Some classes of antimagic graphs with regular subgraphs

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Abstract A labeling f of a graph G is a bijection from its edge set $E(G)$ to the set $\{1, 2, ..., |E(G)|\}$, which is antimagic if for any distinct vertices x and y, the sum of the labels on edges incident to x is different from the sum of the labels on edges incident to y. A graph G is antimagic if G has an f which is antimagic. Hartsfield and Ringel conjectured in 1990 that every connected graph other than K_2 is antimagic. In this paper, we show that some graphs with regular subgraphs are antimagic.

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1. Introduction

Graphs considered in this paper are finite, undirected and loopless. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$, respectively. Let f be a mapping from $E(G)$ onto $\{1,2,...,|E|\}$. For a vertex x of G, we define a mapping on $V(G)$ by $f(x) = \sum_{e\in E_x} f(e)$, where E_x is the set of edges incident with x in G. An antimagic labeling f of G is a bijection from $E(G)$ to $\{1,2,...,|E|\}$ such that for any two distinct vertices x and y, $f(x) \neq f(y)$. Let $E(G) = E_1 \cup E_2$, where E_1 and E_2 are two disjoint subsets. Let f_i be mappings from E_i to the set of integers for $i = 1,2$. Then $(f_1 \cup f_2)$

is a mapping from E and $(f_1 \cup f_2)(x) = (f_1 + f_2)(x) = f_1(x) + f_2(x)$ for $x \in V(G)$. A graph G is antimagic if it admits an antimagic labeling.

The notation of antimagicness of graphs is introduced in 1990 in [1], which conjectured that connected graphs other than K_2 are antimagic. Since then, some classes of graphs are proved to be antimagic, which includes Cartesian products of graphs, regular bipartite graphs, cycles, paths, complete graphs and so on. But the conjecture is open. For more details on antimagic graphs, refer to [1-8].

It is proved in [3] that regular bipartite graphs are antimagic. If we add some edges to the same part of vertices of a bipartite graph, is the obtained graph antimagic? Clearly, the hardness of showing a graph being antimagic is that the antimagic labeling of a graph may be useless for the graph with one more edge, although the difference between the two graphs is very little. In this paper, we use the way of listing edges in [2] and ideas in [3], we show that some classes of graphs derived from regular graphs or regular bipartite graphs are antimagic.

Our main result of this paper can be stated as follows.

Theorem 1.1 *Let $X = \{x_1, x_2, ..., x_n\}$ and $Y = \{y_1, y_2, ..., y_n\}$. Let G_1 be a graph with vertex set Y and maximum degree at most $4r+1(< n)$. Let G_2 be an $m(2 < m < n)$-regular bipartite graph with parts X and Y and G_3 be a $(2r)$-regular graph with vertex set X. Let $G = G_1 \cup G_2 \cup G_3$. Then G is antimagic.*

Before proving Theorem 1.1, we need some lemmas. We put them in Section 2. The proof of Theorem 1.1 is in Section 3.

2. Some lemmas

By the way of listing edges as in [2], we have the following lemma.

Lemma 2.1 *Let G be a connected graph of size e with maximum degree Δ. Let f be a labeling of edges onto $\{1, 2, ..., e\}$ and let $f(z) = max\{f(x_1), f(x_2), ..., f(x_n)\}$. If Δ is odd, then $f(z) \leq \frac{\Delta-1}{2}(e+2)+e$. If Δ is even, then $f(z) \leq \Delta(e+2)/2$.*

Proof Let G^* be the graph with vertex set $V(G) \cup \{u\}$, where u is a new vertex, and edge set $E(G) \cup F$, where $F = \{ux : x \text{ is an odd vertex of } G\}$. Then G^* is a eulerian graph and it has a euler trail R. We order the edges of G along the trail R by r_1, r_2, ..., r_e and label edges in F by zero, label the unlabeled edges along R by 1, e, 2, $e-1$, 3, $e-2$, ..., such that label i is followed by $e+1-i$ for $i \leq e/2$, and label j is followed by $e+2-j$ for $\lceil e/2 \rceil \leq j \leq e$. For an even vertex of degree d, there are $d/2$ pairs of consecutive edges, $f(x) \leq d(e+2)/2$. For an odd vertex of degree d, there are $(d-1)/2$ pairs of consecutive edges and a single edge, $f(x) \leq (d-1)(e+2)/2 + e$. Let $d = \Delta$. The lemma follows. □

Lemma 2.2 is a straightforward result of Lemma 2.1.

