Maximum Minimal k-rankings of Cycles

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Abstract

Given a graph G, a function $f : V(G) \to \{1, 2, ..., k\}$ is a k-ranking of G if $f(u) = f(v)$ implies every $u - v$ path contains a vertex w such that $f(w) > f(u)$. A k-ranking is *minimal* if the reduction of any label greater than 1 violates the described ranking property. The *arank* number of a graph, denoted $\psi_r(G)$, is the maximum k such that G has a minimal k-ranking. We establish new properties for minimal rankings and present new results for the arank number of a cycle.

MSC Classification (Primary): 05C78

1 Introduction

A labeling $f : V(G) \to \{1, 2, ..., k\}$ is a *k-ranking* of a graph G if and only if $f(u) = f(v)$ implies that every $u - v$ path contains a vertex w such that $f(w) > f(u)$. A k-ranking f is *minimal* if for all $x \in V(G)$ if $f(v) \leq g(v)$ for all rankings g. A ranking f has a *drop vertex* x if the labeling defined by $g(v) = f(v)$ when $v \neq x$ and $g(x) < f(x)$ is still a ranking. It was shown by Jamison [10] that a ranking is minimal if and only if it contains no drop vertices. As a result for any ranking f there exists a minimal k-ranking h such that $h(v) \leq f(v)$ for every $v \in V(G)$. When the value of k is unimportant, we will refer to a k-ranking simply as a ranking.

Following along the lines of the chromatic number, the *rank number of a graph* $\chi_r(G)$ can be defined to be the smallest k such that G has a minimal

k-ranking. The *arank number of a graph* $\psi_r(G)$ is defined to be the largest k such that G has a minimal k-ranking.

Early studies involving the rank number of a graph were sparked by its numerous applications including designs for very large scale integration (VLSI) layouts, Cholesky factorizations and scheduling of manufacturing systems [5], [12], and [17]. Numerous papers have since followed [1], [2]-[4], [7]-[11], and [13]-[16]. One of the first results involving minimal rankings was by Bodlaender et al. where they determined the rank number of a path $\chi_r(P_n) = \lfloor \log_2 n \rfloor + 1$. A ranking of this form can be obtained by labeling the vertices $\{v_i \mid 1 \leq i \leq n\}$ with $\alpha + 1$ where 2^α is the highest power of 2 dividing i and this ranking is unique when n is a power of 2 [1]. We will refer to this ranking as the *standard ranking of a path.* Ghoshal, Laskar, and Pillone were the first to investigate minimal rankings from a mathematical standpoint. They obtained precise rank numbers for many classes of graphs and also investigated the problem's complexity and extremal properties [7], [8], [13], and [14]. The rank number of a cycle was determined by Novotny, Ortiz, and Narayan [15]. Hseih studied minimal k-rankings for starlike graphs [9].

The determination of the arank number of a path was first studied by Ghoshal, Laskar, and Pillone [8] and later by Kostyuk, Narayan, and Williams [11]. The arank number of a cycles was first investigated by Fisher, Kostyuk, and Narayan [6] where $\psi_r(C_n)$ was determined for small values of n. Recent works have included the presentation of new methods for determining $\psi_r(G)$ as well as investigations of the extremal and complexity properties of the arank number [8], [10], [11], [13], and [14]. In particular Laskar and Pillone showed that the determination of $\psi_r(G)$ for an arbitrary graph is NP-complete [14].

A motivation for studying the arank number is that it gives a necessary condition for deciding if a given ranking is minimal. That is if a ranking contains a label greater than $\psi_r(G)$ it clearly can not be minimal. Furthermore the determination of $\psi_r(G)$ for various families of graphs may serve to refine algorithms for computing $\chi_r(G)$, since $\chi_r(G) \leq \psi_r(G)$. However despite numerous works, precise arank numbers are only known for a few families of graphs, including split graphs, stars [7], and paths [11].

We next restate a result of Ghoshal, Laskar, Pillone, and Jamison involving the arank number of a graph [7] and [10].

Lemma 1 *Let H be an induced subgraph of a graph G. Then* $\psi_r(H) \leq \psi_r(G)$.

Even for relatively simple families of graphs, the computation of the arank number can be highly nontrivial. Kostyuk, Narayan, and Williams determined the arank number of a path [11] which is restated below.

Theorem 2 *Let P_n be a path on n vertices. Then* $\psi_r(P_n) = \lfloor \log_2 (n+1) \rfloor + \lfloor \log_2 (n+1-(2^{\lfloor \log_2 n \rfloor - 1})) \rfloor$.

In this work we investigate the arank number of a cycle on n vertices C_n. Despite many similarities between paths and cycles, the methods used to calculate $\psi_r(P_n)$ will not apply to the calculation of $\psi_r(C_n)$. The monotonicity property $\psi_r(P_m) \leq \psi_r(P_n)$ when $m < n$, was instrumental in the proof of Theorem 2 and followed directly from Lemma 1. However since a small cycle is not an induced subgraph of a larger cycle a different approach must be used to establish a similar monotonicity property for cycles. However we prove that $\psi_r(C_m) \leq \psi_r(C_n)$ whenever $m < n$ in Theorem 10. Ironically despite the different methods that must be employed we will show the arank numbers of paths and cycles of order n agree in most cases, and differ by at most 1.

