ON γ-LABELINGS OF COMPLETE BIPARTITE GRAPHS

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ABSTRACT. Let G be a connected graph with p vertices and q edges. A γ-labeling of G is a one-to-one function f from $V(G)$ to $\{0, 1, \ldots, q\}$ that induces a labeling f' from $E(G)$ to $\{1, 2, \ldots, q\}$ defined by $f'(e) = |f(u) - f(v)|$ for each edge $e = uv$ of G. The value of a γ-labeling f is defined to be the sum of the values of f' over all edges. Also, the maximum value of a γ-labeling of G is defined as the maximum of the values among all γ-labelings of G, while the minimum value is the minimum of the values among all γ-labelings of G. In this paper, the maximum value and minimum value are determined for any complete bipartite graph.

1. INTRODUCTION

A *γ-labeling* of a graph is introduced in [1]. For a graph G with p vertices and q edges, a γ-labeling f is a one-to-one function $f : V(G) \to \{0, 1, 2, \ldots, q\}$ that induces a labeling $f' : E(G) \to \{1, 2, \ldots, q\}$ defined by $f'(uv) = |f(u) - f(v)|$ for any edge uv of G. Clearly, any connected graph admits a γ-labeling. For such a γ-labeling f, the *value* of f is defined as

$$\text{val}(f) = \sum_{e \in E(G)} f'(e).$$

Furthermore, the *minimum value* of a γ-labeling of G is defined as

$$\text{val}_{\min}(G) = \min\{\text{val}(f) \mid f \text{ is a } \gamma\text{-labeling of } G\}.$$

We have a similar definition for the *maximum value* $\text{val}_{\max}(G)$ of a γ-labeling. A γ-labeling which realizes the minimum (maximum, resp.) value is referred to as a *minimum* (*maximum*, resp.) γ-labeling.

The minimum values and maximum values are determined for paths, stars, cycles, complete graphs in [1], and double stars in [2]. We recall the result for stars here.

Theorem 1.1 ([1]). *For each integer $t \geq 3$,*

$$\text{val}_{\min}(K_{1,t-1}) = \binom{\lfloor \frac{t+1}{2} \rfloor}{2} + \binom{\lceil \frac{t+1}{2} \rceil}{2} \text{ and } \text{val}_{\max}(K_{1,t-1}) = \binom{t}{2}.$$

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The purpose of this paper is to determine the minimum value and maximum value of a γ-labeling for any complete bipartite graph $K_{s,t}$.

Theorem 1.2. *Let $s \le t$.*

(1) *The minimum value of a γ-labeling of $K_{s,t}$ is*

$$\mathrm{val}_{\min}(K_{s,t}) = \frac{s(s+t)(s+t-1)}{2} - \sum_{i=1}^{s} \frac{(4i-s+t-3)^2}{4} + \frac{s}{4}\varepsilon,$$

where $\varepsilon = 0$ if $t-s$ is odd, and $\varepsilon = 1$ otherwise.

(2) *The maximum value of a γ-labeling of $K_{s,t}$ is*

$$\mathrm{val}_{\max}(K_{s,t}) = \sum_{i=1}^{s}\sum_{j=1}^{t}(st+2-i-j).$$

This is a straightforward generalization of Theorem 1.1.

2. Proofs

Lemma 2.1. *Let G be a graph with $|V(G)| = p$ and $|E(G)| = q$. If f is a minimum γ-labeling of G, then the values of f are consecutive.*

Proof. Let f be a minimum γ-labeling of graph G. Suppose that $f(V(G))$ is not consecutive. Then $V(G)$ can be expressed as the union $A \cup B$ such that $A \cap B = \emptyset$ and $f(u) + 1 < f(v)$ for any $u \in A$, $v \in B$. Let us define another γ-labeling g by

$$g(u) = \begin{cases} f(u) & \text{if } u \in A, \\ f(u) - 1 & \text{if } u \in B. \end{cases}$$

Then $\mathrm{val}(g) < \mathrm{val}(f)$, contradicting the minimality of f. □

For a connected graph G with p vertices, let $f : V(G) \to \{0, 1, \dots, p-1\}$ and $g : V(G) \to \{n, n+1, \dots, n+p-1\}$ be γ-labelings of G satisfying $f(v) = g(v) - n$ for any $v \in V(G)$. Then $f'(uv) = g'(uv)$ for any edge $uv \in E(G)$. Hence if f is a minimum γ-labeling of a graph G, then we may assume that $f(V(G)) = \{0, 1, \dots, p-1\}$.

