On Generalizing the "Lights Out" Game and a Generalization of Parity Domination

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Abstract

The Lights Out game on a graph G is played as follows. Begin with a (not necessarily proper) coloring of $V(G)$ with elements of $\mathbb{Z}_2$. When a vertex is toggled, that vertex and all adjacent vertices change their colors from 0 to 1 or vice-versa. The game is won when all vertices have color 0. The winnability of this game is related to the existence of a parity dominating set. We generalize this game to $\mathbb{Z}_k$, $k \geq 2$, and use this to define a generalization of parity dominating sets. We determine all paths, cycles, and complete bipartite graphs in which the game over $\mathbb{Z}_k$ can be won regardless of the initial coloring, and we determine a constructive method for creating all caterpillar graphs in which the Lights Out game cannot always be won.

1 Introduction

The game Lights Out was originally a handheld game by Tiger Electronics. This game has since been generalized to graphs as follows. Let G be a graph with a (not necessarily proper) vertex coloring by the set $\mathbb{Z}_2 = \{0, 1\}$. When a vertex is toggled, that vertex and all of its neighbors change colors (from 0 to 1 or vice-versa). The game is won when all vertices have the color 0.

Strategies for winning this game (when victory is possible) and some variations of the game have been studied in [AF98], [Aua00], [Pel87], [Sto89], and [Sut89]. The Lights Out game has connections with domination theory, specifically parity domination, which has been explored by Amin, Clark,

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Slater, and Zhang (see [AS92], [AS96], [ACS98], and [ASZ02]). J. Goldwasser and W. Klostermeyer were the first to discover the connection between Lights Out and parity domination (see [GK97] and [GKT97]). In particular, they proved that the existence of a parity dominating set is equivalent to whether a corresponding game of Lights Out can be won.

In Section 2, we generalize the game of Lights Out to an arbitrary set of colors $\mathcal{C}$, where the result of toggling a vertex is determined by a function $T : \mathcal{C} \to \mathcal{C}$. We focus on the case where T is a permutation and reduce this problem to the case where T is a cycle.

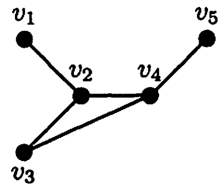
In Section 3, we recast the task of winning Lights Out on a graph whose vertices are labeled by $\mathbb{Z}_k$ as a solution to a linear system of equations over $\mathbb{Z}_k$. This will illustrate the connection between the Lights Out game over $\mathbb{Z}_2$ and parity domination and will allow us to use our generalized Lights Out game to generalize parity domination.

In Section 4, we characterize the labelings for paths, cycles, and complete bipartite graphs in which the Lights Out game over $\mathbb{Z}_k$ can be won. We use these results to determine the paths, cycles, and complete bipartite graphs in which the Lights Out game can be won regardless of the initial labeling. In Section 5, we generalize a result of A. Amin and P. Slater on the construction of caterpillar graphs in which the Lights Out game cannot always be won.

2 Generalized Lights Out

To play Lights Out, one needs to know the graph, the colors, and what the "off" color is. In addition, one needs to know the rule used to change the colors of each toggled vertex and its neighbors. Let G be a graph with vertex coloring $\pi : V(G) \to \mathcal{C}$, where $\mathcal{C}$ is a set, and let $0 \in \mathcal{C}$ be designated as the *off* color. We then define a *toggling function* $T : \mathcal{C} \to \mathcal{C}$ so that if $v \in V(G)$ is toggled, the resulting coloring is π' with $\pi'(w) = T(\pi(w))$ if $w = v$ or $wv \in E(G)$, and $\pi'(w) = \pi(w)$ otherwise. We define the game to be won when the coloring is π_0, where $\pi_0(v) = 0$ for all $v \in V(G)$. In the standard Lights Out game, $\mathcal{C} = \mathbb{Z}_2$, the off color is 0, and the toggling function is $T(c) = c + 1$.

Example 2.1. Consider the following graph:



Let $\mathcal{C} = \mathbb{Z}_5$ with off color 0, and define the toggle function $T : \mathcal{C} \to \mathcal{C}$ by $T(0) = 2$, $T(1) = 0$, $T(2) = 3$, $T(3) = 1$, $T(4) = 3$. Let the initial coloring be $\pi(v_1) = 2$, $\pi(v_2) = 4$, $\pi(v_3) = 1$, $\pi(v_4) = 0$, and $\pi(v_5) = 2$. If we toggle v_2 once, we change the color of v_1 to $T(2) = 3$, v_2 to $T(4) = 3$, v_3 to $T(1) = 0$, and v_4 to $T(0) = 2$. We then toggle v_1 twice, giving v_1 and v_2 the color $T(T(3)) = 0$. After toggling v_5 thrice, all vertices have color 0, and the game is won.