In addition to presenting new results involving arank numbers for cycles, we present new results for minimal rankings in general. One of our main results is that in any minimal ranking of a cycle, at least half of the labels must be labeled 1 or 2. This gives an immediate necessary condition for deciding whether or not a given ranking of a cycle is minimal.

2 Background

We use P_n to denote the path $v_1, v_2, ..., v_n$ and $\langle f(v_1), f(v_2), ..., f(v_n) \rangle$ to explicitly describe the labels in a ranking f. For a given ranking, S_i will represent the independent set of all vertices labeled i.

Definition 3 *For a graph G and a set $S \subseteq V(G)$ the **reduction** of G denoted $G_S^\flat$ is a subgraph of G induced by $V - S$ with an extra edge uv in $E(G_S^\flat)$ if there exists a $u - v$ path in G with all internal vertices belonging to S.*

Unless otherwise stated the set S will consist of vertices labeled 1. For a ranking f of a graph G, $f^\flat_{|G_S^\flat}$ will represent the ranking of $G_S^\flat$ where $f^\flat_{|G_S^\flat}(v) = f(v) - 1$ for all $v \in V(G)$ with $f(v) > 1$. Following the work of Ghoshal, Laskar, and Pillone [14] we define the *reduction of the reduction* of G. Using our notation we denote the reduction of $G_S^\flat$ as $(G_S^\flat)_S^\flat$.

We next restate a result from [7] involving the reduction of a minimal ranking.

Lemma 4 *Let G be a graph and let f be a minimal $\psi_r(G)$-ranking of G. Then $f^\flat_{|G^\flat_S}$ is a minimal $\psi_r(G^\flat_S)$-ranking of $G^\flat_S$.*

For example if $\langle 5,1,2,3,2,1,4\rangle$ is a minimal ranking of P_7 and S is the set of vertices labeled 1, then $\langle 4,1,2,1,3\rangle$ is minimal ranking of $P_7^\flat$.

Definition 5 *Given a graph G, an **expansion** of G is a graph $G^\#$such that $(G^\#)^\flat_S = G$.*

The reduction of a graph G is unique, but a graph may have many expansions. For example, the reduction of $\langle 1,2,3,2,1\rangle$ is $\langle 1,2,1\rangle$, but $\langle 1,2,1\rangle$ can be expanded to any of the following: $\langle 1,2,1,3,1,2\rangle$, $\langle 2,1,3,1,2,1\rangle$, or $\langle 1,2,1,3,1,2,1\rangle$.

Lemma 6 *Let G be a graph and let f be a minimal ranking of G. If $v \in V(G)$ and $f(v) = 2$ then there exists a vertex u adjacent to v such that $f(u) = 1$.*

It is not difficult to show that if P' is an induced subpath of a path P, then $\psi_r(P') \leq \psi_r(P)$. We restate a lemma from [7] and [10] which shows that this monotonicity property holds in general.

3 The arank number of a path

The arank number of a path $\psi_r(P_n)$ has been determined for all values of n [11]. We review some of their results here, as they will be relevant for our calculations for $\psi_r(C_n)$. We give $\psi_r(P_n)$ and $\psi_r(C_n)$ for small values of n in Table 1.

A recursive construction was given in [14] for creating a minimal $(2m-1)$-ranking of $P_{2^m-1}$and a minimal $(2m-2)$-ranking of $P_{2^m-2^{m-2}-1}$. It was later shown that this construction in fact gives ψ_r-rankings [11].

The case $m = 1$ is trivial and when $m = 2$, a minimal 3-ranking of a P_3 can be constructed simply by labeling the vertices $\langle 3,1,2\rangle$. Starting with a k-ranking of a path on w vertices, first delete the two end vertices. We next join two copies of the resulting path with a P_3 with labels, $\langle k-1,k,k-1\rangle$.

Finally add one vertex to each end of the path and label one of these vertices $k+1$ and the other $k+2$. There are actually two families of paths: one where the number of vertices if $2^n - 1$ and the other $\frac{2^n+2^{n-1}}{2} - 1$. An example showing the construction of a minimal 5-ranking of P_7 is shown in Figure 1 (a). An example showing the construction of a minimal 6-ranking of P_{11} is shown in Figure 1 (b).

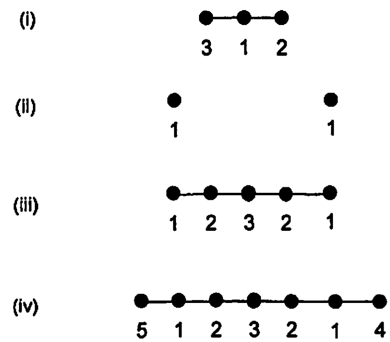

Figure 1 (a). Construction of a minimal 5-ranking from a minimal 3-ranking.

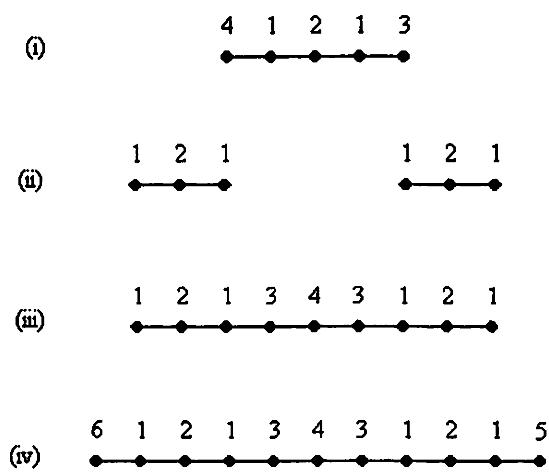

Figure 1. (b). Construction of a minimal 6-ranking from a minimal 4-ranking.