Lemma 2.2 *If G is a $2t$-regular bipartite graph with A and B of size n, then G has a mapping f from the edges of G onto $\{1, 2, ..., 2tn\}$ such that the sum at each vertex of A is a constant $(2tn+1)t$ and the sum of the vertices of B is at most $(2nt+2)t$.*

Proof We assume that G is connected. Since G is eulerian, G has a euler trail R. Starting from a vertex of B, labeling the edges as in Lemma 2.1, we have that for a vertex x in A, $f(x) = t(2nt+1)$ and for a vertex y in B, $f(y) \leq t(2nt+2)$. Actually, $f(y) = t(2nt+2)$ except for the starting vertex with weight $(t-1)(2nt+2) + (2+nt)$.

If G is not connected, we consider each component separately. We assume that $G = G_1 \cup G_2$, where G_1 is connected. Further, we assume that $|E(G)| = e$ and $|E(G_1)| = e_1$. We use 1, 2, ..., $e_1/2$ and e, $e-1$, ..., $e - e_1/2 + 1$ to label edges of G_1 by the way of connectedness. □

Lemma 2.3 *For $n \geq 3$, there is a $3 \times n$ matrix A with entries $\{1, 2, ..., 3n\}$ such that either the sum of the entries of each column is a constant or except for one column, the sum of the entries of each column is a constant.*

Proof We construct the matrix $A_{3\times n}$ as follows:

Case 1. $n = 2k+1$ for $k \geq 1$. $a_{1,i} = 3i$ for $1 \leq i \leq n$. $a_{3,2i+1} = 6k+1-3(i-1)$ for $1 \leq i \leq k+1$, $a_{3,2i} = 3(k-i)+1$ for $1 \leq i \leq k$. $a_{2,2i+1} = 3(k-i)+2$ for $0 \leq i \leq k$, $a_{2,2i} = 6k+2-3(i-1)$ for $1 \leq i \leq k$.

It is easy to know that the constant is $(9n+3)/2$. The following is the required matrix.

$$A = \begin{pmatrix} 3 & 6 & 9 & 12 & \cdots & 6k & 6k+3 \\ 3k+2 & 6k+2 & 3k-1 & 6k-1 & \cdots & 3k+5 & 2 \\ 6k+1 & 3k-2 & 6k-2 & 3k-5 & \cdots & 1 & 3k+1 \end{pmatrix}.$$

Case 2. $n = 2k+2$ for $k \geq 1$. Let vector $\beta = (6k+4, 6k+5, 6k+6)$. Let $B = A_{3\times(n-1)}$, where B is constructed as in Case 1. Let $A = (B\beta^T)$. It is easy to know that the constant is $(9n-6)/2$. The following is the required matrix.

$$A = \begin{pmatrix} 3 & 6 & 9 & 12 & \cdots & 6k & 6k+3 & 6k+4 \\ 3k+2 & 6k+2 & 3k-1 & 6k-1 & \cdots & 3k+5 & 2 & 6k+5 \\ 6k+1 & 3k-2 & 6k-2 & 3k-5 & \cdots & 1 & 3k+1 & 6k+6 \end{pmatrix}.$$

□

Lemma 2.3 is constructive. Since the edge set will be decomposed into a few pieces, one of which is a 3-regular bipartite graph. We will use the entries of the matrix to label these edges such that the entries of each column are the labelings of the edges incident with the same vertex.

The idea of the following lemma is drawn from [4].