Proof of Theorem 1.2(1). We can color the vertices of $K_{s,t}$ with black and white so that no two adjacent vertices receive the same color. Also, we can assume that the number of black vertices is s, and that of white vertices is t. Let f be a minimum γ-labeling of $K_{s,t}$. From Lemma 2.1, the vertices are labeled with $0, 1, \dots, s+t-1$. Let us denote the labels of black vertices by $a_1, a_2, \dots, a_s$, where $a_i \in \{0, 1, \dots, s+t-1\}$ and $a_i < a_{i+1}$ $(i = 1, 2, \dots, s-1)$. Then, the white vertices are labeled with $\{0, 1, \dots, s+t-1\} - \{a_1, a_2, \dots, a_s\}$

Let $A = \{a_1, a_2, \dots, a_s\}$, $B = \{0, 1, \dots, s+t-1\} - \{a_1, a_2, \dots, a_s\}$.

Then,

$$
\begin{aligned}
\mathrm{val}(f) &= \sum_{a\in A}\sum_{b\in B} \mid a-b \mid \\
&= \sum_{i=1}^{s}\sum_{j=0}^{s+t-1} \mid a_i - j \mid - \sum_{i=1}^{s}\sum_{j=1}^{s} \mid a_i - a_j \mid \\
&= \sum_{i=1}^{s}\left\{\sum_{j=0}^{a_i}(a_i-j) + \sum_{j=a_i+1}^{s+t-1}(j-a_i)\right\} \\
&\quad -\sum_{i=1}^{s}\left\{\sum_{j=1}^{i}(a_i-a_j) + \sum_{j=i+1}^{s}(a_j-a_i)\right\} \\
&= \sum_{i=1}^{s}\left\{\sum_{j=0}^{a_i} j + \sum_{j=1}^{s+t-1-a_i} j\right\} \\
&\quad -\sum_{i=1}^{s}\left\{\sum_{j=1}^{i-1}(a_i-a_j) + \sum_{j=i+1}^{s}(a_j-a_i)\right\} \\
&= \sum_{i=1}^{s}\left\{\frac{a_i(a_i+1)}{2} + \frac{(s+t-a_i)(s+t-1-a_i)}{2}\right\} \\
&\quad -\sum_{i=1}^{s}\left\{(i-1)a_i - \sum_{j=1}^{i-1}a_j + \sum_{j=i+1}^{s}a_j - (s-i)a_i\right\} \\
&= \sum_{i=1}^{s}\left\{\frac{a_i(a_i+1)}{2} + \frac{(s+t-a_i)(s+t-1-a_i)}{2}\right\} \\
&\quad +\sum_{i=1}^{s}\{(s+1-2i)a_i + (a_1+\cdots+a_{i-1}) - (a_{i+1}+\cdots+a_s)\}.
\end{aligned}
$$

We can calculate $\sum_{i=1}^{s}(a_1+\cdots+a_{i-1})$ and $\sum_{i=1}^{s}(a_{i+1}+\cdots+a_s)$ as follows.

$$
\begin{aligned}
\sum_{i=1}^{s}(a_1+\cdots+a_{i-1}) &= (s-1)a_1 + (s-2)a_2 + \cdots + a_{s-1} \\
&= \sum_{i=1}^{s}(s-i)a_i,
\end{aligned}
$$

and

$$\begin{aligned}\sum_{i=1}^{s}(a_{i+1}+\cdots+a_s) &= (s-1)a_s+(s-2)a_{s-1}+\cdots+2a_3+a_2\\ &= \sum_{i=1}^{s}(i-1)a_i.\end{aligned}$$

Hence,

$$\begin{aligned}\mathrm{val}(f) &= \sum_{i=1}^{s}\left\{\frac{a_i(a_i+1)}{2}+\frac{(s+t-a_i)(s+t-1-a_i)}{2}\right.\\ &\quad \left.+(s+1-2i)a_i+(s-i)a_i-(i-1)a_i\right\}\\ &= \sum_{i=1}^{s}\left\{a_i^2+(-4i+s-t+3)a_i+\frac{(s+t)(s+t-1)}{2}\right\}\\ &= \sum_{i=1}^{s}\left(a_i-\frac{4i-s+t-3}{2}\right)^2-\sum_{i=1}^{s}\frac{(4i-s+t-3)^2}{4}\\ &\quad +\frac{s(s+t)(s+t-1)}{2}.\end{aligned}$$