It was conjectured in [14] that the rankings produced by this construction were in fact ψ_r-rankings. The construction from [14] was proven in [11] to give ψ_r-rankings for P_n where $n = 2^m - 2^{m-2} - 1$ and $n = 2^m - 1$. The monotonicity property from Lemma 1 can then be used to determine $\psi_r(P_n)$ for all other values of n. We restate the following result established in [11].

Theorem 7 **(The** ***arank*** **number of** P_n**)**

(i) $\psi_r(P_s) = 2m - 2$ for all integers $s \geq 2$, $2^m - 2^{m-2} - 1 \leq s \leq 2^m - 2$.

(ii) $\psi_r(P_t) = 2m - 1$ for all integers $t \geq 2$, $2^m - 1 \leq t \leq 2^{m+1} - 2^{m-1} - 2$.

These formulas can be simplified to form the formula given in Theorem 2, $\psi_r(P_n) = \lfloor \log_2 (n+1) \rfloor + \lfloor \log_2 (n + 1 - (2^{\lfloor \log_2 n \rfloor - 1})) \rfloor$.

4 Minimal k-rankings of cycles

We continue with a lemma that will be used for constructing minimal k-rankings for cycles. It simply says that if we have two vertices in a graph with distinct labels each greater than 1, then we may insert a vertex labeled 1 in between them, and the ranking property is preserved.

Lemma 8 *Let f be a minimal k-ranking of G with adjacent vertices u and v where $f(u) > 1$, $f(v) > 1$, and $f(u) \neq f(v)$. Let $G^{\#}$ be the graph created by subdividing (u, v) and inserting a vertex w between u and v. Then let the ranking $f^{\#}$ of $G^{\#}$ be defined so that $f^{\#}(w) = 1$ and $f^{\#}(x) = f(x)$ for all $x \neq w$. Then $f^{\#}$ is a minimal k-ranking of $G^{\#}$.*

Proof. We first note that $f^{\#}$ is a ranking since inserting a vertex labeled 1 next to vertices with labels larger than 1, will not violate the ranking property. The ranking is minimal since the label of 1 can not be reduced, and all other labels were part of the minimal ranking f. ■

Lemma 9 *Let f be a minimal k-ranking of G. A graph G' is created by subdividing edges of G and adding a set of vertices S that dominates G'. Then the labeling f' where $f'(x) = f(x) + 1$ for all $x \in V(G)$ and $f'(x) = 1$ for all $x \notin V(G)$ is a minimal $(k + 1)$-ranking of G'.*

Proof. None of the labels equal to 1 may be reduced. If label $f'(u)$ can be reduced then the label $f(u)$ could have been reduced, which would contradict the minimality of f'. ∎

4.1 A monotonicity property for the arank number of cycles

One of the main tools used in determining $\psi_r(P_n)$ was the monotonicity property $\psi_r(P_m) \leq \psi_r(P_n)$ [8] and [11]. This follows from Lemma 1 and the fact that P_m is an induced subgraph of P_n whenever $m \leq n$. Of course the same approach cannot be used for cycles since C_m is not an induced subgraph of C_n for any $m < n$. However we will show in Theorem 10 that the monotonicity property $\psi_r(C_m) \leq \psi_r(C_n)$ holds whenever $m \leq n$.

To establish this result we will start with a (minimal) $\psi_r = k$ ranking of a cycle on m vertices and use it to create a minimal $k' \geq k$ ranking on $m+1$ vertices. Since a ψ_r-ranking has the maximum label over all minimal rankings, it will follow that $\psi_r(C_m) \leq \psi_r(C_{m+1})$.

Theorem 10 *Let $m \leq n$. Then $\psi_r(C_m) \leq \psi_r(C_n)$.*

Proof. We will show that $\psi_r(C_m) \leq \psi_r(C_{m+1})$ for all $m \geq 3$. Suppose the prime factorization of m is $2^j t$ where j is the number of factors of 2 in m. There are three cases to consider: (i) $j = 0$, (ii) $j > 0$ and $t = 1$, and $j > 0$ and $t > 1$.

Case (i) ($j = 0$). Since n is odd there must be two adjacent vertices with both labels greater than 1. We can apply Lemma 8 to extend a minimal k-ranking of C_n to a minimal k-ranking for C_{n+1}. Clearly $\left(G^{\#}\right)_S^{\flat} = G$. The expansion results in a minimal ranking of the supergraph $G^{\#}$.

Case (ii) ($j > 0$ and $t = 1$). We start with a minimal k-ranking of $C_{2^j t}$. We may assume that there is no pair of adjacent vertices with labels larger than 1, or else we could apply Lemma 8 to extend the cycle length by 1. Starting with G we perform up to j reductions to obtain the graph G'. If at any stage there are two adjacent vertices both with labels larger than 1 we apply Lemma 8. Note that after j reductions we will have a minimal $(k-j)$-ranking of G' and G' will contain an odd number t of vertices. Hence it must be the case that after some j' reductions where $0 < j' \leq j$ the reduced graph G' must contain a pair of adjacent vertices with labels larger than 1.

