A Characterization on Potentially $K_6 - E(K_3)$−graphic Sequences*

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Abstract: For given a graph H, a graphic sequence $\pi = (d_1, d_2, \ldots, d_n)$ is said to be potentially H-graphic if there is a realization of π containing H as a subgraph. In this paper, we characterize potentially $K_6 - E(K_3)$-graphic sequences without zero terms, where $K_6 - E(K_3)$ is the graph obtained from a complete graph on 6 vertices by deleting three edges which form a triangle. This characterization implies the values of $\sigma(K_6 - E(K_3), n)$.

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1. Introduction

The set of all sequences $\pi = (d_1, d_2, \ldots, d_n)$ of non-negative, non-increasing integers with $d_1 \leq n-1$ is denoted by NS_n. A sequence $\pi \in NS_n$ is said to be *graphic* if it is the degree sequence of a simple graph G on n vertices, and such a graph G is called a *realization* of π. The set of all graphic sequences in NS_n is denoted by GS_n. If each term of a graphic sequence π is nonzero, then π is said to be *positive graphic*. For a sequence $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$, denote $\sigma(\pi) = d_1 + d_2 + \cdots + d_n$. For given a

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graph H, a sequence $\pi \in GS_n$ is said to be *potentially H-graphic*, if there is a realization of π containing H as a subgraph. Eschen and Niu [2] characterized potentially $K_4 - e$-graphic sequences. Yin and Li [11] gave two sufficient conditions for $\pi \in GS_n$ to be potentially $K_r - e$-graphic. In the following, the symbol x^y in a sequence stands for y consecutive terms, each equal to x.

Theorem 1.1 [2] Let $n \geq 4$ and $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ be a positive sequence. Then π is potentially $K_4 - e$-graphic if and only if the following conditions hold:

(1) $d_2 \geq d_2 \geq 3, d_4 \geq 2$.

(2) $\pi \neq (3^6), (3^2, 2^4), (3^2, 2^3)$.

Theorem 1.2 [11] Let $n \geq r+1$ and $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ with $d_{r+1} \geq r-1$. If $d_i \geq 2r - i$ for $i = 1, 2, \ldots, r-1$, then π is potentially $K_{r+1} - e$-graphic.

Theorem 1.3 [11] Let $n \geq 2r+2$ and $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ with $d_{r-1} \geq r$. If $d_{2r+2} \geq r-1$, then π is potentially $K_{r+1} - e$-graphic.

Recently, Hu and Lai [4] characterized potentially $K_5 - E(C_4)$-graphic sequences, where C_4 is a cycle on four vertices. In [5], they characterized potentially $K_5 - E(H)$-graphic sequences, where H is a path on five vertices or a tree on five vertices and three leaves. Besides, they characterized potentially $K_5 - E(K_3)$-graphic sequences and potentially 3-regular graph graphic sequences in [6] and [7], where 3-regular graph is $K_6 - E(C_6)$ or a complete bipartite graph $K_{3,3}$. In [6], they gave the following

Theorem 1.4 [6] Let $\pi = (d_1, d_2, \ldots, d_n)$ be a graphic sequence with $n \geq 5$. Then π is potentially $K_5 - E(K_3)$-graphic if and only if the following conditions hold:

(1) $d_2 \geq 4$ and $d_5 \geq 2$.

(2) $\pi \neq (4^2, 2^4), (4^2, 2^5), (4^3, 2^3)$ and (4^6).

In [3], Gould et al. posed an extremal problem on potentially H-graphic sequences as follows: determine the smallest even integer $\sigma(H, n)$ such that every positive sequence $\pi \in GS_n$ with $\sigma(\pi) \geq \sigma(H, n)$ is potentially H-graphic.

In this paper, we first characterize potentially $K_6 - E(K_3)$-positive graphic sequences. That is the following

Theorem 1.5 Let $n \geq 6$ and $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ be a positive sequence. Then π is potentially $K_6 - E(K_3)$-graphic if and only if π satisfies the following conditions:

(1) $d_3 \geq 5$ and $d_6 \geq 3$;

(2) π is not one of the following sequences:

$$(n-1, 5^2, 3^4, 1^{n-7})(n \geq 7), \quad (n-1, 5^2, 3^5, 1^{n-8})(n \geq 8),$$
$$(n-1, 5^3, 3^3, 1^{n-7})(n \geq 7), \quad (n-1, 5^6, 1^{n-7})(n \geq 7),$$

$n = 7 : (5^6, 4), (5^4, 4^3), (5^3, 3^3, 2), (5^4, 3^2, 2), (5^3, 4, 3^3), (5^4, 4, 3^2),$
$n = 8 : (5^4, 4^4), (5^3, 4^2, 3^3), (6^4, 3^4), (5^7, 3), (5^3, 3^3, 2^2), (5^3, 4, 3^3, 2),$
$(5^3, 3^4, 1), (5^4, 3^3, 1), (5^7, 1), (5^4, 3^3, 1), (5^8), (6, 5^2, 4^5), (5^3, 3^5), (5^3, 4^4, 3),$
$(6^2, 5, 3^5), (6, 5^2, 4, 3^4), (5^4, 3^4), (6, 5^2, 3^4, 2),$
$n = 9 : (5^8, 4), (5^3, 4, 3^5), (6, 5^8), (5^3, 3^5, 2), (5^3, 3^4, 2, 1), (5^3, 4, 3^4, 1),$
$(5^3, 4^5, 1), (6, 5^2, 3^5, 1), (6, 5^2, 4^6), (5^4, 4^5), (6, 5^2, 3^6), (5^3, 4^5, 3),$
$n = 10 : (5^3, 3^6, 1), (5^3, 3^5, 1^2), (5^9, 1), (5^3, 4^6, 1), (5^{10}).$

As an application of this characterization, it is straightforward to find the values of $\sigma(K_6 - E(K_3), n)$.

2. Proof of Theorem 1.5

In order to prove Theorem 1.5, we also need the following known results. Let $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$ and $1 \le k \le n$. Let

$$\pi_k'' = \begin{cases} (d_1 - 1, \ldots, d_{k-1} - 1, d_{k+1} - 1, \ldots, d_{d_k+1} - 1, d_{d_k+2}, \ldots, d_n), \\ \text{if } d_k \ge k, \\ (d_1 - 1, \ldots, d_{d_k} - 1, d_{d_k+1}, \ldots, d_{k-1}, d_{k+1}, \ldots, d_n), \\ \text{if } d_k < k. \end{cases}$$

Let $\pi_k' = (d_1', d_2', \ldots, d_{n-1}')$, where $d_1' \ge \cdots \ge d_{n-1}'$ is the rearrangement in non-increasing order of the $n-1$ terms of π_k''. π_k' is called the *residual sequence* obtained by laying off d_k from π.

Theorem 2.1 [8] Let $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$ and $1 \le k \le n$. Then $\pi \in GS_n$ if and only if $\pi_k' \in GS_{n-1}$.

Theorem 2.2 [1] Let $\pi = (d_1, d_2, \ldots, d_n)$ be a non-increasing sequence of nonnegative integer with even $\sigma(\pi)$. Then $\pi \in GS_n$ if and only if for any $t, 1 \le t \le n-1$,

$$\sum_{i=1}^{t} d_i \le t(t-1) + \sum_{i=t}^{n} min\{t, d_i\}$$

Theorem 2.3 [10] Let $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$, $d_1 = m$ and $\sigma(\pi)$ be even. If there exists an integer $n_1, n_1 \le n$ such that $d_{n_1} \ge h \ge 1$ and $n_1 \ge \frac{1}{h} \left\lfloor \frac{(m+h+1)^2}{4} \right\rfloor$, then $\pi \in GS_n$.