Now suppose the toggling function T is a permutation, so we can write T as a product of disjoint cycles. If $T(0) = 0$, the game can be won if and only if all vertices are initially colored 0. Otherwise, let σ be the cycle in T with $\sigma(0) \neq 0$. If any vertex of the graph has a color that is fixed by σ, then the game cannot be won. If the vertices are colored only by colors that are not fixed by σ, then the other disjoint cycles have no effect on the game. This gives us the following.

Proposition 2.2. Let G be a graph whose vertices are colored by $\mathcal{C}$, and whose toggling function is a permutation $T = \sigma_1\sigma_2\cdots\sigma_m$, where the σ_i's are disjoint cycles and $\sigma_1(0) \neq 0$. Let $\mathcal{C}' \subseteq \mathcal{C}$ be the set of colors that are not fixed by σ_1. Then the Lights Out game can be won if and only if

1. All vertices are colored by elements of $\mathcal{C}'$ and
2. The Lights Out game can be won with toggling function σ_1.

Thus, the question of whether a Lights Out game can be won when the toggling function is a permutation can be reduced to the case where the toggling function is a cycle. If the cycle T has order k, we identify $T^c(0)$ with $c \in \mathbb{Z}_k$, and we can thus let $\mathcal{C} = \mathbb{Z}_k$ with toggling function $T(c) = c+1$. The traditional Lights Out game operates this way with $k = 2$. Note that we can consider $\pi : V(G) \to \mathbb{Z}_k$ a labeling of $V(G)$.

3 Matrix Methods and Parity Domination

As before, let G be a graph with labeling $\pi : V(G) \to \mathbb{Z}_k$ and toggling function $T(c) = c + 1$. Let $V(G) = \{v_1, \ldots, v_n\}$, with $\pi(v_i) = b_i$. In this section, we address the question of whether, given this initial labeling, the Lights Out game can be won.

We proceed as in [AF98]. One can easily check that the order in which the vertices are toggled has no impact on the resulting labeling. All that matters is how many times each vertex is toggled. Let x_i be the number of times that v_i is toggled, and let $\mathbf{x}$ be the n-dimensional vector with $\mathbf{x}[i] = -x_i$. Similarly, let $\mathbf{b}$ be the n-dimensional vector with $\mathbf{b}[i] = b_i$.

Let A be the adjacency matrix of G. Then $N = A + I_n$ is the *neighborhood matrix* or *augmented adjacency matrix* of G. Notice that the label of v_i is increased by one each time either v_i or a neighbor of v_i is toggled. Thus, the label of v_i after the toggling given by $\mathbf{x}$ is $b_i + \sum_{j=1}^{n} N_{ij}x_j$. This gives us the following.

Lemma 3.1. The toggling given by $\mathbf{x}$ can be used to win the Lights Out game if and only if $N\mathbf{x} = \mathbf{b}$ over $\mathbb{Z}_k$.

Example 3.2. Let $G = C_4$, and suppose that we have an initial labeling $\pi : V(G) \to \mathbb{Z}_8$ given by $\pi(v_1) = 4$, $\pi(v_2) = 1$, $\pi(v_3) = 5$, $\pi(v_4) = 3$. We then have

$$N = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ 5 \\ 3 \end{bmatrix}.$$

Suppose $k = 8$. We then solve the equation $N\mathbf{x} = \mathbf{b}$ by row reduction modulo 8 to get $\mathbf{x}[1] = 2$, $\mathbf{x}[2] = 4$, $\mathbf{x}[3] = 3$, and $\mathbf{x}[4] = 6$. Thus, the game can be won. Note that if we row reduce in $\mathbb{Z}_3$, there is no solution. In this case, the game cannot be won.

These methods are reminiscent of those used in domination theory (see, for example, [AS92] and [ASZ02]). The classical domination problem is to find a set $S \subseteq V(G)$ (called a *dominating set*) of minimum cardinality such that every vertex of G is either in S or adjacent to a vertex in S.

For each $v \in V(G)$, define $N[v] = \{w \in V : vw \in E(G) \text{ or } w = v\}$. Then $S \subseteq V(G)$ is a dominating set if and only if $|N[v] \cap S| \geq 1$ for all $v \in V(G)$. Other types of domination have been studied by placing various restrictions on $|N[v] \cap S|$. Note that if we let $\mathbf{x}$ be the n-dimensional vector with $x_i = 1$ if $v_i \in S$ and $x_i = 0$ otherwise, then

$$|N[v] \cap S| = \sum_{j=1}^{n} N_{ij}x_j \tag{1}$$

In *parity domination*, we begin with a labeling $\pi : V(G) \to \mathbb{Z}_2$. We call a set $S \subseteq V(G)$ a parity dominating set of π if $|N[v] \cap S| \equiv \pi(v) \pmod 2$ for all $v \in V(G)$. Using (1), this is equivalent to S satisfying the equation

$N\mathbf{x} = \mathbf{b}$ over $\mathbb{Z}_2$, where $\mathbf{x}(i) = 1$ if $v_i \in S$ and $\mathbf{x}(i) = 0$ otherwise. Thus, we have a parity domination set S for π if and only if the Lights Out game with initial labeling π can be won by toggling precisely the vertices in S.