Earlier we defined the term expansion, which is a graph $G^{\#}$ whose reduction is G. We will consider a particular expansion where $V(G^{\#})$

consists of the vertices of G along with another set of vertices S which is a vertex dominating set of G. Starting with the graph G'_0 we perform j'expansions of the graph G'_0 obtaining graphs G'_i, for $1 \le i \le j'$. At each stage $1 \le i \le j' - 1$ we take G'_i and add a set of $|V(G'_i) - 1|$ vertices each labeled 1 that dominates the expanded graph G'_{i+1}.

This yields a minimal k-ranking of $C_{2^j t+1}$. An example with $j' = 1$ is illustrated in Figure 2.

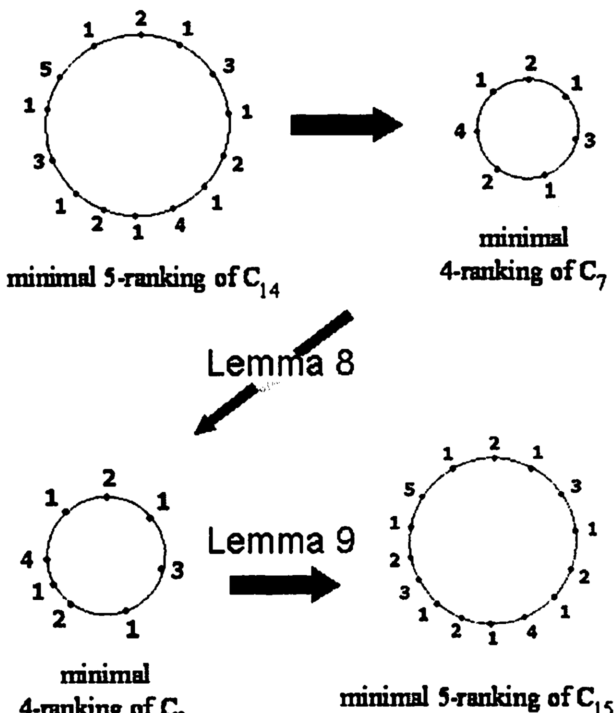

Figure 2. Extenstion from a minimal 5-ranking of C_{14} to a minimal 5-ranking of C_{15}.

Case (iii) ($j > 0$ and $t > 1$). If G or some reduction of G has two adjacent vertices with labels greater than 1, then we revert back to cases (i) or (ii). Next we consider case (ii), where the cycle length equals 2^j. If some reduction of G has a pair of vertices each with labels larger than 1, then we can resort to one of our two previous cases. If no reduction of G has a pair of vertices with labels larger than 1, then G must have the form where there are vertices labeled 1 in every other position along the cycle and after additional reduction(s) the ranking has labels equal to 1 in every other position. This ranking corresponds to the standard ranking of a path. We note that when $n = 2^j$ the last vertex has label $j + 1$. Connecting the two end vertices gives a minimal $(j + 1)$-ranking of C_{2^j}.

In our previous cases we inserted a new vertex into a ranking of a cycle to create a larger cycle and a corresponding ranking. In this case we replace this minimal k-ranking with a new minimal ranking from scratch that has a larger k. Here we use the construction described in Figure 1 to build a minimal ψ_r-ranking of C_{2^j-1}. We then insert a with label 1, resulting in a minimal ψ_r-ranking of C_{2^j} with the same largest label. This ranking contains two adjacent vertices both labeled with integers greater than 1, so we may insert a vertex labeled 1 between them to obtain a minimal k-ranking of C_{2^j} where $k = \psi_r(P_n) \geq \chi_r(P_n)$.

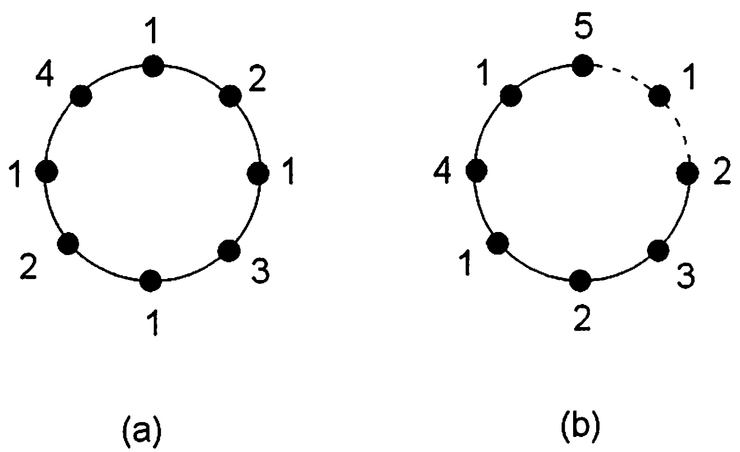

Figure 3. Replacing a minimal χ_r-ranking of C_{2^j} with a minimal ψ_r-ranking of C_{2^j}

■

We note that as part of the above theorem we established that $\psi_r(C_{m+1}) - \psi_r(C_m) \leq 1$.

4.2 Frequency and location of vertices with the same label

We next restate a result from [17] involving the maximum distance between two vertices with the same label.

Lemma 11 *If f is a minimal ranking of P_n then any subpath of order 2^{m+1} has a vertex v such that $f(v) = m$.*

Proof. The proof is by induction on m. The case where $m = 1$ was shown in [14]. The inductive step follows using reduction. ∎

The case of $m = 1$ states that the maximum distance between two vertices labeled 1 is 4, and the case of $m = 2$ states that the maximum distance between two vertices labeled 2 is 8, etc.