Lemma 2.4 *Let $S = \{a+1, a+2, ..., a+n\}$and H be a 2-regular graph with $V(H) = \{x_1,\ x_2,\ ...,\ x_n\}$. Let $f: V(H) \to \{b+1, b+2, ..., b+n\}$ be a bijective mapping. Then there is a bijective mapping $g: E(H) \to S$ such that $\{(f+g)(x_i): 1 \leq i \leq n\} = \{2a+n+1+b+i: 1 \leq i \leq n\}$.*

Proof Without loss of generality, we assume $f(x_i) = b+i$, for $1 \leq i \leq n$.We construct the labeling g as follows:

Firstly, we orient the edges of H such that for each vertex x, the in-degree is one and so is the out-degree. Then we assign the out-edge of x_i by $a+n+1-i$ for $1 \leq i \leq n$.

Assume the weight of the in-edge of x_i is α_i, then

$$\begin{aligned}(f+g)(x_i) &= f(x_i) + a + n + 1 - i + \alpha_i \\ &= b + i + a + n + 1 - i + \alpha_i \\ &= a + b + n + 1 + \alpha_i.\end{aligned}$$

And so, $\{(f+g)(x_i) : 1 \le i \le n\} = a+b+n+1+\{\alpha_i : 1 \le i \le n\}$. By $\{\alpha_i : 1 \le i \le n\} = S$. This completes the lemma. □

By Lemma 2.4, we have Lemmas 2.5 and 2.6.

Lemma 2.5 *Let $S = \{a+1,\ a+2, \dots,\ a+rn\}$ and H be a $2r$-regular graph with $V(H) = \{x_1,\ x_2,\ \dots,\ x_n\}$. Let $f : V(H) \to \{b+1,\ b+2, \dots,\ b+n\}$ be a bijective mapping. Then there is a bijective mapping $g : E(H) \to S$ such that $\{(f+g)(x_i) : 1 \le i \le n\} = \{r(2a+rn+1)+b+i : 1 \le i \le n\}$.*

Proof We decompose the $2r$-regular graph H into r 2-factors H_1, H_2, ..., H_r. Let $f_1 = f$, for 2-factor H_1, by Lemma 2.4, we may construct a mapping g_1 from $E(H_1)$ to $S_1 = \{a+1,\ a+2, \dots,\ a+n\}$ such that $\{(f_1+g_1)(x_i) : 1 \le i \le n\} = \{2a+n+1+b+i : 1 \le i \le n\}$. Let $f_{j+1} = f_j + g_j$,for 2-factor H_{j+1}, by Lemma 2.4, we may construct a mapping g_{j+1} from $E(H_{j+1})$ to $S_{j+1} = \{a+jn+1,\ a+jn+2, \dots,\ a+(j+1)n\}$ such that $\{(f_{j+1}+g_{j+1})(x_i) : 1 \le i \le n\} = \{(j+1)(2a+jn+n+1)+b+i : 1 \le i \le n\}$. Let $j = r-1$ and $g = g_1 + g_2 + \dots + g_r$, the conclusion of the lemma holds. □

Lemma 2.6 *Let $S = \{a+1, a+2, \dots, a+rn\}$ and H be a $2r$-regular graph with $V(H) = \{x_1,\ x_2,\ \dots,\ x_n\}$. Let f be a vertex mapping on $V(H)$ such that $f(x_1) < f(x_2) < \cdots < f(x_n)$, and $\{f(x_1),\ f(x_2),\ \cdots,\ f(x_{n-1})\} \subset \{b+1,\ b+2,\ \dots,\ b+n\}$. If $f(x_n) \ge 2n+b$. Then there is a bijective mapping $g : E(H) \to S$ such that $(f+g)(x_i) \ne (f+g)(x_j)$ for distinct i and j, and $(f+g)(x_i) \ge b+1+r(2a+rn+1)$for $1 \le i \le n$.*

Proof Firstly, let $f_1(x_i) = f(x_i)$ for $1 \le i \le n-1$, and $f_1(x_n) = b+l$, where $1 \le l \le n$, such that$\{f(x_1), f(x_2), \cdots, f(x_n)\} = \{b+1, b+2, \dots, b+n\}$. Using the way in Lemma 2.5, we have an edge mapping g on $E(H)$ such that $\{(f_1+g)(x_i) : 1 \le i \le n\} = \{r(2a+rn+1)+b+i : 1 \le i \le n\}$. Let $(f_1 \cup g)(x) = \max\{(f_1 \cup g)(x_1),\ (f_1 \cup g)(x_2),\ \cdots,\ (f_1 \cup g)(x_n)\}$, then $(f_1 \cup g)(x) - (f_1 \cup g)(x_n) \le n-1$.