To extend parity domination to labelings $\pi : V(G) \to \mathbb{Z}_k$, $k \geq 3$, we must address the possibility that a solution to $N\mathbf{x} = \mathbf{b}$ over $\mathbb{Z}_k$ may have an entry that is neither 0 nor 1. We resolve this issue by using multisets. Recall that a *multiset* is a pair $M = (S, m)$, where S is a set (called the *underlying set*), and $m : S \to \mathbb{N}$ is a function. For $s \in S$, we call $m(s)$ the *multiplicity of s in M*.

Definition 3.3. For each $v \in V(G)$, let $N_k[v]$ be the multiset with underlying set $N[v]$ and with each element having multiplicity $k-1$. Let M be a multiset whose underlying set is a subset of $V(G)$ and whose elements each have multiplicity at most $k-1$. For a labeling $\pi : V(G) \to \mathbb{Z}_k$, we call M a *$\mathbb{Z}_k$-dominating multiset* for π if $|N_k[v] \cap M| \equiv \pi(v) \pmod{k}$ for all $v \in V(G)$.

Note that a $\mathbb{Z}_2$-dominating multiset is merely a parity dominating set. The following describes the relationship between $\mathbb{Z}_k$-domination and the Lights Out game.

Theorem 3.4. Let G be a graph with labeling $\pi : V(G) \to \mathbb{Z}_k$. If $V(G) = \{v_1, v_2, \ldots, v_n\}$, let $\mathbf{b} \in \mathbb{R}^n$ such that $\mathbf{b}[i] = \pi(v_i)$. The following are equivalent.

1. There exists a $\mathbb{Z}_k$-dominating multiset for π.
2. There is a solution to the equation $N\mathbf{x} = \mathbf{b}$ over $\mathbb{Z}_k$.
3. The Lights Out game with initial labeling π can be won.

Proof. By Lemma 3.1, 2 and 3 are equivalent. For $1 \Rightarrow 2$, let S be a $\mathbb{Z}_k$-dominating multiset. Define $\mathbf{x}$ so that $\mathbf{x}[i]$ is the multiplicity of v_i in S. It is not hard to show that $\sum_{j=1}^{n} N_{ij}x_j = |N_k[v] \cap S|$. Since also S is a $\mathbb{Z}_k$-dominating multiset, we have, for each $v \in V$,

$$\sum_{j=1}^{n} N_{ij}x_j = |N_k[v] \cap S| \equiv \pi(v) \pmod{k}$$

and so $\mathbf{x}$ is a solution to $N\mathbf{x} = \mathbf{b}$. The proof of $2 \Rightarrow 1$ is similar. □

4 Winnable Labelings and AW Graphs

We use notation as in previous sections, with our graph G, labeling set $\mathbb{Z}_k$, and toggling function $T(c) = c + 1$. We say the labelings π and π' are *equivalent under T* (or merely *equivalent* if the context is clear) if, given the initial labeling π, there is a sequence of toggles such that the terminal labeling is π'. We denote this relation by $\mathcal{R}_G^k$. It is easy to see that $\mathcal{R}_G^k$ is an equivalence relation. We call a labeling π *winnable* if π is equivalent to π_0, where $\pi_0(v) = 0$ for all $v \in V(G)$. We say that G is *always winnable over* $\mathbb{Z}_k$ (or simply *always winnable* or *AW* if the context is clear) if all labelings $\pi : V(G) \to \mathbb{Z}_k$ are winnable.

Example 4.1. P_3 has neighborhood matrix $N = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. Since $\det(N) = -1$, N is invertible, and so the equation $N\mathbf{x} = \mathbf{b}$ can always be solved. By Lemma 3.1, P_3 is AW over all $\mathbb{Z}_k$, $k \geq 2$.

On the other hand, K_n is non-AW for all $k \geq 2$. Every labeling $\pi \neq \pi_0$ is not winnable, since toggling any vertex has the effect of increasing the label of every vertex by one.