We will use S_i to denote the set of vertices labeled i in a ranking. We next recall a Lemma from [8].

Lemma 12 *In any minimal k-ranking $|S_1| \geq |S_2| \geq \cdots \geq |S_k|$.*

We next show that at least half of the labels in a minimal ranking of a cycle are either 1 or 2.

Theorem 13 *For any minimal ranking of C_n, $|S_1 \cup S_2| \geq \frac{n}{2}$.*

Proof. Let $V(C_n) = \{v_1, v_2, ..., v_n\}$ and $E(C_n) = \{v_1v_2, v_2v_3, ..., v_nv_1\}$. We use the vertices labeled 2 to partition the vertices of C_n into parts $F_1, F_2, ..., F_M$ in the following manner. Each vertex labeled 2 is the last vertex in some part F_i, $1 \leq i \leq M - 1$. The final part F_M consists of the remaining vertices. This partition is illustrated in Figure 4.

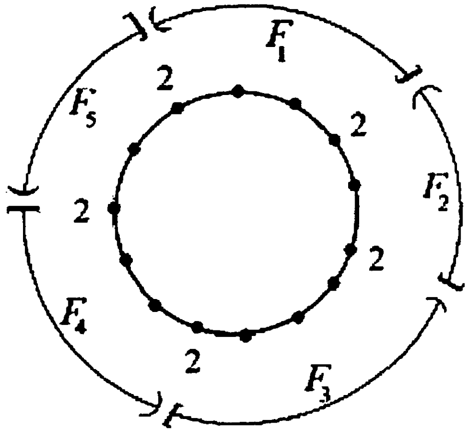

Figure 4. An example of a partition of C_n with $M = 5$.

By Lemma 11, $|V(F_i)| \leq 8$ for all $i = 2, 3, ..., M$. We consider different cases depending on the cardinality of the segment F_i. For completeness we include the details.

- Case (i) $|F_i| = 2$. Since F_i contains two vertices one of which is labeled 2, the result is immediate.

- Case (ii) $|F_i| = 3$ or 4. By Lemma 11 F_i must contain at least one vertex labeled 1. The last vertex in the segment has label 2. Hence $|V(F_i) \cap (S_1 \cup S_2)| \geq \frac{|V(F_i)|}{2}$.

- Case (iii) $|F_i| = 5$. By Lemma 11 $|F_i \cap S_1| \geq 1$ and the vertex labeled 1 can not be the first or fourth vertex of F_i. Assume without loss of generality, the second vertex is labeled 1. We use a, b, and c to denote the first, third and fourth vertices of F_i respectively. If $f(c) > f(b)$, then $f(b)$ can be set to 2 and f still is a ranking; thus $f(c) < f(b)$, which implies $f(c)$ can only equal 1 if the ranking f is minimal. Hence$|F_i \cap (S_1 \cup S_2)| \geq 3 \geq \frac{|F_i|}{2}$.

- Case (iv) $6 \leq |F_i| \leq 8$. If $|F_i \cap S_1| < |F_i| - 4$ then F_i contains at least four vertices with labels higher than 2. Then $f^b_{|S_1}$ contains labels for four consecutive vertices that are all greater than 1. By

Lemma 11 $f^{\flat}_{|S_1}$ can not be a minimal ranking, a contradiction. Hence $|F_i \cap S_1| \geq |F_i| - 4$ and $|F_i \cap (S_1 \cup S_2)| \geq |V(F_i)| - 3 \geq \frac{|F_i|}{2}$.

Hence $|V(F_i) \cap (S_1 \cup S_2)| \geq \frac{|V(F_i)|}{2}$ for all F_i and $|S_1 \cup S_2| \geq \frac{n}{2}$. ∎

It was shown in [11] that for any minimal k-ranking of a path it holds that $|S_1 \cup S_2| > \frac{n}{2}$. The bound for cycles is almost the same, but here there is the possibility of equality. We give an example of a minimal ranking of C_{10} in the Figure 5 that shows the inequality presented in Theorem 13 is tight.

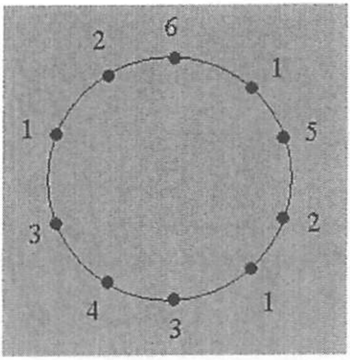

Figure 5. A minimal ranking of C_{10} with $|S_1| + |S_2| = \frac{10}{2}$

5 The arank number of a cycle

We next give results involving the arank number of a cycle. We begin with some small cases.

Lemma 14 *We have the following:* $\psi(C_3) = 3$; $\psi(C_4) = 3$; $\psi(C_5) = 4$; $\psi(C_6) = 4$; $\psi(C_n) = 5$ *for* $7 \leq n \leq 9$, *and* $\psi(C_n) = 6$ *for* $10 \leq n \leq 13$.

Proof. We prove each case separately:

♦ $\psi(C_3) = 3$. A minimal 3-ranking is $1 - 2 - 3$.

