Total chromatic number of folded hypercubes *

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Abstract

A total coloring of a simple graph G is a coloring of both the edges and the vertices. A total coloring is proper if no two adjacent or incident elements receive the same color. The minimum number of colors required for a proper total coloring of G is called the total chromatic number of G and denoted by $\chi_t(G)$. The Total Coloring Conjecture (TCC) states that for every simple graph G, $\Delta(G)+1 \leq \chi_t(G) \leq \Delta(G)+2$. G is called Type 1 (resp. Type 2) if $\chi_t(G) = \Delta(G)+1$ (resp. $\chi_t(G) = \Delta(G)+2$). In this paper, we prove that the folded hypercubes FQ_n is of Type 1 when $n \geq 4$.

Keywords: Total coloring; Total chromatic number; Folded hypercubes

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1 Introduction

All graphs considered in this paper are finite, simple and undirected. Terminology and notation not defined here are followed [3]. Let G be a graph, we use $V(G)$, $E(G)$ and $\Delta(G)$ (or simply V, E and Δ) to denote the vertex set, the edge set and the maximum degree of G, respectively.

A *k-total coloring* $h : V \cup E \to \{1, 2, \cdots, k\}$ of a graph $G = (V, E)$ is an assignment of k colors to both the edges and the vertices of G. The total coloring h is called *a proper k-total coloring* if no incident or adjacent elements (vertices or edges) receive the same color. The *total chromatic number* of G, $\chi_t(G)$, is the least integer k for which G admits a proper k-total coloring. Behzad [1] and Vizing [11] proposed independently the following famous conjecture, which is known as the *Total Coloring Conjecture* (TCC).

Conjecture 1. For any graph G, $\Delta(G) + 1 \leq \chi_t(G) \leq \Delta(G) + 2$. □

The lower bound of this conjecture is obvious, the upper bound remains to be proved. If G satisfies TCC and $\chi_t(G) = \Delta(G) + 1$ (resp. $\chi_t(G) = \Delta(G) + 2$), then G is of Type 1 (resp. Type 2).

The n-dimensional hypercube Q_n is an undirected graph. Any vertex $x \in V(Q_n)$ is denoted by a 0-1 sequence $x_1 x_2 \cdots x_n$ of length n. Hence, there are 2^n vertices in Q_n. Two vertices $x, y \in V(Q_n)$ are joined by an edge if and only if x and y differ at exactly one position. If $x = x_1 \cdots x_i \cdots x_n$, denote the vertex $x_1 \cdots \overline{x}_i \cdots x_n$ by $x + e_n^i$, where $\overline{x}_i = 1 - x_i$. Then the set of edges incident with x is $\{(x, x + e_n^i) : i \in \{1, 2, \cdots, n\}\}$. For any vertex $x = x_1 x_2 \cdots x_n \in V(Q_n)$, let $x \cdot 0 = x_1 x_2 \cdots x_n 0$ and $x \cdot 1 = x_1 x_2 \cdots x_n 1$ denote the vertices in $V(Q_{n+1})$ corresponding to x in $V(Q_n)$.

As a variant of the hypercube, the n-dimensional folded hypercube FQ_n, proposed first by El-Amawy and Latifi [4], is a graph obtained from the hypercube Q_n by adding an edge, called a complementary edge, between any two vertices $x = x_1 x_2 \cdots x_n$ and $\overline{x} = \overline{x}_1 \overline{x}_2 \cdots \overline{x}_n$. Therefore, FQ_n has 2^{n-1} more edges than a Q_n. It is easy to know that the complementary edges forms a perfect matching of FQ_n. It has been shown that FQ_n is an $(n + 1)$-regular graph. The properties of folded hypercube was studied extensively. The pancyclicity and fault-free cycles in faulty folded hypercubes were studied in [10] and [5], respectively. The fault-tolerance of folded hypercubes were analyzed in [6-8]. The Hamilton-connectivity of folded hypercubes was showed in [9].

In this paper, we investigate the total chromatic number of the folded hypercubes FQ_n. If $n = 2$ (resp. $n = 3$) then FQ_2 (resp. FQ_3) is isomorphic to the complete graph K_4 (resp. the complete bipartite graph $K_{4,4}$). The total chromatic number of K_4 and $K_{4,4}$ have been determined, see [13] and [2] respectively. So we only need to consider the case for $n \geq 4$. In this work, we obtain that $\chi_t(FQ_n) = \Delta(FQ_n) + 1 = n + 2$ which attains the lower bound of TCC. We get the result by the following method: first, color the complementary edges of the folded hypercube with one color; second, decompose the hypercube into 2^{n-3} 3-dimensional cubes, color the edges and the vertices of each of these 3-dimensional cubes properly by four colors such that any two adjacent vertices in folded hypercube are colored differently; third, the uncolored edges form an $n-3$ regular bipartite graph, by König's theorem, it can be colored by $n-3$ colors.