In this section, we study the winnable labelings of paths, cycles and complete bipartite graphs. Let $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(P_n) = \{v_i v_{i+1} : 1 \leq i \leq n-1\}$. We let $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(C_n) = \{v_1 v_n, v_i v_{i+1} : 1 \leq i \leq n-1\}$. Finally, we let $V(K_{m,n}) = \{v_i, w_j : 1 \leq i \leq m, 1 \leq j \leq n\}$ and $E(K_{m,n}) = \{v_i w_j : 1 \leq i \leq m, 1 \leq j \leq n\}$. We begin by proving that every labeling of the vertex sets of each of these graphs is equivalent to a "nice" labeling.

Lemma 4.2.
1. Each labeling of $V(P_n)$ by $\mathbb{Z}_k$ is equivalent to some π where $\pi(v_i) = 0$ for all $i \neq 1$.
2. Each labeling of $V(C_n)$ by $\mathbb{Z}_k$ is equivalent to some π, where $\pi(v_i) = 0$ for all $i \neq 1, 2$.
3. Each labeling of $V(K_{m,n})$ by $\mathbb{Z}_k$ is equivalent to some π, where $\pi(w_j) = 0$ for all $1 \leq j \leq n$.

Proof. For 1, if all vertices have label 0, then we are done. Otherwise, let m be maximum such that the label of v_m is $c \neq 0$. If $m = 1$, we are done. If $m \geq 2$, we toggle v_{m-1} $k - c$ times (or $-c$ times modulo k). The resulting labeling has the label of 0 for v_i, $i \geq m$. By induction, this labeling is equivalent to one in which all vertices except perhaps v_1 have label 0. Part 2 follows similarly.

For 3, let the label of w_j be c_j for each $1 \leq i \leq n$. We merely toggle w_j $k - c_j$ times to get the desired labeling. □

Lemma 4.2 motivates the following labelings. For each $z \in \mathbb{Z}_k$, we define $\pi_z : V(P_n) \to \mathbb{Z}_k$ to be $\pi_z(v_i) = \delta_{i,1} z$. For each $y, z \in \mathbb{Z}_k$, we define $\pi_{y,z} : V(C_n) \to \mathbb{Z}_k$ to be $\pi_{y,z}(v_i) = \delta_{1,i}y + \delta_{2,i}z$. Finally, for each $z_1, \ldots, z_m \in \mathbb{Z}_k$, let $\mathbf{z} \in \mathbb{Z}_k^m$ with $\mathbf{z}(i) = z_i$. We define $\pi_{\mathbf{z}} : V(K_{m,n}) \to \mathbb{Z}_k$ by $\pi_{\mathbf{z}}(v_i) = z_i$ and $\pi_{\mathbf{z}}(w_j) = 0$, for all $1 \leq i \leq m$, $1 \leq j \leq n$. Our main result tells precisely when π_z, $\pi_{y,z}$, and $\pi_{\mathbf{z}}$ are winnable.

Theorem 4.3. Let π_z, $\pi_{y,z}$, and $\pi_{\mathbf{z}}$ be as above

1. For P_n, π_z is winnable if and only if either $n \equiv 0, 1 \pmod 3$ or $z = 0$.
2. For C_n, $\pi_{y,z}$ is winnable if and only if one of the following holds.
   (a) $n \equiv 1, 2 \pmod 3$ and $\gcd(3, k) = 1$
   (b) $n \equiv 1 \pmod 3$, $3|k$, and the equivalence $3x \equiv -y - z \pmod k$ has a solution.
   (c) $n \equiv 2 \pmod 3$, $3|k$, and the equivalence $3x \equiv y - 2z \pmod k$ has a solution.
   (d) $y = z = 0$.
3. For $K_{m,n}$, $\pi_{\mathbf{z}}$ is winnable if and only if $(mn-1)x \equiv \sum_{i=1}^{m} \mathbf{z}(i) \pmod k$ has a solution.

Proof. For 1, suppose we toggle the vertices in the order $v_1, v_2, \ldots, v_n$. Let t_i be the number of times v_i is toggled, and let d_i be the label of v_i after v_i is toggled. Clearly $t_i = -d_{i-1}$ for all $i \geq 2$. We then have $d_i = t_i + t_{i-1} = -d_{i-1} - d_{i-2}$, and so $d_i + d_{i-1} + d_{i-2} = 0$. This, along with $d_1 = t_1 + z$ and $d_2 = -z$, gives us

$$d_i = \begin{cases} -t_1, & i \equiv 0 \pmod 3 \\ t_1 + z, & i \equiv 1 \pmod 3 \\ -z, & i \equiv 2 \pmod 3 \end{cases} \tag{2}$$

Note that π_z is winnable if and only if there exists $t_1 \in \mathbb{Z}_k$ such that $d_n \equiv 0 \pmod k$. If $n \equiv 0 \pmod 3$, let $t_1 = 0$; if $n \equiv 1 \pmod 3$, let $t_1 = -z$. For $n \equiv 2 \pmod 3$, π_z is winnable precisely when $z = 0$.