Secondly, we only change $f_1(x_n)$ of vertex x_n for $f(x_n)$. By $f_1(x_n) - b = l \le n$ and $f(x_n) - b \ge 2n$, then $f(x_n) - f_1(x_n) \ge n$. For $(f_1 \cup g)(x) - (f_1 \cup$

$g)(x_n) \leq n-1$, we have $(f \cup g)(x_n) > (f_1 \cup g)(x_i)$, for $1 \leq i \leq n-1$. This completes the lemma.□

3. Proof of Theorem 1.1

Now we are ready for proving Theorem 1.1. We decompose the edge set into three parts, and label each part by the above lemmas. After checking the last labeling, we finish the proof.

Let $X = \{x_1, x_2, ..., x_n\}$ and $Y = \{y_1, y_2, ..., y_n\}$. We construct an antimagic mapping of G by three steps.

Let $e_i = |E(G_i)|$ for $i = 1, 2, 3$.

Step 1 We label the edges of G_1 by $\{1, 2, ..., e_1\}$ as doing in Lemma 2.1, and denote the labeling by f_1. We assume that $f_1(y) = \max\{f_1(y_1), f_1(y_2), ..., f_1(y_n)\}$. Since $\Delta(G_1) \leq 4r+1$, by Lemma 2.1,

$$f_1(y) \leq 2r(e_1+2)+e_1.$$

Step 2 We label edges of G_2. Let f_2 be the mapping on $E(G_2)$. We split this step by two cases.

Case 1 $m = 2t+1$ $(t \geq 1)$. Then $E(G_2)$ can be decomposed into a $2t$-regular bipartite subgraph G_2^1 and a perfect matching M_2. Let $e_2^1 = |E(G_2^1)|$. By Lemma 2.2, there is a labeling f_2^1 from $E(G_2^1)$ to $e_1 + \{1, 2, ..., e_2^1\}$ such that for $x \in X$, $f_2^1(x)$ is a constant $2te_1 + (e_2^1+1)t$, and for $y \in Y$, $f_2^1(y) \leq 2te_1 + (e_2^1+2)t$.

We may order the vertices of Y such that $(f_1+f_2^1)(y_1) \leq (f_1+f_2^1)(y_2) \leq \cdots \leq (f_1+f_2^1)(y_n)$. Now we label the edges of M_2 by labels $e_1+e_2^1+1$, ..., e_1+e_2. Define the mapping f_2^2 by $f_2^2(y_i x_{j_i}) = e_1+e_2^1+i$ for $1 \leq i \leq n$, where $\{j_1, j_2, ..., j_n\} = \{1, 2, ..., n\}$.

Let $f_2(E(G_2)) = (f_2^1 \cup f_2^2)(E(G_2))$, we have

$$\begin{aligned}(f_1 \cup f_2)(y_1) \quad &< (f_1 \cup f_2)(y_2) < \cdots < (f_1 \cup f_2)(y_n) \leq 2r(e_1+2)+e_1 \\ &+2te_1 + (e_2^1+2)t + e_1 + e_2,\end{aligned}$$

and

$$\{f_2(x_1), f_2(x_2), ..., f_2(x_n)\} = 2te_1 + (e_2^1+1)t + e_1 + e_2^1 + \{1, 2, ..., n\}.$$

Case 2. $m = 2t + 4$ ($t \geq 0$). Then $E(G_2)$ can be decomposed into a $2t$-regular bipartite subgraph G_2^1 and four perfect matching M_i for $i = 1, 2, ..., 4$. Let $G_2^2 = M_1 \cup M_2 \cup M_3$ and $G_2^3 = M_4$. Let $e_2^i = |E(G_2^i)|$ for $i = 1, 2, 3$. By Lemma 2.2, there is a labeling f_2^1 from $E(G_2^1)$ to $e_1 + \{1, 2, ..., e_2^1\}$ such that for $x \in X$, $f_2^1(x)$ is a constant $2te_1 + (e_2^1 + 1)t$, and for $y \in Y$, $f_2^1(y) \leq 2te_1 + (e_2^1 + 2)t$.