♦ $\psi(C_4) = 3$. A minimal 3-ranking can be constructed using the labeling $1 - 2 - 3 - 1$. Note that if distinct labels are used one the vertex that is not adjacent to a vertex labeled 1 can have its label reduced to a 1. Hence any 4-ranking is not minimal.

♦ $\psi(C_5) = 4$. A minimal 4-ranking can be constructed using the labeling $4-1-2-1-3$. Assume $\psi(C_5) \geq 5$. Then each of the labels must have a different label. Hence one of the vertices not adjacent to a vertex labeled 1 can have its label reduced to a 1. This any 5-ranking of C_5 cannot be minimal.

♦ $\psi(C_6) = 4$. The labeling $4-1-2-1-3-1$ of C_6 is a minimal 4-ranking. Assume $\psi(C_6) \geq 5$. Then by Lemma 12 we must have $|S_1| = 2$, and $|S_i| = 1$ for all $2 \leq i \leq 5$. Since the labels $2, 3, 4$, and 5 are only used once the vertex that is not adjacent to a vertex labeled 1, can have its label reduced to a 1. Hence any 5-ranking of C_6 cannot be minimal.

♦ $\psi(C_7) = 5$. The labeling $5-1-2-3-2-1-4$ of C_7 is a minimal 5-ranking. Suppose $\psi_r(C_7) \geq 6$. Then reducing twice would result in a minimal k-ranking of C_t where $k \geq 4$ and $t \leq 3$ which is clearly impossible.

♦ $\psi(C_8) = \psi(C_9) = 5$. We first show that $\psi(C_9) = 5$. The labeling $5-1-2-3-2-1-4-1-2$ of C_9 is a minimal 5-ranking. Suppose $\psi_r(C_9) \geq 6$. Then reducing twice would result in a minimal k-ranking of C_t where $k \geq 4$ and $t \leq 4$ which is clearly impossible. We can then apply Lemma 10 to conclude that $\psi(C_8) = 5$.

♦ $\psi(C_{10}) = 6$. A minimal 6-ranking of C_{10} is given in Figure 5. If $\psi_r(C_{10}) \geq 7$ then reducing twice would give us a minimal k-ranking of C_t where $k \geq 5$ and $t \leq 5$ which is impossible.

♦ $\psi(C_{11}) = 6$. We can obtain a minimal 6-ranking of C_{11} by joining the first and last vertices of P_{11} with its ψ-ranking: $6-1-2-1-3-4-3-1-2-1-5$. Suppose $\psi_r(C_{11}) \geq 7$. Then reducing twice would result in a minimal k-ranking of C_t where $k \geq 5$ and $t \leq 5$ which is clearly impossible.

♦ $\psi(C_{12}) = \psi(C_{13}) = 6$. By Lemma 10 we have $\psi(C_{13}) \geq \psi(C_{12}) \geq \psi(C_{11}) = 6$. Assume $\psi(C_{13}) \geq 7$. Then there would exist a minimal k-ranking of C_{13} with $k \geq 7$. Reducing twice would result in a minimal k-ranking of C_t where $k \geq 5$ and $t \leq 6$. However this is a contradiction since we have $\psi(C_7) = 5$ and Lemma 10. Hence $\psi(C_{13}) \leq 6$ and finally $6 \leq \psi(C_{12}) \leq \psi(C_{13}) \leq 6$ gives the desired result. ■

The combination of Lemmas 19-24 will give precise values for most values of n, and differs by one only in the following two cases: (i) when n is slightly less than a power of 2 and (ii) when n is slightly less than the average of two consecutive powers of 2.

We next present a series of three 'wrapping lemmas' which are used to convert minimal rankings of paths to minimal rankings of cycles, by joining the end vertices. Each of the wrapping lemmas will use ψ_r-ranking of a path to give a lower bound for the arank number of a cycle.

5.1 Lower bounds and the wrapping lemmas

Lemma 15 (Wrapping Lemma #1) *Let* $n = 2^m - 1$. *Then* $\psi_r(C_n) \geq 2m - 1$.

Proof. We can use the construction described in Figure 1(a) to build a minimal k-ranking of P_{2^m-1} that has the form shown in the Figure 6. We can then connect the two end vertices to form a labeling of a cycle. We note that this is a ranking since the labeling of the path was a ranking and the two largest labels are on the end vertices. To see that this ranking is minimal note that if the ranking of the cycle were to contain a drop vertex v then v would have been a drop vertex of the path.

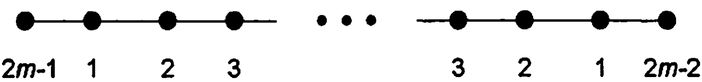

Figure 6. Constructing a $(2m-1)$-ranking of C_{2^m-1}

The two end vertices in the graph in Figure 6 can then joined. The labeling forms a minimal $(2m-2)$-ranking of C_n. ∎

Lemma 16 (Wrapping Lemma #2) *Let* $n = 2^m - 2^{m-2} - 1$. *Then* $\psi_r(C_n) \geq 2m - 2$.

Proof. We can use the construction described in Figure 1 (b) to build the minimal k-ranking of $P_{2^m-2^{m-2}-1}$ shown in Figure 7. We can then connect the two end vertices to form a labeling of a cycle. We note that this is a ranking since the labeling of the path was a ranking and the two largest labels are on the end vertices. To see that this ranking is minimal note that if the ranking of the cycle were to contain a drop vertex v then v would have been a drop vertex of the path.