We proceed similarly for 2. Let t_i be the number of times v_i is toggled, and let d_i be the label of v_i after v_i is toggled. Similarly as before, we have $t_i = -d_{i-1}$ for $3 \leq i \leq n$ and $d_i + d_{i-1} + d_{i-2} = 0$ for $4 \leq i \leq n-1$. This is not the case with $i = 3$ since t_2 is not necessarily $-d_1$ (v_1 is adjacent to both v_2 and v_n). We have $d_2 = t_1 + t_2 + z$ and $d_3 = -t_1 - z$, and so

$$d_i = \begin{cases} -t_1 - z, & i \equiv 0 \pmod 3 \\ -t_2, & i \equiv 1 \pmod 3 \\ t_1 + t_2 + z, & i \equiv 2 \pmod 3 \end{cases} \tag{3}$$

After v_n is toggled, we have $d_n = t_{n-1} + t_n + t_1$ and v_1 has label $t_n + t_1 + t_2 + y$. If $n \equiv 0 \pmod 3$, we have $d_n = -z$ and v_1 has label $y - z$, so $\pi_{y,z}$ is winnable if and only if $y = z = 0$. If $n \equiv 1 \pmod 3$, we have $d_n = t_1 - t_2$ and v_1 has label $2t_1 + t_2 + y + z$, and so $\pi_{y,z}$ is winnable if and only if $t_2 = t_1$ and $3t_1 \equiv -y - z \pmod k$. Finally, if $n \equiv 2 \pmod 3$, we have $d_n = 2t_1 + t_2 + z$ and v_1 has label $t_1 + 2t_2 + y$. Eliminating t_2 gives us $3t_1 \equiv y - 2z \pmod k$.

For 3, let $z_i = \mathbf{z}(i)$, and let x_i be the number of times v_i is toggled. Once the v_i's have been toggled, v_i has label $x_i + z_i$ and each w_j has label $\sum_{\ell=1}^{m} x_\ell$. In order to have a final label of 0, each w_j must be toggled $-\sum_{\ell=1}^{m} x_\ell$ times. This leaves v_i with the label $z_i + x_i - n\sum_{\ell=1}^{m} x_\ell = z_i + (1-n)x_i - n\sum_{\ell\neq i}^{m} x_\ell$. We must then have $(n-1)x_i + n\sum_{\ell\neq i}^{m} x_\ell = z_i$, a linear system over $\mathbb{Z}_k$. For each $p \in \mathbb{N}$, let $B(p) = [b_{ij}]$ be the $p \times p$ matrix with $b_{ii} = n - 1$ for $1 \le i \le p$ and $b_{ij} = n$ otherwise. Then the augmented matrix for our linear system is $[B(m)|\mathbf{z}]$.

We now seek to put $[B(m)|\mathbf{z}]$ in row echelon form. If we subtract row 2 from row 1 and add n times the new row 1 to every other row, we get the matrix

$$\begin{bmatrix} -1 & 1 & 0 & \cdots & 0 & z_1 - z_2 \\ 0 & 2n-1 & n & \cdots & n & nz_1 - nz_2 + z_2 \\ 0 & 2n & & & & nz_1 - nz_2 + z_3 \\ \vdots & \vdots & & B(m-2) & & \vdots \\ 0 & 2n & & & & nz_1 - nz_2 + z_m \end{bmatrix}$$

If we iterate this process $j-1$ times, $2 \le j \le m$, we get

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 & \cdots & 0 & z_1 - z_2 \\ 0 & \ddots & \ddots & 0 & 0 & \cdots & 0 & \vdots \\ 0 & \cdots & -1 & 1 & 0 & \cdots & 0 & z_{j-1} - z_j \\ 0 & \cdots & 0 & jn-1 & n & \cdots & n & \left(\sum_{\ell=1}^{j-1} nz_\ell\right) - (j-1)nz_j + z_j \\ 0 & \cdots & 0 & jn & & & & \left(\sum_{\ell=1}^{j-1} nz_\ell\right) - (j-1)nz_j + z_{j+1} \\ \vdots & \ddots & \vdots & \vdots & & B(m-j) & & \vdots \\ 0 & \cdots & 0 & jn & & & & \left(\sum_{\ell=1}^{j-1} nz_\ell\right) - (j-1)nz_j + z_m \end{bmatrix}$$