Theorem 2.4 [9] Let $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$ and $\sigma(\pi)$ be even. If $d_1 - d_n \le 1$ and $d_1 \le n-1$, then $\pi \in GS_n$.

Theorem 2.5 [3] If $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ has a realization G containing H as a subgraph, then there exists a realization G' of π containing H as a subgraph so that the vertices of H have the largest degrees of π.

In order to prove our main result, we need the following definition.

Let $n \geq r+1$ and $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$ with $d_{r+1} \geq r$. We define sequences $\pi_0, \ldots, \pi_{r+1}$ as follows. Let $\pi_0 = \pi$. Let

$$\pi_1 = (d_2 - 1, \ldots, d_{r+1} - 1, d_{r+2}^{(1)}, \ldots, d_n^{(1)}),$$

where $d_{r+2}^{(1)} \geq \cdots \geq d_n^{(1)}$ is the rearrangement in non-increasing order of $d_{r+2} - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n$. For $2 \leq i \leq r+1$, given $\pi_{i-1} = (d_i - i + 1, \ldots, d_{r+1} - i + 1, d_{r+2}^{(i-1)}, \ldots, d_n^{(i-1)})$, let

$$\pi_i = (d_{i+1} - i, \ldots, d_{r+1} - i, d_{r+2}^{(i)}, \ldots, d_n^{(i)}),$$

where $d_{r+2}^{(i)} \geq \cdots \geq d_n^{(i)}$ is the rearrangement in non-increasing order of $d_{r+2}^{(i-1)} - 1, \ldots, d_{d_i+1}^{(i-1)} - 1, d_{d_i+2}^{(i-1)}, \ldots, d_n^{(i-1)}$.

The following lemma is obvious from the definition of π_i.

Lemma 2.1 Let $n \geq 6$ and $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ with $d_3 \geq 5, d_6 \geq 3$ and $d_n \geq 1$. If π_3 is graphic, then π is potentially $K_6 - E(K_3)$-graphic.

Lemma 2.2 Let $\pi = (3^x, 2^y, 1^z)$, where $x + y + z = n \geq 1$ and $\sigma(\pi)$ is even. Then $\pi \in GS_n$ if and only if $\pi \notin S$, where

$$S = \{(2), (2^2), (3,1), (3^2), (3,2,1), (3^2,2), (3^3,1), (3^2,1^2)\}.$$

Proof. For $n = 1$, since $\sigma(\pi)$ is even, π must be (2), which belongs to S. For $n \geq 2$, we consider the following cases.

Case 1. $n = 2$. Then π is one of the following sequences:

$$(3,1), (2^2), (3^2), (1^2).$$

It is easy to check that only one sequence (1^2) is graphic.

Case 2. $n = 3$. Since $\sigma(\pi)$ is even, π must be one of the following sequences:

$$(3,2,1), (3^2,2), (2^3), (2,1^2).$$

We can see that the sequences (2^3) and $(2,1^2)$ are graphic.

Case 3. $n = 4$. Then π is one of the following sequences:

$$(3^3,1), (3,1^3), (3^4), (2^4, (3,2^2,1), (2^2,1^2), (3^2,2^2), (1^4), (3^2,1^2).$$

Theses sequences are all graphic except $(3^2,1^2)$ and $(3^3,1)$.

Case 4. $n = 5$. We know that π must be one of the following sequences:

$$(2,1^4), (3,2,1^3), (3^2,2,1^2), (3^3,2,1), (3,2^3,1), (2^5), (3^2,2^3), (2^3,1^2), (3^4,2).$$

These sequences are all graphic.

Case 5. $n \geq 6$. If $x > 0$ and $z > 0$, then $6 = \lfloor \frac{(3+1+1)^2}{4} \rfloor \leq n$. Hence, π is graphic from Theorem 2.3. Otherwise, π is graphic by Theorem 2.4. □

Lemma 2.3 Let $\pi = (4^x, 3^y, 2^z, 1^m)$ with even $\sigma(\pi), x + y + z + m = n \geq 5$ and $x \geq 1$. Then $\pi \in GS_n$ if and only if $\pi \notin A$, where $A = \{(4, 3^2, 1^2), (4, 3, 1^3), (4^2, 2, 1^2), (4^2, 3, 2, 1), (4^3, 1^2), (4^3, 2^2), (4^3, 3, 1), (4^4, 2), (4^2, 3, 1^3), (4^2, 1^4), (4^3, 2, 1^2), (4^4, 1^2), (4^3, 1^4)\}$.

Proof. It is easy to see that the sequences of the set A are not graphic. Now we verify the sufficient condition. We need consider the following five cases.

Case 1. $n = 5$. Since $\sigma(\pi)$ is even, π must be one of the following sequences:
$(4, 1^4), (4, 2^2, 1^2), (4, 3^2, 1^2), (4, 3, 1^3), (4, 3, 2^2, 1), (4, 3^3, 1), (4, 2^4), (4, 3^2, 2^2), (4, 3^4), (4^2, 2, 1^2), (4^2, 3, 2, 1), (4^2, 2^3), (4^2, 3^2, 2), (4^3, 1^2), (4^3, 2^2), (4^3, 3^2), (4^3, 3, 1), (4^4, 2), (4^5)$. Except $(4, 3^2, 1^2)$, $(4, 3, 1^3), (4^2, 2, 1^2), (4^2, 3, 2, 1), (4^3, 1^2), (4^3, 2^2), (4^3, 3, 1)$ and $(4^4, 2)$, the others are all graphic.

Case 2. $n = 6$. Since $\sigma(\pi)$ is even and $x \geq 1$, $m \neq 5$.

If $m = 0$, then $n = 6 \geq max\{\frac{1}{3}\lfloor \frac{(4+3+1)^2}{4} \rfloor, \frac{1}{2}\lfloor \frac{(4+2+1)^2}{4} \rfloor\}$. By Theorem 2.3, π is graphic.

If $m = 1$, then π must be one of the following sequences:

$$(4^4, 3, 1), (4^2, 3^3, 1), (4, 3, 2^3, 1), (4^2, 3, 2^2, 1), (4^3, 3, 2, 1), (4, 3^3, 2, 1).$$

It is easy to check that these sequences are all graphic.

If $m = 2$, then π must be one of the following sequences:

$$(4, 3^2, 2, 1^2), (4^2, 2^2, 1^2), (4^2, 3^2, 1^2), (4, 2^3, 1^2), (4^3, 2, 1^2), (4^4, 1^2).$$

Except for $(4^3, 2, 1^2)$ and $(4^4, 1^2)$, the others are graphic.

If $m = 3$, then π is $(4, 3, 2, 1^3)$ or $(4^2, 3, 1^3)$. The sequence $(4, 3, 2, 1^3)$ is graphic and the sequence $(4^2, 3, 1^3)$ belongs to A.

If $m = 4$, then π is $(4, 2, 1^4)$ or $(4^2, 1^4)$. The sequence $(4, 2, 1^4)$ is graphic and the sequence $(4^2, 1^4)$ belongs to A.

Case 3. $n = 7$.

If $m = 0$, then $n = 7 \geq max\{\frac{1}{3}\lfloor \frac{(4+3+1)^2}{4} \rfloor, \frac{1}{2}\lfloor \frac{(4+2+1)^2}{4} \rfloor\}$. By Theorem 2.3, π is graphic.