Figure 7. Constructing a $(2m-2)$-ranking of $C_{2^m-2^{m-2}-1}$

The two end vertices in the graph in Figure 7 can then joined. The labeling forms a minimal $(2m-2)$-ranking of C_n. ∎

Lemma 17 (Wrapping Lemma #3) *Let* $n = 2^m - 2^{m-2} - 2$. *Then* $\psi_r(C_n) \geq 2m - 2$.

Proof. In Figure 6 we have a minimal k-ranking of $P_{2^m-2^{m-2}-1}$. For the sake of clarity this figure is shown again in Figure 8 (a). We can then rearrange the labels on the right side to form the labeling shown in Figure 8 (b). This is done by first interchanging the order of the first two vertices appearing on the left, and then removing the second vertex from the right, and finally adding an edge between the vertex on the right end and its neighbor. As a result, we have one less vertex and have a ranking of $P_{2^m-2^{m-2}-2}$ where the only vertex that can be reduced and still maintain the ranking property is the end vertex labeled $2m-3$. Then the endpoints of the path may be joined to form a minimal $(2m-2)$-ranking of $C_{2^m-2^{m-2}-2}$.

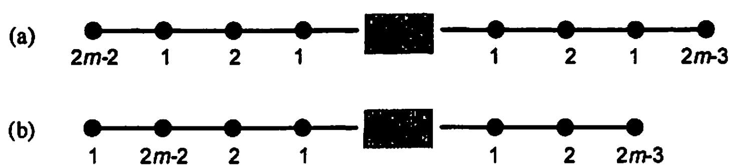

Figure 8. Constructing a $(2m-2)$-ranking of $C_{2^m-2^{m-2}-2}$

The two end vertices in the graph in Figure 7 (b) can then joined. The labeling forms a minimal $(2m-2)$-ranking of C_n. ∎

6 Arank numbers for larger cycles

The results of this section give the arank number of a cycle for certain values of n and determine the arank number within 1 for remaining values of n. Most of the proofs follow by induction and tie together two of the main ideas presented in earlier sections. The first is the use of the reduction operation. The second is Theorem 13 which states that at least half of the labels in a minimal ranking of a cycle are labeled either 1 or 2. The combination of these two tools shows that applying the reduction operation twice reduces the size of the cycle by approximately one half, and largest label in the ranking drops by 2.

Before we get to the main lemmas of this section, we give an example that uses the techniques described above.

Example 18 *We show that* $\psi_r(C_{22}) = 8$. *By Lemma 17 we have* $\psi_r(C_{22}) \geq 8$. *Suppose* $\psi_r(C_{22}) \geq 9$. *Then we have a minimal k-ranking of* C_{22} *where*

$k \geq 9$. *Since at least half of the vertices are labeled* 1 *or* 2 *by Theorem 13, reducing this ranking twice results in a k-ranking of* C_t *where* $k \geq 7$ *and* $t \leq 11$. *However this is a contradiction since Lemma 14 and Theorem 10 imply* $\psi_r(C_{11}) = 6$ *and* $\psi_r(C_t) \leq 6$.

Lemma 19 *Let* $m \geq 4$. *If* $2^m + 2^{m-2} \leq n \leq 2^m + 2^{m-1} - 3$, *then* $\psi_r(C_n) = 2m - 1$ *or* $2m$.

Proof. We proceed by induction on m. For the base case we show that $7 \leq \psi_r(C_{20})$ and $\psi_r(C_{21}) \leq 8$. By Lemma 10 we have $\psi_r(C_{20}) \leq \psi_r(C_{21})$. For the first lower we note that $7 = \psi_r(P_{19}) \leq \psi_r(C_{20})$. To show $\psi_r(C_{21}) \leq 8$, we assume that $\psi_r(C_{21}) \geq 9$ and seek a contradiction. Reducing twice would result in a minimal k-ranking of C_t where $k \geq 7$ and $t \leq 10$. This is impossible since $\psi_r(C_{10}) = 6$ as shown in Lemma 14. Next we let $m \geq 5$ and assume our hypothesis is true for m. Let $2^m + 2^{m-2} \leq n \leq 2^m + 2^{m-1} - 3$. Then $\psi_r(C_n) \geq \psi_r(P_{n-1}) = 2m - 1$. For the upper bound, we assume $\psi_r(C_{2^m+2^{m-1}-3}) \geq 2m + 1$ and seek a contradiction. Then there exists a minimal k-ranking f of $C_{2^m+2^{m-1}-3}$ for some $k \geq 2m + 1$. Then the ranking $\left(f_S^b\right)_S^b$ is a minimal k-ranking of C_t where $k \geq 2m - 1$ and $t < \frac{1}{2}\left(2^m + 2^{m-1} - 3\right)$, which is impossible, since $\psi_r(C_{2^{m-1}+2^{m-2}-2})$ $= 2m - 2$. ■

Lemma 20 *Let* $m \geq 4$ *and* $n = 2^m - 2^{m-2} - 2$ *or* $2^m - 2^{m-2} - 1$. *Then* $\psi_r(C_n) = 2m - 2$.