We get our echelon form by setting $j = m$. Since all row operations used to obtain the echelon form are invertible, this echelon form is equivalent to the original system. Moreover, the first $m-1$ leading entries are -1 and the last leading entry is $mn - 1$, so $\pi_{\mathbf{z}}$ is winnable if and only if

$$(mn-1)x = \left(\sum_{\ell=1}^{m-1} nz_\ell\right) - (m-1)nz_m + z_m = \left(\sum_{\ell=1}^{m} nz_\ell\right) - (mn-1)z_m$$

has a solution in $\mathbb{Z}_k$. This is equivalent to $(mn-1)x \equiv \sum_{\ell=1}^{m} nz_\ell$ having a solution. Since $\gcd(mn-1, n) = 1$, this is equivalent to $(mn-1)x \equiv \sum_{\ell=1}^{m} z_\ell$ having a solution, which completes the proof. □

Let G be any graph. Recall the equivalence relation $\mathcal{R}_G^k$ between labelings of $V(G)$ by $\mathbb{Z}_k$, and note that the number of winnable labelings is $||[\pi_0]||$. There is a natural group action of $\mathbb{Z}_k{}^{|V(G)|}$ on the labelings of $V(G)$ in which the equivalence classes of $\mathcal{R}_G^k$ are the orbits of the action. It follows that all equivalence classes of $\mathcal{R}_G^k$ have the same size. Furthermore, Lemma 4.2 implies that π_z, $\pi_{y,z}$, and $\pi_{\mathbf{z}}$ represent all equivalence classes of $\mathcal{R}_{P_n}^k$, $\mathcal{R}_{C_n}^k$, and $\mathcal{R}_{K_{m,n}}^k$, although not necessarily uniquely.

Armed with this information, we can now count the winnable labelings for P_n, C_n, and $K_{m,n}$.

Theorem 4.4. 1. The number of winnable labelings of $V(P_n)$ by $\mathbb{Z}_k$ is

(a) k^n if $n \equiv 0, 1 \pmod 3$.

(b) k^{n-1} if $n \equiv 2 \pmod 3$

2. The number of winnable labelings of $V(C_n)$ by $\mathbb{Z}_k$ is

(a) k^n if $n \equiv 1, 2 \pmod 3$ and $\gcd(3, k) = 1$.

(b) k^{n-2} if $n \equiv 0 \pmod 3$.

(c) $\frac{k^n}{3}$ if $3|k$ and $n \equiv 1, 2 \pmod 3$.

3. $K_{m,n}$ has $\dfrac{k^{m+n}}{\gcd(k, mn-1)}$ winnable labelings by $\mathbb{Z}_k$.

Proof. For 1, Lemma 4.2(1) implies that the collection of all π_z, $z \in \mathbb{Z}_k$ represents all equivalence classes. If $k \equiv 0, 1 \pmod 3$, all labelings are winnable, so we have k^n winnable labelings. If $k \equiv 2 \pmod 3$, suppose that π_y and π_z are equivalent. Then the same toggling sequence that takes π_z to π_y will take π_{z-y} to π_0. By Theorem 4.3(1), we have $z - y = 0$, so $z = y$. Thus, there are k equivalence classes. Since each has the same cardinality, each equivalence class has $\frac{k^n}{k} = k^{n-1}$ labelings.

For 2, Lemma 4.2(2) implies that the $\pi_{y,z}$ represent all equivalence classes. Suppose π_{y_1,z_1} and π_{y_2,z_2} are equivalent. As before, if $y = y_2 - y_1$ and $z = z_2 - z_1$, then $\pi_{y,z}$ is winnable. Let $n \equiv 1$ or $2 \pmod 3$. If $\gcd(3, k) = 1$, then Theorem 4.3(2a) implies that all k^n labelings are winnable. If $n \equiv 0 \pmod 3$, then Theorem 4.3(2) implies that $\pi_{y,z}$ is winnable if and only if $y = z = 0$. It follows that there are k^2 equivalence classes, and therefore k^{n-2} winnable labelings. If $3|k$ and $n \equiv 1 \pmod 3$, then by Theorem 4.3(2b), $\pi_{y,z}$ is winnable if and only if $3x \equiv -y - z \pmod k$ has a solution. This occurs precisely when $3|y + z$, or, equivalently, $y_1 + z_1 \equiv y_2 + z_2 \pmod 3$. There are then 3 equivalence classes and therefore $\frac{k^n}{3}$ winnable labelings. The $n \equiv 2 \pmod 3$ case follows similarly, using Theorem 4.3(2c).