2 Main Result

In this section, we would like to decompose the hypercube into 2^{n-3} 3-dimensional cubes first.

We define some notations. If $P = u_1 - u_2 - \cdots - u_m$ is a path in Q_n from the vertex u_1 to the vertex u_m, then $P^{-1} = u_m - u_{m-1} - \cdots - u_2 - u_1$ is a path in Q_n from the vertex u_m to the vertex u_1. Let $P \cdot 0 = u_1 \cdot 0 - u_2 \cdot 0 - \cdots - u_m \cdot 0$ be a path from the vertex $u_1 \cdot 0$ to the vertex $u_m \cdot 0$ in Q_{n+1}. The symbol $P \cdot 1$ is defined similarly.

We know that Q_3 contains a Hamiltonian path $P_3 = 000 - 100 - 110 - 010 - 011 - 111 - 101 - 001$. If $n \geq 4$, then define P_n as: $P_n = P_{n-1} \cdot 0 - P_{n-1}^{-1} \cdot 1$. Clearly, P_n is a Hamiltonian path of Q_n. Denote the i-th vertex (from left to right) of P_n by v_n^i $(1 \leq i \leq 2^n)$. By definition of P_n, the following properties are obvious:

(1) The vertices v_n^{4t+1} and v_n^{4t+4} are adjacent in Q_n for $n \geq 4$ and $t \in \{0, 1, \cdots, 2^{n-2} - 1\}$;

(2) $v_n^{2^n+1-l} = v_n^l + e_n^n$ for $n \geq 4$ and $l \in \{1, \cdots, 2^n\}$, *i.e.*, v_n^l and $v_n^{2^n+1-l}$ are adjacent in Q_n.

For $n \geq 4$, by the above properties and definition of FQ_n, we can verify that for any $k \in \{0, 1, \cdots, 2^{n-3}-1\}$, the vertices v_n^{4k+1}, v_n^{4k+2}, v_n^{4k+3}, v_n^{4k+4} and $v_n^{2^n-4k} = v_n^{4(2^{n-2}-k-1)+4}$, $v_n^{2^n-4k-1}$, $v_n^{2^n-4k-2}$, $v_n^{2^n-4k-3} = v_n^{4(2^{n-2}-k-1)+1}$ induce a 3-dimensional cube. Denote the cube by Q_n^k. Figure 1 shows the Q_4^0 and Q_4^1. Notice that the edges of Q_n^k are edges in Q_n.

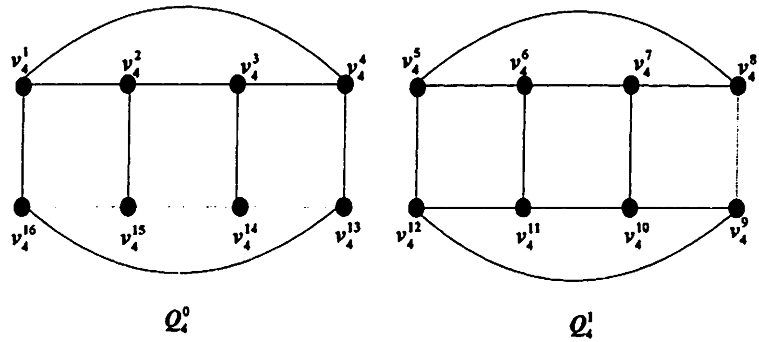

Figure 1: The two 3-dimensional cube in Q_4.

In fact, we can color the cube Q_n^k properly with four colors by the following manner. Assume there are four colors 1, 2, 3, 4. Let $f(v_n^{4k+1}) = 1$, $f(v_n^{4k+2}) = 2$, $f(v_n^{4k+3}) = 3$, $f(v_n^{4k+4}) = 4$. Assign the same color to two diagonal vertices and to three non-incident edges. In other words, let $f(v_n^{2^n-4k-2}) = 1$, $f(v_n^{2^n-4k-3}) = 2$, $f(v_n^{2^n-4k}) = 3$, $f(v_n^{2^n-4k-1}) = 4$; $f(v_n^{4k+3}, v_n^{4k+4}) = f(v_n^{2^n-4k-3}, v_n^{2^n-4k}) = f(v_n^{4k+2}, v_n^{2^n-4k-1}) = 1$, $f(v_n^{4k+4}, v_n^{4k+1}) = f(v_n^{2^n-4k}, v_n^{2^n-4k-1}) = f(v_n^{4k+3}, v_n^{2^n-4k-2}) = 2$, $f(v_n^{4k+1}, v_n^{4k+2}) = f(v_n^{2^n-4k-1}, v_n^{2^n-4k-2}) = f(v_n^{4k+4}, v_n^{2^n-4k-3}) = 3$, $f(v_n^{2^n-4k-3}, v_n^{2^n-4k-2}) = f(v_n^{4k+2}, v_n^{4k+3}) = f(v_n^{4k+1}, v_n^{2^n-4k}) = 4$. See Figure 2. We can check that f is a proper 4-total coloring of Q_n^k. Moreover, we can color the edges and the vertices of the 2^{n-3} three-dimensional cubes $\bigcup_{k=0}^{2^{n-3}-1} Q_n^k$ properly with four colors such that any two vertices adjacent in FQ_n are colored differently.