For edges of G_2^2, we define a mapping from $E(G_2^2)$ to $e_1 + e_2^1 + \{1, 2, ..., 3n\}$. We use the sum of $e_1 + e_2^1$ and the entries of a row of the matrix A constructed in Lemma 2.3 to label the edges of a matching. The mapping f_2^2 is defined as follows.

For $1 \leq i \leq n$ and $1 \leq j \leq n$, and if $x_i y_j \in M_1$,

$$f_2^2(x_i y_j) = e_1 + e_2^1 + a_{1,i};$$

if $x_i y_j \in M_2$,

$$f_2^2(x_i y_j) = e_1 + e_2^1 + a_{2,i};$$

and if $x_i y_j \in M_3$,

$$f_2^2(x_i y_j) = e_1 + e_2^1 + a_{3,i}.$$

If n is odd, $f_2^2(x_i) = 3(e_1 + e_2^1) + \frac{9n+3}{2}$, $f_2^2(y_i) \leq 3(e_1 + e_2^1) + 9n - 3$ for $1 \leq i \leq n$. If n is even, $f_2^2(x_i) = 3(e_1 + e_2^1) + \frac{9n-6}{2}$, $f_2^2(y_j) \leq 3(e_1 + e_2^1) + 9n - 3$ for $1 \leq i \leq n-1$, $1 \leq j \leq n$. And $f_2^2(x_n) = 3(e_1 + e_2^1) + 9n - 3$.

We may order the vertices of Y such that

$$(f_1 + f_2^1 + f_2^2)(y_1) \leq (f_1 + f_2^1 + f_2^2)(y_2) \leq \cdots \leq (f_1 + f_2^1 + f_2^2)(y_n).$$

Now we label the edges of G_2^3 by labels $e_1 + e_2^1 + e_2^2 + \{1, 2, ..., n\}$. Define the mapping f_2^3 by $f_2^3(y_i x_{j_i}) = e_1 + e_2 - n + i$ for $1 \leq i \leq n$, where $\{j_1, j_2, ..., j_n\} = \{1, 2, ..., n\}$.

Let $f_2(E(G_2)) = (f_2^1 \cup f_2^2 \cup f_2^3)(E(G_2))$, then

$$\begin{aligned}(f_1 \cup f_2)(y_1) &< (f_1 \cup f_2)(y_2) < \cdots < (f_1 \cup f_2)(y_n) \leq 2r(e_1 + 2) + e_1 \\ &\quad + 2te_1 + (e_2^1 + 2)t + 3(e_1 + e_2^1) + 9n - 3 + e_1 + e_2.\end{aligned}$$

For n is odd, we have

$$\begin{aligned}\{f_2(x_1), f_2(x_2), ..., f_2(x_n)\} &= 2te_1 + (e_2^1 + 1)t + 3(e_1 + e_2^1) + \tfrac{9n+3}{2} \\ &\quad + e_1 + e_2^1 + e_2^2 + \{1, 2, ..., n\}.\end{aligned}$$

For n is even,let $f_2^3(x_n) = e_1 + e_2^1 + e_2^2 + l$, where $1 \le l \le n$.We have the following

$f_2(x_n) = 2te_1 + (e_2^1 + 1)t + 3(e_1 + e_2^1) + 9n - 3 + e_1 + e_2^1 + e_2^2 + l$,

and

$$\begin{aligned}\{f_2(x_1), f_2(x_2), ..., f_2(x_{n-1})\} &= 2te_1 + (e_2^1 + 1)t + 3(e_1 + e_2^1) + \tfrac{9n-6}{2} \\ &+e_1 + e_2^1 + e_2^2 + \{1, 2, ..., n\} \setminus \{l\}.\end{aligned}$$

Step 3 We label the edges of $2r$-regular graph G_3 by $\{e_1 + e_2 + 1, e_1 + e_2 + 2, ..., e_1 + e_2 + rn\}$, and denote the labeling by f_3.