For 3, Lemma 4.2(3) implies that the $\pi_{\mathbf{z}}$ represent all equivalence classes. Let $\mathbf{y} \in \mathbb{Z}_k^m$ be fixed. As before, if $\pi_{\mathbf{y}}$ and $\pi_{\mathbf{z}}$ are equivalent, then $\pi_{\mathbf{z}-\mathbf{y}}$

is winnable. By Theorem 4.3(3), this occurs precisely when $(mn-1)x = \sum_{i=1}^{m}[\mathbf{z}(i) - \mathbf{y}(i)] = \sum_{i=1}^{m}\mathbf{z}(i) - \sum_{i=1}^{m}\mathbf{y}(i)$ has a solution in $\mathbb{Z}_k$. If $d = \gcd(k, mn-1)$, then there are $\frac{k}{d}$ values of $\sum_{i=1}^{m}\mathbf{z}(i)$ for which a solution to this congruence exists. For each r of these $\frac{k}{d}$ values, there are k^{m-1} different $\mathbf{z} \in \mathbb{Z}_k^m$ whose entries add up to r. Thus, there are $\frac{k^m}{d}$ $\pi_{\mathbf{z}}$'s in each equivalence class, and so there are $\frac{k^m}{k^m/d} = d$ equivalence classes. The result follows. □

From this, we can easily determine the AW paths, cycles, and complete bipartite graphs.

Corollary 4.5. 1. P_n is AW over $\mathbb{Z}_k$ if and only if $n \equiv 0$ or $1 \pmod 3$.

2. C_n is AW over $\mathbb{Z}_k$ if and only if $n \equiv 1$ or $2 \pmod 3$ and $\gcd(3, k) = 1$.

3. $K_{m,n}$ is AW over $\mathbb{Z}_k$ if and only if $\gcd(mn-1, k) = 1$

5 Non-AW Caterpillar Graphs

One of the results in [AS96] gives a constructive method for generating all caterpillar graphs G for which there exists a labeling $\pi : V(G) \to \mathbb{Z}_2$ that does not admit a parity domination set. In this section, we derive a similar constructive method for generating non-AW caterpillar graphs that will give Amin and Slater's result as a special case.

Recall that a caterpillar graph is a graph in which the vertices that are not leaves (called the *spine*) induce a path. Let $v_1(G), v_2(G), \ldots, v_n(G)$ be the vertices of the spine with $v_i(G)v_{i+1}(G) \in E(G)$, and let $\ell_i(G)$ be the number of leaves adjacent to $v_i(G)$. We can leave out the argument G if G is known. We begin with a result similar to Lemma 4.2.

Lemma 5.1. If G is a caterpillar graph, then every labeling of G is equivalent to some labeling π such that $\pi(v) = 0$ for all $v \in V(G) - \{v_1\}$.

Proof. Follows from an argument similar to Lemma 4.2(1) and (3). □

For each $z \in \mathbb{Z}_k$, let π_z be the labeling of the caterpillar graph G given by $\pi_z(v_1) = z$ and $\pi_z(v) = 0$ for all $v \neq v_1$. Let C be the set of all equivalence classes of labelings of $V(G)$. If $y, z \in \mathbb{Z}_k$, one can easily verify that the binary operation $[\pi_y] + [\pi_z] = [\pi_{y+z}]$ is well-defined, making C an additive group. Moreover, the map $\Psi : \mathbb{Z}_k \to C$ given by $\Psi(z) = [\pi_z]$ is an epimorphism whose kernel is the set of all $z \in \mathbb{Z}_k$ such that π_z is winnable. We then use standard group theory to get the following.

Lemma 5.2. Let G be a caterpillar graph.

1. If π_d is winnable, then π_{md} is winnable for all $m \in \mathbb{Z}$.

2. π_y and π_z are winnable if and only if $\pi_{\gcd(y,z)}$ is winnable.

3. G is AW if and only if π_1 is winnable.

Thus, in our study of AW (and non-AW) graphs, we will begin with the labeling π_1. To make the computations more convenient, let $m_i(G) = \ell_i(G) - 1$. We proceed as with P_n, toggling the vertices of the spine in the order $v_1, v_2, \ldots, v_n$. After each v_i is toggled, we toggle the leaves adjacent to v_i so that they each have label 0. Let t_i be the number of times v_i is toggled with $t_1 = x$, and let $d_i^G(x)$ (or simply $d_i(x)$ when G is known) be the label of v_i after v_i and all its adjacent leaves are toggled. After toggling v_1, we must toggle each adjacent leaf $-x$ times to get $d_1(x) = 1+x-\ell_1 x = -m_1x+1$. We toggle v_2 and its adjacent leaves similarly to get $d_2(x) = (1-m_1m_2)x+m_2$. For the remaining vertices, we get

$$d_i(x) = t_i-\ell_it_i+t_{i-1} = -d_{i-1}(x)+\ell_id_{i-1}(x)-d_{i-2}(x) = m_id_{i-1}(x)-d_{i-2}(x)$$

This gives us the following.