Suppose that $t-s$ is odd. By the minimality of f, $a_i=\frac{4i-s+t-3}{2}$ for any i. Then

$$\mathrm{val}(f)=\frac{s(s+t)(s+t-1)}{2}-\sum_{i=1}^{s}\frac{(4i-s+t-3)^2}{4}.$$

Suppose that $t-s$ is even. Similarly, the minimality of f implies that $a_i=\frac{4i-s+t-4}{2}$ or $\frac{4i-s+t-2}{2}$ for each i. Then

$$\mathrm{val}(f)=\frac{s(s+t)(s+t-1)}{2}-\sum_{i=1}^{s}\frac{(4i-s+t-3)^2}{4}+\frac{s}{4}.$$

This completes the proof of Theorem 1.2(1). □

We introduce a new labeling to prove Theorem 1.2(2).

Fix a positive integer x with $x \geq st$. A *γ'-labeling* f of $G=K_{s,t}$ is a one-to-one function $f: V(G)\rightarrow\{0,1,\ldots,x\}$, that induces a labeling $f': E(G)\rightarrow\{1,2,\ldots,x\}$ of the edges of G defined by $f'(e)=|f(u)-f(v)|$ for each edge $e=uv$ of G. The value of a γ'-labeling f is defined as $\mathrm{val}(f)=\sum_{e\in E(G)}f'(e)$.

Lemma 2.2. *Let $G=K_{s,t}$ be a complete bipartite graph with stable sets X and Y. Let $X=\{u_1,u_2,\ldots,u_s\}$ and $Y=\{v_1,v_2,\ldots,v_t\}$. For any*

γ'*-labeling* f *of* G,

$$\sum_{i=1}^{s}\sum_{j=1}^{t}|f(u_i)-f(v_j)| \le \sum_{i=1}^{s}\sum_{j=1}^{t}(x+2-i-j).$$

Proof. Without loss of generality, we may assume that a vertex of X, u say, is assigned label 0 by f. We prove the lemma by induction on s.

(i) Assume $s=1$.

It is clear that

$$\sum_{j=1}^{t}|f(u)-f(v_j)| = \sum_{j=1}^{t}|f(v_j)| \le \sum_{j=1}^{t}(x+1-j).$$

(ii) Suppose $s=k+1$. We assume the conclusion is true when $s \le k$.

Then the sum of the labels on the edges incident to u is

$$\sum_{j=1}^{t}|f(u)-f(v_j)| \le \sum_{j=1}^{t}(x+1-j)$$

as in (i).

Let $H=G-u$. Then f induces a labeling g of H, which is equal to the labeling of $K_{k,t}$ by the set $\{1,2,\ldots,x\}$.

Let g' be the induced edge labeling of g. Since it is invariant by subtracting one from each label on the vertices, g' is equal to the induced edge labeling when the vertices of $K_{k,t}$ is labeled by the set $\{0,1,\ldots,x-1\}$. Note that $x-1 \ge (k+1)t-1 = kt+(t-1) \ge kt$. By inductive hypothesis,

$$\begin{aligned}\sum_{i=1}^{k}\sum_{j=1}^{t}|f(u_i)-f(v_j)| &\le \sum_{i=1}^{k}\sum_{j=1}^{t}(x+1-i-j)\\ &= \sum_{i=2}^{k+1}\sum_{j=1}^{t}(x+2-i-j).\end{aligned}$$

Hence, when $s=k+1$,

$$\begin{aligned}\sum_{i=1}^{s}\sum_{j=1}^{t}|f(u_i)-f(v_j)| &\le \sum_{j=1}^{t}(x+1-j)+\sum_{i=2}^{k+1}\sum_{j=1}^{t}(x+2-i-j)\\ &= \sum_{i=1}^{k+1}\sum_{j=1}^{t}(x+2-i-j).\end{aligned}$$

□

Proof of Theorem 1.2(2). Let X and Y be the stable sets of $K_{s,t}$ with $|X|=s$ and $|Y|=t$. Consider a labeling f such that $f(X)=\{0,1,\ldots,s-1\}$

and $f(Y) = \{st - t + 1, st - t + 2, \ldots, st\}$. Then,

$$\mathrm{val}(f) = \sum_{i=1}^{s} \sum_{j=1}^{t} (st + 2 - i - j).$$

By Lemma 2.2 with $x = st$, f is a maximum γ-labeling, and so

$$\mathrm{val}_{\max}(K_{s,t}) = \sum_{i=1}^{s} \sum_{j=1}^{t} (st + 2 - i - j).$$

□

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