If $1 \leq m \leq 3$, then π must be one of the following graphic sequences:
$(4, 3, 2^4, 1), (4^2, 3, 2^3, 1), (4^3, 3, 2^2, 1), (4^4, 3, 2, 1), (4^5, 3, 1), (4, 3^3, 2^2, 1), (4^2, 3^3, 2, 1), (4^3, 3^3, 1), (4, 3^5, 1), (4, 2^4, 1^2), (4^2, 2^3, 1^2), (4^3, 2^2, 1^2), (4^4, 2, 1^2), (4^5, 1^2), (4, 3^2, 2^2, 1^2), (4^2, 3^2, 2, 1^2), (4^3, 3^2, 1^2), (4, 3^4, 1^2), (4^3, 3, 1^3), (4, 3, 2^2, 1^3), (4^2, 3, 2, 1^3), (4, 3^3, 1^3)$.

If $m = 4$, then π is $(4^3, 1^4), (4^2, 2, 1^4), (4, 2^2, 1^4)$ or $(4, 3^2, 1^4)$. The sequence $(4^3, 1^4) \in A$ and the others are graphic.

If m is 5 or 6, π is $(4, 3, 1^5)$ or $(4, 1^6)$. They are both graphic.

Case 4. $n=8$. Since $x \geq 1$ and $\sigma(\pi)$ is even, $m \neq 7$.

If $m \leq 2$, then π is graphic by Theorem 2.3 and 2.4.

If $m \geq 3$, then π is one of the following graphic sequences: $(4,3,2^3,1^3),(4^2,3,2^2,1^3),(4^3,3,2,1^3),(4^4,3,1^3),(4,3^3,2,1^3),(4^2,3^3,1^3),$ $(4,2^3,1^4),(4^2,2^2,1^4),(4^3,2,1^4),(4^4,1^4),(4,3^2,2,1^4),(4^2,3^2,1^4),$ $(4,3,2,1^5),(4^2,3,1^5),(4,2,1^6),(4^2,1^6)$.

Case 5. $n \geq 9$. For $n \geq 9 \geq max\{\frac{1}{2}\lfloor\frac{(4+2+1)^2}{4}\rfloor, \lfloor\frac{(4+1+1)^2}{4}\rfloor\}$, π is graphic by Theorem 2.3 and 2.4. □

The proof of Theorem 1.5 Assume that π is potentially $K_6-E(K_3)$-graphic. (1) is obvious. It is easy to compute the corresponding π_3(zero omitted) of the excepted sequences is one of the following sequences: $(2),(2^2),(4,2^3),(4,2,1^2),(3,1),(3,2,1),(4^2,2,1^2),(3^2,1^2),(5,3,2^3),(5,2^3,1),$ $(5^2,2^3),(4,3,1^3),(3^2),(3^2,2),(5^2,4,2^3),(3^3,1),(4^2,1^4),(4^3,2,1^2),(4^2,3,1^3),$ $(5^3,2^3,1),(4^3,1^4)$, none of which is graphic.

To prove the sufficiency, we use induction on n. From Lemma 2.1, it is enough to show that π_3 is graphic. Assume that $n=6$ and $\pi=(d_1,d_2,\dots,d_n) \in GS_n$ satisfies (1) and (2). Then π is one of the following sequences:

$$(5^3,3^3),(5^3,4^2,3),(5^4,4^2),(5^6).$$

It is easy to see that these sequences are all potentially $K_6-E(K_3)$-graphic. Now suppose that the sufficiency holds for $n-1 (n \geq 7)$, and let $\pi=(d_1,d_2,\dots,d_n) \in GS_n$ satisfies (1) and (2). In the following, we will use Theorem 2.5, repeatedly. We now prove that π is potentially $K_6-E(K_3)$-graphic in terms of the following cases

Case 1. $d_n \geq 5$. Then $\pi'_n=(d'_1,d'_2,\dots,d'_{n-1})$ satisfies (1) since $\pi \neq (6,5^6),(5^8)$. If π'_n also satisfies (2), then by the induction hypothesis, π'_n is potentially $K_6-E(K_3)$-graphic, and hence so is π. If π'_n does not satisfy (2), i.e. π'_n is one of the following since $\pi \neq (5^{10}),(6,5^8)$

$$(6,5^6),(5^6,4),(5^4,4^3),(5^8,4),(6,5^8),(5^8).$$

Hence π is one of the following

$$(6^6,5^2),(7,6^4,5^3),(6^4,5^4),(6^4,5^6),(7,6^4,5^5),(6^6,5^4),(6^5,5^4),(6^2,5^6).$$

It is easy to compute the corresponding π_3 is one of the following graphic sequences

$$(4,3^4),(3^4,2),(4,3^2,2^2),(5,4^3,3,2^2),(4^4,3^2,2),(5,4^3,3^3),(4^3,3^2,2),(4^2,2^3).$$

So π is potentially $K_6-E(K_3)$-graphic.

Case 2. $d_n=4$. Consider $\pi'_n=(d'_1,d'_2,\dots,d'_{n-1})$ where $d'_{n-1} \geq 3$ and $d'_{n-2} \geq 4$. If π'_n satisfies (1) and (2), then by the induction hypothesis, π'_n is potentially $K_6-E(K_3)$-graphic, and hence so is π.

Assume that π'_n does not satisfy (1), i.e. $d'_3 = 4$. Then $d_3 = 5$. Thus, the general form for π must be one of t he following four types

$$(d_1, d_2, 5, 4^{n-3}), (d_1, d_2, 5^2, 4^{n-4}), (d_1, d_2, 5^3, 4^{n-5}), (d_1, d_2, 5^4, 4^{n-6}).$$

If $\pi = (d_1, d_2, 5, 4^{n-3})$ and $d_1 + d_2 \leq n + 4$, then

$$\pi_3 = (4^{n+4-d_1-d_2}, 3^{d_1+d_2-10}, 1^3)$$

and $|\pi_3| \geq 4$, where $|\pi|$ means the number of the positive term of π. If $\pi_3 = 4$, then $n = 7$ and π is $(6, 5^2, 4^4)$, which is potentially $K_6 - E(K_3)$-graphic since π'_1 is potentially $K_5 - E(K_3)$-graphic by Theorem 1.4. If $\pi_3 \geq 5$ and $\pi_3 \neq (4, 3, 1^3), (4^2, 3, 1^3)$, then π_3 is graphic by Lemma 2.2 and Lemma 2.3. If π_3 is $(4, 3, 1^3)$ or $(4^2, 3, 1^3)$, then π is $(6, 5^2, 4^5)$ or $(6, 5^2, 4^6)$, which is a contradiction. If $d_1 + d_2 \geq n + 5$, then $\pi_3 = (3^x, 2^y, 1^3)(y \geq 1)$, which is graphic by Lemma 2.2.

If $\pi = (d_1, d_2, 5^2, 4^{n-4})$ and $d_1 + d_2 \leq n + 4$, then

$$\pi_3 = (4^{n+4-d_1-d_2}, 3^{d_1+d_2-10}, 2, 1^2)$$

and $|\pi_3| \geq 4$. If $\pi_3 = 4$, then $n = 7$ and π is $(6^2, 5^2, 4^3)$ since $\pi \neq (5^4, 4^3)$. The sequence π_3 is graphic sequence $(2^2, 1^2)$, so π is potentially $K_6 - E(K_3)$-graphic. If $\pi_3 \geq 5$ and $\pi_3 \neq (4^2, 2, 1^2), (4^3, 2, 1^2)$, then π_3 is graphic by Lemma 2.2 and Lemma 2.3. If π_3 is $(4^2, 2, 1^2)$ or $(4^3, 2, 1^2)$, then π is $(5^4, 4^4)$ or $(5^4, 4^5)$, a contradiction. If $d_1 + d_2 \geq n + 5$, then $\pi_3 = (3^x, 2^y, 1^2)(y \geq 2)$, which is graphic by Lemma 2.2.