Proof. We proceed by induction on m. We first establish the base case. By Lemma 14 we have $\psi_r(C_{10}) = \psi_r(C_{11}) = 6$. Then let $m \geq 5$. Assume our hypothesis holds for m. By Lemmas 16 and 17 we have $\psi_r(C_n) \geq 2m - 2$. For the upper bound we assume that $\psi_r(C_n) \geq 2m - 1$ and seek a contradiction. Then there exists a minimal k-ranking of C_n where $k \geq 2m - 1$. Then reducing twice gives a minimal k-ranking of C_t where $k \geq 2m - 3 = 2(m-1) - 1$ and $t \leq \left\lfloor \frac{1}{2}\left(2^m - 2^{m-2} - 1\right)\right\rfloor \leq 2^{m-1} - 2^{m-3} - 1$. ■

Lemma 21 *Let* $m \geq 4$. *If* $2^m - 2^{m-3} \leq n \leq 2^m - 2$, *then* $\psi_r(C_n) = 2m - 2$ *or* $2m - 1$.

Proof. For the base case note that $\psi_r(C_{14}) \geq \psi_r(P_{13}) = 6$. If $\psi_r(C_{14}) \geq 8$ then reducing twice would give us a minimal k-ranking of C_t where $k \geq 6$ and $t \leq 7$ which is impossible since we know $\psi_r(C_7) = 5$

by Lemma 14 .We next show that $\psi_r(C_n) < 2m$ using induction. Assume $\psi_r(C_{2^m-2}) \geq 2m$. Then there exists a minimal k-ranking f where $k \geq 2m$. Then $(f_S^b)_S^b$ is a minimal k-ranking of C_t, $k \geq 2m-2$ and $t \leq 2^{m-1}-1$. This is impossible since $\psi_r(C_{2^{m-1}-1}) = 2m-3$.

Next we establish the lower bound. By Lemma 2 we have that $\psi_r(P_{2^m-3}) = 2m-2$. Since C_{2^m-2} contains P_{2^m-3} as an induced subgraph, it follows by Lemma 1 that $\psi_r(C_{2^m-2}) \geq \psi_r(P_{2^m-2^{m-3}}) = 2m-2$. ■

Lemma 22 *Let* $m \geq 4$ *and* $2^m - 2^{m-2} - 2 \leq n \leq 2^m - 2^{m-3} - 1$. *Then* $\psi_r(C_n) = 2m-2$.

Proof. We showed that $\psi_r(C_{2^m-2^{m-2}-2}) = 2m-2$ in Lemma 20. We next show that $\psi_r(C_{2^m-2^{m-3}-1}) = 2m-2$. Assume that $\psi_r(C_{2^m-2^{m-3}-1}) \geq 2m-1$. Then there exists a minimal k-ranking f of $C_{2^m-2^{m-3}-1}$ for some $k \geq 2m-1$. Then the ranking $(f_S^b)_S^b$ is a minimal k-ranking of C_t where $k \geq 2m-3$ and $t \leq 2^{m-1} - 2^{m-4} - 1$, which is impossible, since $\psi_r(C_{2^{m-1}-2^{m-4}-1}) = 2m-4$. Finally, application of Theorem 10 gives the arank numbers for all intermediate values. ■

Lemma 23 *If* $n = 2^m - 1$, *then* $\psi_r(C_n) = 2m-1$.

Proof. The proof is by induction on m starting with $m = 3$. The base case follows from Lemma 14. Let $n = 2^m - 1$. Next we show the upper bound $\psi_r(C_n) \leq 2m-1$. Assume $\psi_r(C_n) \geq 2m$. Then there exists a minimal k-ranking f of C_n for some $k \geq 2m$. Then the ranking $(f_S^b)_S^b$ is a minimal k-ranking of C_t where $k \geq 2m-2$ and $t \leq 2^{m-1} - \frac{1}{2}$. But since t must be an integer we have $t \leq 2^{m-1} - 1$. This is impossible since $\psi_r(C_{2^{m-1}-1}) = 2m-3$. ■

Lemma 24 *Let* $m \geq 4$. *If* $2^m - 1 \leq n \leq 2^m + 2^{m-2} - 1$, *then* $\psi_r(C_n) = 2m-1$.

Proof. The proof is by induction on m starting with $m = 4$. By Lemma 23, we have $\psi_r(C_{2^m-1}) = 2m-1$. Then we will show that $\psi_r(C_{2^m+2^{m-2}-1}) = 2m-1$. Let $n = 2^m + 2^{m-2} - 1$. Assume $\psi_r(C_n) \geq 2m$. Then there exists a minimal k-ranking f of C_n for some $k \geq 2m$. Then the ranking $(f_S^b)_S^b$ is a minimal k-ranking of C_t where $k \geq 2m$ and $t \leq 2^{m-1} + 2^{m-3} - 1$, which is impossible since $\psi_r(C_{2^{m-1}+2^{m-3}-1}) = 2m-3$. Finally, application of Theorem 10 gives the arank numbers for all intermediate values. ■

A comparison of the results in this paper involving the arank number of a cycle and the known results involving the arank number of a path show that the arank numbers for the two families are identical in some cases and differ by at most one in the remaining cases. We formally state their close relationship in the following corollary.

Corollary 25 *For any $n \geq 3$ we have $\psi_r(P_n) = \psi_r(C_n)$ or $\psi_r(C_n) \leq \psi_r(P_n) + 1$.*

7 Conclusion

It would be an interesting problem to close the bounds given in Lemmas 19 and 21. Our conjecture is that the lower bounds hold.

We conclude by posing the following problem.

Problem 26 *Characterize minimal rankings of paths and cycles.*

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