Since $f_1(x_i) = 0$, $(f_1 + f_2)(x_i) = f_2(x_i)$ for $1 \le i \le n$.

If $m = 2t + 1$, we assume that $f_2(x_i)$ is the vertex labeling of $x_i$for $1 \le i \le n$,by Lemma 2.5,we may find a labeling f_3 of $E(G_3)$ onto $e_1 + e_2 + \{1, 2, ..., rn\}$ such that $\{(f_2 \cup f_3)(x_i) : 1 \le i \le n\} = \{r(2a + rn + 1) + b + i : 1 \le i \le n\}$, where $a = e_1 + e_2$ and $b = 2te_1 + (e_2^1 + 1)t + e_1 + e_2^1$.

If $m = 2t + 4$ and n is odd. Similarly,by Lemma 2.5,we can find a labeling f_3 of $E(G_3)$ onto $e_1 + e_2 + \{1, 2, ..., rn\}$ such that $\{(f_2 \cup f_3)(x_i) : 1 \le i \le n\} = \{r(2a + rn + 1) + b + i : 1 \le i \le n\}$, where $a = e_1 + e_2$ and $b = 2te_1 + (e_2^1 + 1)t + 3(e_1 + e_2^1) + \frac{9n+3}{2} + e_1 + e_2^1 + e_2^2$.

If $m = 2t+4$ and n is even. Let $b = 2te_1 + (e_2^1+1)t + 3(e_1 + e_2^1) + \frac{9n-6}{2} + e_1 + e_2^1 + e_2^2$,we have $f_2(x_n) \ge 2n+b$. By Lemma 2.6, we may find a labeling f_3 of $E(G_3)$ onto $e_1+e_2+\{1, 2, ..., rn\}$ such that $(f_2 \cup f_3)(x_i) \ne (f_2 \cup f_3)(x_j)$ for distinct vertices of X,and $(f_2 \cup f_3)(x_i) \ge b + 1 + r(2a + rn + 1)$ for $1 \le i \le n$,where $a = e_1 + e_2$.

Now we check $f_1 \cup f_2 \cup f_3$ is an antimagic mapping of G.

For $m = 2t + 1$, we have the following.

$$(f_1 \cup f_2 \cup f_3)(y) \quad \le 2r(e_1 + 2) + e_1 + 2te_1 + (e_2^1 + 2)t + e_1 + e_2,$$

and

$$\begin{aligned}(f_1 \cup f_2 \cup f_3)(x) &\ge 2te_1 + (e_2^1 + 1)t + e_1 + e_2^1 + 1 \\ &+2r(e_1 + e_2) + r^2 n + r.\end{aligned}$$

For $m = 2t + 4$, we have the following.

$$\begin{aligned}(f_1 \cup f_2 \cup f_3)(y) &\le 2r(e_1 + 2) + e_1 + 2te_1 + (e_2^1 + 2)t + 3(e_1 + e_2^1) \\ &+9n - 3 + e_1 + e_2,\end{aligned}$$

and

$$\begin{aligned}(f_1 \cup f_2 \cup f_3)(x) \quad & \geq 2te_1 + (e_2^1 + 1)t + 3(e_1 + e_2^1) + (9n-6)/2 \\ & +e_1 + e_2^1 + e_2^2 + 1 + 2r(e_1 + e_2) + r^2 n + r.\end{aligned}$$

Note that $e_1 \leq 2rn + n/2$, $e_2 = mn$, $e_2^1 = 2nt$, $e_2^2 = 3n$ and $2 < m < n$. Clearly, we have $(f_1 \cup f_2 \cup f_3)(y) < (f_1 \cup f_2 \cup f_3)(x)$. And G is antimagic.□

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