If $\pi = (d_1, d_2, 5^3, 4^{n-5})$ and $d_1 + d_2 \leq n + 4$, then

$$\pi_3 = (4^{n+4-d_1-d_2}, 3^{d_1+d_2-10}, 2^2, 1)$$

and $|\pi_3| \geq 4$. If $\pi_3 = 4$, then $\pi_3 = (3, 2^2, 1)$ since $\sigma(\pi_3)$ is even. So π is potentially $K_6 - E(K_3)$-graphic by Lemma 2.1. If $\pi_3 \geq 5$, then π_3 is graphic by Lemma 2.2 and Lemma 2.3. Thereby, π is potentially $K_6 - E(K_3)$-graphic by Lemma 2.1. If $d_1 + d_2 \geq n + 5$, then $\pi_3 = (3^x, 2^y, 1)(y \geq 3)$, which is graphic by Lemma 2.2.

If $\pi = (d_1, d_2, 5^4, 4^{n-6})$ and $d_1 + d_2 \leq n + 4$, then

$$\pi_3 = (4^{n+4-d_1-d_2}, 3^{d_1+d_2-10}, 2^3)$$

and $|\pi_3| \geq 4$. If $\pi_3 = 4$, then $\pi_3 = (4, 2^3)$ since $\sigma(\pi_3)$ is even. So π is $(5^6, 4)$, which is a contradiction. If $\pi_3 \geq 5$, then π_3 is graphic by Lemma 2.2 and Lemma 2.3. Thereby, π is potentially $K_6 - E(K_3)$-graphic by Lemma 2.1. If $d_1 + d_2 \geq n + 5$, then $\pi_3 = (3^x, 2^y)(y \geq 4)$, which is graphic by Lemma 2.2.

Assume that π'_n does not satisfy (2), i.e. π'_n is one of the following

$$(6,5^6),(5^6,4),(5^4,4^3),(5^4,4^4),(5^8,4),(6,5^8),(5^8),(6,5^2,4^5),(6,5^2,4^6),$$

$$(5^4,4^5),(5^3,4^4,3),(5^3,4^5,3),(5^{10}).$$

Since $\pi \neq (5^8,4)$, π is one of the following
$(7,6^3,5^3,4),(6^3,5^4,4),(6^4,5^2,4^2),(6^4,4^4),(6^3,5^2,4^3),(6^2,5^4,4^2),$
$(6^5,5^2,4),(6^3,4^6),(6^4,5^6,4),(6,5^6,4^2),(6^2,5^4,4^3),(6^3,5^2,4^4),$
$(6^4,4^5),(6^3,5^6,4),(6^4,5^4,4^2),(7,6^3,5^5,4),(6^3,4^7),(6,5^6,4),$
$(6^5,5^4,4),(6^4,5^4,4),(7,6^2,5,4^5),(7,6,5^3,4^4),(7,5^5,4^3),(7,5^5,4^4),$
$(7,6,5^3,4^5),(6^4,4^6),(7,6^2,5,4^6),(6^3,5^2,4^5),(6^2,5^4,4^4),(6,5^6,4^3),(5^8,4^2).$
It is easy to compute the corresponding π_3 is one of the following graphic sequences
$(3^4,2),(3^2,2^3),(3^2,2,1^2),(3^3,2^2,1),(4^2,2^3),(4,3^2,2^3),$
$(3^4,1^2),(4^4,2^3),(4^3,3^2,2^2),(4^4,3^2,2),(4^2,3^2,2^2),(3^2,2^2,1^2),$
$(3^4,2,1^2),(4^4,2^3),(4,3^3,2^2,1),(4^2,3^2,2^3),(4,3^4,1^2),(5^2,4^2,2^3),$
$(3,2^3,1),(3^3,1^3),(4,3^3,1^3),(5,4^4,3,2^2).$
So π is potentially $K_6-E(K_3)$-graphic by Lemma 2.1.

Case 3. $d_n=3$. If $\pi'_n=(d'_1,d'_2,\ldots,d'_{n-1})$ satisfies (1) and (2), then by the induction hypothesis, π'_n is potentially $K_6-E(K_3)$-graphic, and hence so is π.

Assume that π'_n does not satisfy (1), i.e. $d'_3=4$. Then $d_3=5$ and $3\le d_6\le 4$.

If $d_6=d_4=3$, then $\pi=(d_1,d_2,5,3^{n-3})$. If $d_1=5$, then $\pi=(5^3,3^{n-3})$ and n is even. Hence $\pi_3=(3^{n-6})$. If $n\ge 10$, then π_3 is graphic by Theorem 2.4. Thus π is potentially $K_6-E(K_3)$-graphic by Lemma 2.1. If $n=8$, then $\pi=(5^3,3^5)$, a contradiction. If $6\le d_1\le n-2$ and $d_2=5$, then $\pi_3=(3^{n-1-d_1},2^{d_1-5})$. If $\pi_3\neq(3^2,2)$, then π_3 is graphic by Lemma 2.2. If $\pi_3=(3^2,2)$, then $\pi=(6,5^2,3^6)$, a contradiction. If $d_1=n-1$ and $d_2=5$, then $\pi_3=(2^{n-6})$. If $\pi_3\neq(2^2),(2)$, then π_3 is graphic by Lemma 2.2. If π_3 is (2^2) or (2), then π is $(6,5^2,3^4)$ or $(7,5^2,3^5)$, a contradiction. If $d_2\ge 6$ and $d_1+d_2\le n+3$, then $\pi_3=(3^{n+4-d_1-d_2},2^{d_1+d_2-10})$, which is graphic by Lemma 2.2. If $d_2\ge 6$ and $d_1+d_2=n+4$, then $\pi_3=(2^{n-6})$ and $n\ge 8$. If $\pi_3\neq(2^2)$, then π_3 is graphic by Lemma 2.2. If π_3 is (2^2), then π is $(6^2,5,3^5)$, a contradiction. If $d_2\ge 6$ and $d_1+d_2\ge n+5$, then $\pi_3=(2^x,1^y)(y\ge 1)$, which is graphic by Lemma 2.2.

If $d_6=d_5=3$ and $d_4=4$, then $\pi=(d_1,d_2,5,4,3^{n-4})$. If $d_1=5$, then $\pi=(5^3,4,3^{n-4})$ and n is odd. Hence $\pi_3=(3^{n-6},1)$. If $\pi_3\neq(3,1),(3^3,1)$, then π_3 is graphic by Lemma 2.2. If π_3 is $(3,1)$ or $(3^3,1)$, then π is $(5^3,4,3^3)$ or $(5^3,4,3^5)$, a contradiction. If $d_1\ge 6$ and $d_1+d_2\le n+3$, then $\pi_3=(3^{n+4-d_1-d_2},2^{d_1+d_2-10},1)$. If $\pi_3\neq(3,2,1)$, then π_3 is graphic by Lemma 2.2. If $\pi_3=(3,2,1)$, then $\pi=(6,5^2,4,3^4)$, a contradiction. Since $\sigma(\pi_3)$

is even, then $d_1 + d_2 \neq n + 4$. If $d_1 \geq 6$ and $d_1 + d_2 \geq n + 5$, then $\pi_3 = (2^x, 1^y)(y \geq 1)$, which is graphic by Lemma 2.2.

