-1})$ for all $n\in\mathbb{Z}$. Also, $x^2-7xy+y^2=-45$ has a solution $(x,y)=(1,11)$ but $(1,11)\neq(v_n,v_{n-1})$ for all $n\in\mathbb{Z}$. More generally, taking $k=L_m$, where $m\neq 0$, we can give the following theorem.

Theorem 2.16. *All integer solutions of the equation $x^2-L_mxy+(-1)^my^2=\mp 5F_m^2$ are given by $(x,y)=\mp(L_{m+n},L_n)$ with $n\in\mathbb{Z}$.*

Proof. Assume that $x^2-L_mxy+(-1)^my^2=\mp 5F_m^2$. Then $(2x-L_my)^2-(L_m^2-4(-1)^m)y^2=\mp 20F_m^2$ and therefore $(2x-L_my)^2-5F_m^2y^2=\mp 20F_m^2$. Then it follows that $5F_m|(2x-L_my)$. Similarly it is seen that $5F_m|(x+L_{m-1}y)$. Taking

$$u=(x+L_{m-1}y)/5F_m,$$
$$v=(2x-L_my)/5F_m$$

and considering the identity

$$L_m^2-L_mL_{m-1}-L_{m-1}^2=5(-1)^m$$

we see that

$$u^2-uv-v^2=\frac{(x+L_{m-1}y)^2-(x+L_{m-1}y)(2x-L_my)-(2x-L_my)^2}{25F_m^2}$$
$$=\frac{-5(x^2-L_mxy+(-1)^my^2)}{25F_m^2}=\mp 1$$

Then it follows from Corollary 2 that $(u,v)=\mp(F_{n+1},F_n)$ for some $n\in\mathbb{Z}$. That is,

$$(x+L_{m-1}y)/5F_m=\mp F_{n+1}$$

and

$$(2x-L_my)/5F_m=\mp F_n.$$

A simple computation shows that $(x,y)=\mp(L_{n+m},L_n)$. Conversely, it can be shown that

$$L_{n+m}^2-L_mL_{n+m}L_n+(-1)^mL_n^2=(-1)^{n+1}5F_m^2.$$

This completes the proof. ■

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