Lemma 2. There exists a proper 4-total coloring f_n for $\bigcup_{k=0}^{2^{n-3}-1} Q_n^k$ such that any two vertices adjacent in FQ_n are colored differently, where $n \geq 4$.

Proof. If $n = 4$, then set $f_4(v_4^1) = 1, f_4(v_4^2) = 2, f_4(v_4^3) = 3, f_4(v_4^4) = 4$;

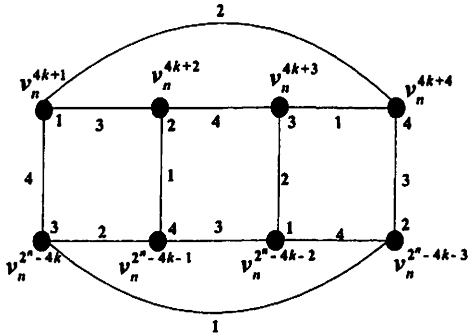

Figure 2: The proper 4-total coloring method for Q_n^k.

$f_4(v_4^5) = 3, f_4(v_4^6) = 4, f_4(v_4^7) = 1, f_4(v_4^8) = 2$. The other vertices and edges are colored by the manner in Figure 2. If $n \geq 5$, then for any $1 \leq j \leq 2^{n-1}$, define $f_n(v_n^j) = f_{n-1}(v_{n-1}^j)$. That is to say, for any $x \in V(Q_{n-1})$, let $f_n(x \cdot 0) = f_{n-1}(x)$. Color the other vertices and edges by the manner in Figure 2. Figure 3 shows the coloring of $\bigcup_{k=0}^{3} Q_5^k$. By the coloring method, each Q_n^k $(0 \leq k \leq 2^{n-3} - 1)$ is colored properly. We only need to prove that any two adjacent vertices in FQ_n are colored differently.

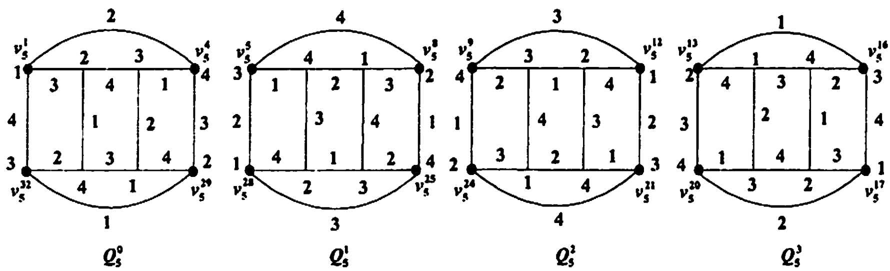

Figure 3: A proper 4-total coloring of $\bigcup_{k=0}^{3} Q_5^k$ distinguishes two adjacent vertices in FQ_5.

By the coloring method, we find that $\{f_n(v), f_n(v + e_n^n)\} = \{1, 3\}$ or $\{2, 4\}$ for any $v \in V(FQ_n)$. For $i \in \{1, 2, 3, 4\}$ and $n \geq 4$, denote $C_n(i) = \{v | f_n(v) = i\}$, $\overline{C_n(i)} = \{\overline{v} | f_n(v) = i\}$, $C_n(i) \cdot 0 = \{v \cdot 0 | f_n(v) = i\}$ (resp. $C_n(i) \cdot 1 = \{v \cdot 1 | f_n(v) = i\}$).

For $n \geq 5$, we find that $C_n(1) = C_{n-1}(1) \cdot 0 \cup C_{n-1}(3) \cdot 1$, $C_n(2) =$

$C_{n-1}(2) \cdot 0 \cup C_{n-1}(4) \cdot 1$, $C_n(3) = C_{n-1}(3) \cdot 0 \cup C_{n-1}(1) \cdot 1$, $C_n(4) = C_{n-1}(4) \cdot 0 \cup C_{n-1}(2) \cdot 1$. It is easy to verify that for even n, $\overline{C_n(1)} = C_n(2)$ and $\overline{C_n(3)} = C_n(4)$; for odd n, $\overline{C_n(1)} = C_n(4)$ and $\overline{C_n(2)} = C_n(3)$.