If $d_6 = d_5 = 3$ and $d_4 = 5$, then $\pi = (d_1, 5^3, 3^{n-4})$ since $d'_3 = 4$. It is easy to compute the sequence π_3 is $(3^{n-1-d_1}, 2^{d_1-4})$. If $|\pi_3| \geq 4$, that is $n \geq 9$, then $|\pi_3|$ is graphic by Lemma 2.2. If $n = 7$, then π is $(6, 5^3, 3^3)$, a contradiction. If $n = 8$, then $\pi = (7, 5^3, 3^4)$ since $\pi \neq (5^4, 3^4)$. The sequence π_3 is graphic sequence (2^3).

If $d_6 = 3$ and $d_4 = d_5 = 4$, then $\pi = (d_1, d_2, 5, 4^2, 3^{n-5})$. If $d_1 = 5$, then $\pi = (5^3, 4^2, 3^{n-5})$ and n is even. Hence $\pi_3 = (3^{n-6}, 1^2)$. If $\pi_3 \neq (3^2, 1^2)$, then π_3 is graphic by Lemma 2.2. If π_3 is $(3^2, 1^2)$, then π is $(5^3, 4^2, 3^3)$, a contradiction. If $d_1 \geq 6$ and $d_1 + d_2 \leq n + 3$, then $\pi_3 = (3^{n+4-d_1-d_2}, 2^{d_1+d_2-10}, 1^2)$. According to Lemma 2.2, π_3 is graphic. If $d_1 \geq 6$ and $d_1 + d_2 \geq n + 4$, then π_3 is graphic sequence $(2^x, 1^y)(y \geq 2)$.

If $d_6 = 3$ and $d_4 = d_5 = 5$, then $\pi = (5^5, 3^{n-5})$ since $d'_3 = 4$. It is easy to compute the sequence π_3 is $(3^{n-6}, 2^2)$. Since $\sigma(\pi_3)$ is even, π_3 is graphic by Lemma 2.2. According to Lemma 2.1, π is potentially $K_6 - E(K_3)$-graphic.

If $d_6 = 3, d_4 = 5$ and $d_5 = 4$, then $\pi = (d_1, 5^3, 4, 3^{n-5})$ since $d'_3 = 4$. If $d_1 = 5$, then $\pi = (5^4, 4, 3^{n-5})$ and n is odd. Hence $\pi_3 = (3^{n-6}, 2, 1)$. If $\pi_3 \neq (3, 2, 1)$, then π_3 is graphic by Lemma 2.2. If π_3 is $(3, 2, 1)$, then π is $(5^4, 4, 3^2)$, a contradiction. If $d_1 \geq 6$, then $\pi_3 = (3^{n-1-d_1}, 2^{d_1-4}, 1)$, which is graphic by Lemma 2.2.

If $d_6 = d_4 = 4$, then $\pi = (d_1, d_2, 5, 4^3, 4^x, 3^{n-6-x})(n - 6 - x \geq 1)$. If $d_1 = 5$, then $\pi = (5^3, 4^3, 4^x, 3^{n-6-x})$ and the parities of n differs from x. Hence $\pi_3 = (4^x, 3^{n-6-x}, 1^3)$. If $\pi_3 \neq (4, 3, 1^3), (4^2, 3, 1^3)$, then π_3 is graphic by Lemma 2.2 and Lemma 2.3. If π_3 is $(4, 3, 1^3)$ or $(4^2, 3, 1^3)$, then π is $(5^3, 4^4, 3)$ or $(5^3, 4^5, 3)$, a contradiction. If $d_1 \geq 6$ and $d_1 + d_2 \leq x + 9$, then $\pi_3 = (4^{x+10-d_1-d_2}, 3^{n-16-x+d_1+d_2}, 1^3)$, which is graphic by Lemma 2.3. If $d_1 \geq 6$ and $d_1 + d_2 = x + 10$, then π_3 is graphic sequence $(3^{n-6}, 1^3)$. If $d_1 \geq 6$ and $d_1 + d_2 \geq x + 11$, then π_3 is graphic sequence $(3^{x'}, 2^{y'}, 1^3)(y' \geq 1)$.

If $d_6 = d_5 = 4$ and $d_4 = 5$, then $\pi = (d_1, 5^3, 4^2, 4^x, 3^{n-6-x})(n - 6 - x \geq 1)$. If $d_1 \leq x + 4$, then π_3 is graphic sequence $(4^{x-d_1+5}, 3^{n-11-x+d_1}, 2, 1^2)$. If $d_1 \geq x + 5$, then $\pi_3 = (3^{x'}, 2^{y'}, 1^2)(x' \geq 1, y' \geq 1)$, which is graphic by Lemma 2.2.

If $d_6 = 4$ and $d_4 = d_5 = 5$, then $\pi = (5^5, 4, 4^x, 3^{n-6-x})(n - 6 - x \geq 1)$ since $d'_3 = 4$. It is easy to compute the sequence π_3 is $(4^x, 3^{n-6-x}, 2^2, 1)$, which is graphic by Lemma 2.2 and Lemma 2.3.

Assume that π'_n does not satisfy (2). Then π'_n is one of the following sequences:

$(6, 5^2, 3^4), (7, 5^2, 3^5), (6, 5^3, 3^3), (6, 5^6), (5^6, 4), (5^4, 4^3), (5^4, 4^4),$
$(5^8, 4), (5^3, 4^2, 3^3), (6^4, 3^4), (5^7, 3), (5^3, 4, 3^5), (6, 5^8), (5^8),$
$(6, 5^2, 4^5), (6, 5^2, 4^6), (5^4, 4^5), (5^{10}), (5^3, 3^5), (6, 5^2, 3^6), (6^2, 5, 3^5),$
$(5^3, 4, 3^3), (6, 5^2, 4, 3^4), (5^4, 3^4), (5^4, 4, 3^2), (5^3, 4^4, 3), (5^3, 4^5, 3).$

Since $\pi \neq (6^4, 3^4), (5^7, 3)$, π is one of the following sequences:

$(7,6^2,3^5),(8,6^2,3^6),(7,6^2,5,3^4),(6^4,5^3,3),(7,6^2,5^4,3),(6^3,5^3,4,3),$
$(6^2,5^5,3),(6^3,5,4^3,3),(6,5^5,4,3),(6^2,5^3,4^2,3),(5^7,4,3),(6^3,5,4^4,3),$
$(6,5^5,4^2,3),(6^2,5^3,4^3,3),(6^3,5^5,4,3),(6^2,5^7,3),(6^3,4^2,3^4),(6^2,5^2,4,3^4),$
$(6,5^4,3^4),(6^3,4,3^6),(6^2,5^2,3^6),(7^3,6,3^5),(7,5^4,4^4,3),(6^3,5^4,3^2),$
$(6^4,5^5,3),(7,6^2,5^6,3),(6^3,5^5,3),(7,6^2,4^5,3),(7,6^2,4^6,3),(6^3,5,4^5,3),$
$(6^2,5^3,4^4,3),(6,5^5,4^3,3),(5^7,4^2,3),(6^3,5^7,3),(7,6,5^2,4^4,3),(7,5^4,4^3,3),$
$(7,6^2,3^7),(6^3,3^6),(7^2,6,3^6),(6^3,4,3^4),(6^2,5^2,3^4),(7,6^2,4,3^5),$
$(7,6,5^2,3^5),(6^3,5,3^5),(6^3,5,4,3^3),(6^2,5^3,3^3),(6^3,4^4,3^2),(5^6,4,3^2),$
$(6^2,5^2,4^3,3^2),(6,5^4,4^2,3^2),(6^3,4^5,3^2),(5^6,4^2,3^2),(6^2,5^2,4^4,3^2),$
$(6,5^4,4^3,3^2),(7,6,5^2,4^5,3)$.

It is easy to compute the corresponding π_3(zero omitted) is one of the following graphic sequences:
$(1^2),(2,1^2),(2^5),(3^2,2^3),(2^3,1^2),(3,2^3,1),(5,4,3,2^3),(3^2,2^2,1^2),$
$(4,3^2,2^3),(3^3,2^2,1),(4^2,3^2,2^3),(5,4^2,3,2^3),(3,1^3),(3^2,2^4),(4^3,3^2,2^2),$
$(3,2^2,1^3),(3^3,2,1^3),(3^4,2,1^2),(4,3^3,2^2,1),(5,4^3,3,2^3),(2^3),(2^4),(2^2,1^2)$.

Case 4. $d_n = 2$. If $\pi'_n = (d'_1, d'_2, \ldots, d'_{n-1})$ satisfies (1) and (2), then by the induction hypothesis, π'_n is potentially $K_6 - E(K_3)$-graphic, and hence so is π.

Assume that π'_n does not satisfy (1), i.e. $d'_3 = 4$. Then $d_2 = d_3 = 5$ and $d_5 \leq 4$.

If $d_5 = d_4 = 3$, then $\pi = (d_1, 5^2, 3^3, 3^x, 2^{n-6-x})(n-6-x \geq 1)$. If $d_1 = 5$, then $\pi_3 = (3^x, 2^{n-6-x})$. If $\pi_3 \neq (3^2, 2)$, then π_3 is graphic by Lemma 2.2. If $\pi_3 = (3^2, 2)$, then π is $(5^3, 3^5, 2)$, a contradiction. If $6 \leq d_1 \leq x+4$, then π_3 is graphic sequence $(3^{x+5-d_1}, 2^{n-6-x+d_1-5})$. If $d_1 = x+5$, then π_3 is (2^{n-6}). If π_3 is not (2) or (2^2), then π_3 is graphic. If π_3 is (2) or (2^2), then π is $(5^3, 3^3, 2)$ or $(5^3, 3^3, 2^2)$ or $(6, 5^2, 3^4, 2)$, which is contradict. If $d_1 \geq x+6$, then π_3 is graphic sequence $(2^{x'}, 1^{y'})(y' \geq 1)$.

If $d_5 = 3$ and $d_4 = 4$, then $\pi = (d_1, 5^2, 4, 3^2, 3^x, 2^{n-6-x})(n-6-x \geq 1)$. If $d_1 = 5$, then $\pi_3 = (3^x, 2^{n-6-x}, 1)$. If $\pi_3 \neq (3,2,1)$, then π_3 is graphic by Lemma 2.2. If $\pi_3 = (3,2,1)$, then π is $(5^3, 4, 3^3, 2)$, a contradiction. If $6 \leq d_1 \leq x+4$, then π_3 is graphic sequence $(3^{x+5-d_1}, 2^{n-6-x+d_1-5}, 1)$. Since $\sigma(\pi_3)$ is even, $d_1 \neq x+5$. If $d_1 \geq x+6$, then π_3 is graphic sequence $(2^{x'}, 1^{y'})(y' \geq 2)$.

If $d_5 = 3$ and $d_4 = 5$, then $\pi = (5^4, 3^2, 3^x, 2^{n-6-x})(n-6-x \geq 1)$ since $d'_3 = 4$. It is easy to compute $\pi_3 = (3^x, 2^{n-5-x})$. Since $\sigma(\pi_3)$ is even, x is even. If $\pi_3 \neq (2^2)$, then π_3 is graphic by Lemma 2.2. If $\pi_3 = (2^2)$, then π is $(5^4, 3^2, 2)$, a contradiction.

If $d_5 = d_4 = 4$ and $d_6 = 3$, then $\pi = (d_1, 5^2, 4^2, 3, 3^x, 2^{n-6-x})(n-6-x \geq 1)$. If $d_1 = 5$, then it is easy to compute $\pi_3 = (3^x, 2^{n-6-x}, 1^2)$, which is graphic. If $6 \leq d_1 \leq x+4$, then π_3 is graphic sequence

$$(3^{x+5-d_1}, 2^{n-6-x+d_1-5}, 1^2).$$

If $d_1 \geq x+5$, then π_3 is graphic sequence $(2^{x'}, 1^{y'})(y' \geq 2)$.

If $d_5 = d_4 = d_6 = 4$, then $\pi = (d_1, 5^2, 4^3, 4^x, 3^y, 2^{n-6-x-y})(n-6-x-y \geq 1)$. If $d_1 = 5$, then $\pi_3 = (4^x, 3^y, 2^{n-6-x-y}, 1^3)$. According to Lemma 2.2 and Lemma 2.3, π_3 is graphic. If $6 \leq d_1 \leq x+4$, then $\pi_3 = (4^{x-d_1+5}, 3^{y+d_1-5}, 2^{n-6-x-y}, 1^3)$, which is graphic by Lemma 2.3. If $d_1 \geq x+5$, then $\pi_3 = (3^{x'}, 2^{y'}, 1^{z'})(z' \geq 3)$, which is graphic by Lemma 2.2.

If $d_5 = 4, d_4 = 5$ and $d_6 = 3$, then $\pi = (5^4, 4, 3, 3^x, 2^{n-6-x})(n-6-x \geq 1)$ since $d'_3 = 4$. It is easy to compute $\pi_3 = (3^x, 2^{n-5-x}, 1)$. According to Lemma 2.2, π_3 is graphic.

If $d_5 = d_6 = 4$ and $d_4 = 5$, then $\pi = (5^4, 4^2, 4^x, 3^y, 2^{n-6-x-y})(n-6-x-y \geq 1)$ since $d'_3 = 4$. It is easy to compute $\pi_3 = (4^x, 3^y, 2^{n-5-x-y}, 1^2)$, which is graphic by Lemma 2.2 and Lemma 2.3.

Assume that π'_n does not satisfy (2), i.e. π'_n is one of the following sequences:
$(6,5^2,3^4), (7,5^2,3^5), (6,5^3,3^3), (6,5^6), (5^6,4), (5^4,4^3), (5^4,4^4),$
$(5^8,4), (5^3,4^2,3^3), (6^4,3^4), (5^7,3), (5^3,4,3^5), (6,5^8), (5^3,3^3,2),$
$(5^3,3^3,2^2), (5^3,3^5,2), (5^3,4,3^3,2), (5^4,3^2,2), (5^8), (6,5^2,4^5), (6,5^2,4^6),$
$(5^4,4^5), (6,5^2,3^4,2), (5^3,3^5), (6,5^2,3^6), (6^2,5,3^5), (5^3,4,3^3),$
$(6,5^2,4,3^4), (5^4,3^4), (5^4,4,3^2), (5^3,4^4,3), (5^3,4^5,3), (5^{10})$.