Next, we will prove that any two adjacent vertices in FQ_n are colored differently by induction on n.

If $n = 4$, $C_4(1) = \{0000, 1010, 0111, 1101\}$, $C_4(2) = \{1000, 0010, 1111, 0101\}$, $C_4(3) = \{1100, 0110, 1011, 0001\}$, $C_4(4) = \{0100, 1110, 0011, 1001\}$. Clearly, $C_4(i)$ is independent for any $i \in \{1, 2, 3, 4\}$.

For $n > k \geq 4$, assume $C_k(i)$ is independent for any $i \in \{1, 2, 3, 4\}$. Now it is enough to show that $C_{k+1}(i)$ is independent for any $i \in \{1, 2, 3, 4\}$.

By contrary, without loss of generality, assume $x, y \in C_{k+1}(1)$ and $(x, y) \in E(FQ_{k+1})$. By induction, both $C_k(1)$ and $C_k(3)$ are independent. So both $C_k(1) \cdot 0$ and $C_k(3) \cdot 1$ are independent. Hence, $x \in C_k(1) \cdot 0$, $y \in C_k(3) \cdot 1$ or $y \in C_k(1) \cdot 0$, $x \in C_k(3) \cdot 1$. Without loss of generality, suppose $x \in C_k(1) \cdot 0$, $y \in C_k(3) \cdot 1$. Since $C_k(1) \cap C_k(3) = \emptyset$, so $y = \overline{x}$. Suppose $v \in C_k(1)$ such that $x = v0$. Thus, $y = \overline{x} = \overline{v}1$. We conclude that $\overline{v} \in C_k(3)$, which contradicts that $\overline{C_k(1)} = C_k(2)$ for even k and $\overline{C_k(1)} = C_k(4)$ for odd k. So $C_{k+1}(1)$ is independent. Similarly, we can get $C_{k+1}(i)$ is independent for any $i \in \{2, 3, 4\}$. The proof is completed. □

The *edge chromatic number* of G, $\chi'(G)$, is the least integer k for which G admits a proper k-edge coloring. We recall a classical result on edge coloring.

Theorem 3 [3]. Let G be a simple bipartite graph. Then, $\chi'(G) = \Delta(G)$.

Next is the main result of this paper.

Theorem 4. If $n \geq 4$, then $\chi_t(FQ_n) = \Delta(FQ_n) + 1 = n + 2$.

Proof. First, color the complementary edges of the folded hypercube with one color. Second, by Lemma 2, color the edges and the vertices of each of these 3-dimensional cubes properly by four colors such that any two adjacent vertices in folded hypercube are colored differently. Third, the uncolored edges form an $n-3$ regular bipartite graph since it is a subgraph of hypercube Q_n, by Theorem 3, it can be colored by $n-3$ colors. This yields a proper $(n+2)$-total coloring of FQ_n. Hence, we can conclude that $\chi_t(FQ_n) \leq n + 2$. On the other hand, by definition, we know that

$\chi_t(FQ_n) \geq \Delta(FQ_n) + 1 = n + 2$. Therefore, $\chi_t(FQ_n) = n + 2$. □

3 Remark

Zhang *et al.* introduced [13] the concept of the adjacent vertex-distinguishing edge chromatic number of G. A *k-edge coloring* $f : E \to \{1, 2, \cdots, k\}$ of a graph $G = (V, E)$ is an assignment of k colors to the edges of G. The edge coloring f is proper if no two adjacent edges are assigned a same color. Let $f(uv)$ be the color of the edge $uv \in E(G)$. Denote by $F(v) = \{f(uv) : uv \in E(G)\}$. If f is a proper k-edge coloring, and $F(u) \neq F(v)$ for any edge $uv \in E(G)$, then f is called *a k-adjacent vertex-distinguishing edge coloring* of graph G (abbreviated k-$AVDEC$ of G). The smallest k for which G has a k-$AVDEC$ is *the adjacent vertex-distinguishing edge chromatic number* $\chi'_{av}(G)$ of G.

If G is a r-regular graph then the following lemma reveals the relations between $\chi_t(G)$ and $\chi'_{av}(G)$.

Lemma 5 [12]. Let $G = (V, E)$ be a r-regular graph, where $r \geq 2$. Then $\chi_t(G) \geq r + 1$, $\chi'_{av}(G) \geq r + 1$, and $\chi_t(G) = r + 1$ if and only if $\chi'_{av}(G) = r + 1$. □

By Theorem 4 and Lemma 5, we can get the adjacent vertex-distinguishing edge chromatic number of folded hypercube immediately.

Corollary 6. If $n \geq 4$, then $\chi'_{av}(FQ_n) = \Delta(FQ_n) + 1 = n + 2$. □

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