Hence π is one of the following sequences:
$(7,6,5,3^4,2), (6^3,3^4,2), (8,6,5,3^5,2), (7,6,5^2,3^3,2), (6^3,5,3^3,2), (7,6,5^5,2),$
$(6^2,5^4,4,2), (6^3,5^4,2), (6,5^6,2), (6^2,5^2,4^3,2), (6,5^4,4^2,2), (5^6,4,2),$
$(6^2,5^2,4^4,2), (5^6,4^2,2), (6,5^8,2), (6,5^4,4^3,2), (6^2,5^6,4,2), (6^2,5,4^2,3^3,2),$
$(5^5,3^3,2), (7,6,5^7,2), (6^3,5^6,2), (6^2,5,4,3^5,2), (6,5^3,3^5,2), (7^2,6^2,3^4,2),$
$(6^2,5^5,3,2), (6^2,5,3^3,2^2), (6^2,5,3^3,2^3), (6^2,5,3^5,2^2), (6^2,5,4,3^3,2^2),$
$(6,5^3,3^3,2^2), (6^2,5^2,3^2,2^2), (6^2,5^6,2), (7,6,5,4^5,2), (6^3,4^5,2), (7,5^3,4^4,2),$
$(7,6,5,4^6,2), (7,5^3,4^5,2), (6^3,4^6,2), (6^2,5^2,4^5,2), (5^6,4^3,2), (6,5^4,4^4,2),$
$(7,6,5,3^4,2^2), (6^3,3^4,2^2), (6,5^8,2), (6^2,5,3^5,2), (7,6,5,3^6,2), (6^3,3^6,2),$
$(7^2,5,3^5,2), (7,6^2,3^5,2), (6,5^3,3^3,2), (6^2,5,4,3^3,2), (7,6,5,4,3^4,2),$
$(6,5^3,4,3^3,2), (6^3,4,3^4,2), (7,5^3,3^4,2), (6^2,5^2,3^4,2), (6^2,5^2,4,3^2,2),$
$(6,5^4,3^2,2), (6^2,5,4^4,3,2), (5^5,4^2,3,2), (6,5^3,4^3,3,2), (6^2,5,4^5,3,2),$
$(5^5,4^3,3,2), (6,5^3,4^4,3,2)$.
It is easy to compute the corresponding π_3(zero omitted) is one of the following graphic sequences:
$(1^2), (2,1^2), (2^5), (3^2,2^3), (2^3,1^2), (3,2^3,1), (5^2,4,2^4), (3^2,2^2,1^2),$
$(4^2,2^4), (3^3,2,1^3), (4,3,2^3,1), (4^3,2^4), (3,2^3,1^3), (3,1^3), (3^2,2^4), (2^4),$
$(3,2^2,1^3), (5^2,4^2,2^4), (4,2^4), (4,3^2,2^2,1^2), (4^2,3,2^3,1), (2^3), (2^2,1^2)$.

Case 5. $d_n = 1$. Consider $\pi'_n = (d'_1, d'_2, \ldots, d'_{n-1})$ where $d'_2 \geq 5, d'_3 \geq 4, d'_6 \geq 3$ and $d'_{n-1} \geq 1$. If π'_n satisfies (1) and (2), then by the induction hypothesis, π'_n is potentially $K_6 - E(K_3)$-graphic, and hence so is π.

Assume that π'_n does not satisfy (1), i.e. $d'_3 = 4$. Then $d_1 = d_2 = d_3 = 5$ and $d_4 \leq 4$.

If $d_4 = 3$, then $\pi = (5^3, 3^3, 3^x, 2^y, 1^z)(z \geq 1, x+y+z = n-6)$ and π_3 is $(3^x, 2^y, 1^z)$. If $\pi_3 \neq (3,1), (3,2,1), (3^3,1), (3^2,1^2)$, then π_3 is graphic by

Lemma 2.2. If π_3 is one of the following sequences:

$$(3,1),(3,2,1),(3^3,1),(3^2,1^2),$$

then π is one of the following:

$$(5^3,3^4,1),(5^3,3^4,2,1),(5^3,3^6,1),(5^3,3^5,1^2),$$

which is contradict.

If $d_4=4$ and $d_5=3$, then $\pi=(5^3,4,3^2,3^x,2^y,1^z)(z\geq 1, x+y+z=n-6)$ and $\pi_3=(3^x,2^y,1^{z+1})$. If $\pi_3\neq(3^2,1^2)$, then π_3 is graphic by Lemma 2.2. If $\pi_3=(3^2,1^2)$, then $\pi=(5^3,4,3^4,1)$, a contradiction.

If $d_4=d_5=4$ and $d_6=3$, then $\pi=(5^3,4^2,3,3^x,2^y,1^z)(z\geq 1, x+y+z=n-6)$ and $\pi_3=(3^x,2^y,1^{z+2})$. According to Lemma 2.2, π_3 is graphic.

If $d_4=d_5=d_6=4$, then $\pi=(5^3,4^3,4^{x_1},3^{x_2},2^{x_3},1^{x_4})(x_4\geq 1, x_1+x_2+x_3+x_4=n-6)$ and $\pi_3=(4^{x_1},3^{x_2},2^{x_3},1^{x_4+3})$. If π_3 is not $(4^2,1^4)$ or $(4^3,1^4)$, then π_3 is graphic by Lemma 2.2 and Lemma 2.3. If π_3 is $(4^2,1^4)$ or $(4^3,1^4)$, then π is $(5^3,4^5,1)$ or $(5^3,4^6,1)$, which is contradict.

Assume π'_n does not satisfy (2). Since π is not one of the following sequences

$$(6,5^2,3^5,1),(n-1,5^2,3^4,1^{n-7}),(n-1,5^2,3^5,1^{n-8}),$$
$$(n-1,5^3,3^3,1^{n-7}),(n-1,5^6,1^{n-7}),$$

π'_n is one of the following sequences:
$(5^6,4),(5^4,4^3),(5^4,4^4),(5^8,4),(5^3,4^2,3^3),(6^4,3^4),$
$(5^7,3),(5^3,4,3^5),(6,5^8),(5^3,3^3,2),(5^3,3^3,2^2),(5^3,3^5,2),$
$(5^3,4,3^3,2),(5^4,3^2,2),(5^3,3^6,1),(5^3,3^5,1^2),(5^3,3^4,1),(5^3,3^4,2,1),$
$(5^3,4,3^4,1),(5^4,3^3,1),(5^7,1),(5^9,1),(5^3,4^6,1),(5^3,4^5,1),$
$(6,5^2,4^5),(6,5^2,4^6),(5^4,4^5),(5^4,3^3,1),(6,5^2,3^6),(6^2,5,3^5),$
$(5^3,4,3^3),(6,5^2,4,3^4),(5^4,3^4),(5^4,4,3^2),(5^3,4^4,3),(5^3,4^5,3),$
$(5^8),(6,5^2,3^5,1),(5^{10}),(6,5^2,3^4,2),(6,5^2,3^4),(6,5^3,3^3),(6,5^6)$.

Since $\pi\neq(5^7,1),(5^9,1),(5^4,3^3,1)$, π is one of the following sequences:
$(6,5^5,4,1),(6,5^3,4^3,1),(5^5,4^2,1),(6,5^3,4^4,1),(5^5,4^3,1),(6,5^7,4,1),$
$(6,5^2,4^2,3^3,1),(6^2,5^5,1),(5^4,4,3^3,1),(6,5^2,4,3^5,1),(5^4,3^5,1),$
$(7,6^3,3^4,1),(6,5^6,3,1),(7,5^8,1),(6^2,5^7,1),(6,5^2,3^3,2,1),$
$(6,5^2,3^3,2^2,1),(6,5^2,3^5,2,1),(6,5^2,4,3^3,2,1),(5^4,3^3,2,1),(6,5^3,3^2,2,1),$
$(6,5^2,3^6,1^2),(6,5^7,1),(6,5^2,3^5,1^3),(6,5^2,3^4,1^2),(6,5^2,3^4,2,1^2),$
$(6,5^2,4,3^4,1^2),(5^4,3^4,1^2),(6,5^3,3^3,1^2),(6,5^6,1^2),(6,5^8,1^2),$
$(7,5^2,4^5,1),(6^2,5,4^5,1),(7,5^2,4^6,1),(6^2,5,4^6,1),(6,5^3,4^5,1),$
$(5^5,4^4,1),(6,5^9,1),(6,5^3,3^3,1^2),(7,5^2,4,3^4,1),(6^2,5,4,3^4,1),$
$(7,5^2,3^6,1),(6^2,5,3^6,1),(7,6,5,3^5,1),(6^3,3^5,1),(6,5^2,4,3^3,1),$
$(6,5^3,3^4,1),(6,5^3,4,3^2,1),(5^5,3^2,1),(6,5^2,4^4,3,1),(5^4,4^3,3,1),$
$(6^2,5,3^4,1),(6,5^2,4^5,3,1),(5^4,4^4,3,1),(6,5^2,4^6,1^2),(5^4,4^5,1^2),$

$(6,5^2,4^5,1^2),(5^4,4^4,1^2),(7,5^2,3^5,1^2),(6^2,5,3^5,1^2),(7,5^2,3^4,2,1),$
$(6^2,5,3^4,2,1),(6^2,5^2,3^3,1)$.
It is easy to compute the corresponding π_3(zero omitted) is one of the following graphic sequences:
$(3,2^3,1),(5,4,2^3,1),(3,2,1^3),(4,2^2,1^2),(4,3,2,1^3),(4^2,2^2,1^2),$
$(5,4^2,2^3,1),(3^2,2,1^2),(3^3,2,1),(3,1^3),(4,3,2^3,1),(1^2),(2,1^2),$
$(3,2^2,1),(2^2,1^2),(4,2^3,1^2),(5^2,4,2^3,1^2),(3^2,1^4),(4,3^2,1^4),(4^2,3,2,1^3),$
$(4^3,2^2,1^2),(5^3,4,2^3,1),(4^2,3,1^5),(4^3,2,1^4),(4,3,1^5),(4^2,2,1^4)$. □

We now give an application of Theorem 1.5.

Corollary $\sigma(K_6-E(K_3),6)=28$, $\sigma(K_6-E(K_3),7)=38$, $\sigma(K_6-E(K_3),8)=42$, $\sigma(K_6-E(K_3),9)=48$, $\sigma(K_6-E(K_3),10)=52$ and $\sigma(K_6-E(K_3),n)=6n-8$ for $n\geq 11$.

Proof. For $6\leq n\leq 10$, the following sequences

$$(5^2,4^4),(6,5^6),(5^8),(6,5^8),(5^{10}),$$

none of which is potentially $K_6-E(K_3)$-graphic, thereby, $\sigma(K_6-E(K_3),6)\geq 5\times 2+4\times 4+2=28, \sigma(K_6-E(K_3),7)\geq 6+5\times 6+2=38, \sigma(K_6-E(K_3),8)\geq 5\times 8+2=42, \sigma(K_6-E(K_3),9)\geq 6+5\times 8+2=48, \sigma(K_6-E(K_3),10)\geq 5\times 10+2=52$.

According to Theorem 1.5, $\sigma(K_6-E(K_3),6)=28, \sigma(K_6-E(K_3),7)=38, \sigma(K_6-E(K_3),8)=42, \sigma(K_6-E(K_3),9)=48, \sigma(K_6-E(K_3),10)=52$.

For $n\geq 11$, take $\pi=((n-1)^2,4^{n-2})$. It is easy to see that π is graphic. If π is potentially $K_6-E(K_3)$-graphic, then there are at least three terms in π which are greater or equal to five, a contradiction. Hence, π is not potentially $K_6-E(K_3)$-graphic. In other other words, $\sigma(K_6-E(K_3),n)\geq 2(n-1)+4(n-2)+2=6n-8$.

Let $n\geq 11$ and $\pi=(d_1,d_2,\ldots,d_n)\in GS_n$ be a positive sequence with $\sigma(\pi)\geq 6n-8$. We show that π is potentially $K_6-E(K_3)$-graphic.

(1) By $n\geq 11$ and $\sigma(\pi)\geq 6n-8$, it is easy to check that π is not one of the following sequences:

$$(n-1,5^2,3^4,1^{n-7})(n\geq 7),\quad (n-1,5^2,3^5,1^{n-8})(n\geq 8),$$
$$(n-1,5^3,3^3,1^{n-7})(n\geq 7),\quad (n-1,5^6,1^{n-7})(n\geq 7),$$

(2) We claim that $d_3\geq 5$. By way of contradiction, we assume that $d_3\leq 4$. Then $\sigma(\pi)=d_1+d_2+\cdots+d_n\leq 2(n-1)+4(n-2)=6n-10<6n-8$, a contradiction.

(3) We claim that $d_6\geq 3$. By way of contradiction, we assume that $d_6\leq 2$. Then $\sigma(\pi)=\sum_{i=1}^5 d_i+\sum_{i=6}^n d_i\leq 5(5-1)+\sum_{i=6}^n min\{5,d_i\}+\sum_{i=6}^n d_i=20+2\sum_{i=6}^n d_i\leq 4n<6n-10$, a contradiction.

Thus, π is potentially $K_6-E(K_3)$-graphic by Theorem 1.5. □

References

[1] P. Erdős and T. Gallai, Graphs with given degrees of vertices, *Math. Lapok,* **11** (1960) 264–274.

[2] E.M. Eschen and J.B. Niu, On potentially $K_4 - e$-graphic sequences, *Australasian J. Combinatorics,* **29** (2004) 59–65.

[3] R.J. Gould, M.S. Jacobson and J. Lehel, Potentially G-graphical degree sequences, in: Y. Alavi et al., (Eds.), *Combinatorics, Graph Theory, and Algorithms,* Vol.1, New Issues Press, Kalamazoo Michigan, 1999, 451–460.

[4] L.L. Hu and C.H. Lai, On potentially $K_5 - C_4$-graphic sequences, accepted by *Ars Combinatoria.*

[5] L.L. Hu, C.H. Lai and P. Wang, On potentially $K_5 - H$-graphic sequences, accepted by *Czechoslovak Mathematical Journal.*

[6] L.L. Hu and C.H. Lai, On potentially $K_5 - E_3$-graphic sequences, accepted by *Ars Combinatoria.*

[7] L.L. Hu and C.H. Lai, On potentially 3-regular graph graphic sequences, accepted by *Utilitas Mathematics.*

[8] D.J. Kleitman and D.L. Wang, Algorithm for constructing graphs and digraphs with given valences and factors, *Discrete Math.,* **6** (1973) 79–88.

[9] J.S. Li and J.H. Yin, A variation of an extremal theorem due to Woodall, *Southeast Asian Bulletin of Mathematics,* **25** (2001) 427-434.

[10] J.H. Yin and J.S. Li, An extremal problem on potentially $K_{r,s}$-graphic sequences, *Discrete Math.,* **26** (2003) 295-305.

[11] J.H. Yin and J.S. Li, Two sufficient conditions for a graphic sequence to have a realization with prescribed clique size, *Discrete Math.,* **301** (2005) 218–227.