Lemma 5.3. For each $1 \leq i \leq n$, we have $d_i(x) = a_ix + b_i$, where the sequences $\{d_i(x)\}$, $\{a_i\}$, and $\{b_i\}$ satisfy the homogeneous linear difference equation $y_j = m_jy_{j-1} - y_{j-2}$ with initial values $a_1 = -m_1$, $a_2 = 1-m_1m_2$, $b_1 = 1$, and $b_2 = m_2$.

Note that if the spine of G is P_n, then G is AW if and only if the equivalence $a_nx + b_n \equiv 0 \pmod{k}$ can be solved, which occurs precisely when $\gcd(a_n, k)|b_n$. Using techniques similar to the proof of Lemma 5.3, we get a slight strengthening of Lemma 5.2(1).

Lemma 5.4. Suppose that π_d can be won by toggling v_1 y times. Then π_{md} can be won by toggling v_1 my times.

Amin and Slater use the following construction to generate non-AW caterpillar graphs in the case $k = 2$.

Definition 5.5. Let G_1 and G_2 be caterpillar graphs whose spines are P_{n_1} and P_{n_2}, respectively. We define $G_1.w.G_2(r)$ to be the caterpillar graph with vertex set $V(G_1)\cup V(G_2)\cup\{w, x_1, \ldots, x_r\}$, where $r \geq 0$ ($r = 0$ denotes no x_i's) and edge set $E(G_1)\cup E(G_2)\cup\{v_{n_1}(G_1)w, wv_1(G_2), wx_1, \ldots, wx_r\}$. We call this construction a *pasting* of G_1 and G_2.

Note that the construction depends on which order the vertices of the spine are written. This can be made clear by defining the ℓ_i's. We call $K_{1,n}$ *type* T_1 if n is odd, and we call a caterpillar graph *type* $T_{2,j}$, $j \geq 0$, if it has spine P_{j+2}, if ℓ_1 and ℓ_{j+2} are even, and if ℓ_i are odd for $2 \leq i \leq j+1$. Note that T_1 and $T_{2,j}$ are unique if we consider each ℓ_i modulo 2. We can now state Amin and Slater's result (in Lights Out terminology) as follows.

Theorem 5.6. [AS96]

1. Let G_1 and G_2 be two non-AW caterpillar graphs over $\mathbb{Z}_2$. Then $G_1.w.G_2(r)$ is non-AW over $\mathbb{Z}_2$.

2. A caterpillar graph G is non-AW over $\mathbb{Z}_2$ if and only if either

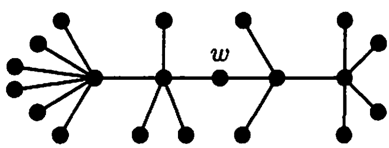
   (a) G is of type T_1 or $T_{2,j}$.

   (b) G can be obtained by repeated pastings of graphs of types T_1 and $T_{2,j}$.

The following example shows that this result does not hold for all $\mathbb{Z}_k$.

Example 5.7. Consider G_1, G_2, and $G_1.w.G_2(0)$ as follows.

G_1 $\qquad$ G_2



$G_1.w.G_2(0)$

We have $m_1(G_1) = 5$, $m_2(G_1) = 2$, $m_1(G_2) = 1$, and $m_2(G_2) = 3$. If $k = 6$, then, $d_2^{G_1}(x) = 3x + 2$, and $d_2^{G_2}(x) = 4x + 3$. Since neither $d_2^{G_i}(x)$ can be 0 modulo 6, G_1 and G_2 are not AW. However, $G_1.w.G_2(0)$ is AW (using $x = 0$ for π_1). Thus, it is possible to paste two non-AW caterpillars together to get an AW caterpillar.

While we do not have an analogue of Theorem 5.6 for arbitrary k, our main result generalizes the theorem to $k = p^e$, where p is prime. We begin with a lemma.

Lemma 5.8. Suppose p is a prime such that $p|k$, and let a_i and b_i be as in Lemma 5.3. Then for each $1 \le i \le n$, p cannot divide both a_i and b_i.

Proof. For contradiction, let i be minimal such that $p|a_i$ and $p|b_i$. By Lemma 5.3, we have $i \ge 3$, and (in $\mathbb{Z}_k$) $a_i = m_i a_{i-1} - a_{i-2}$ and $b_i = m_i b_{i-1} - b_{i-2}$. Since $p|k$, these equations also hold when we consider them over $\mathbb{Z}_p$. In this context, $a_i = b_i = 0$, and so $a_{i-2} = m_i a_{i-1}$ and $b_{i-2} = m_i b_{i-1}$.

We claim that $a_{j+1}b_j \equiv a_jb_{j+1} \pmod{p}$ for all $1 \le j \le i-2$. We induct on $i-j-2$. For $i-j-2=0$, by the minimality of i, either a_{i-1} or b_{i-1} is nonzero in $\mathbb{Z}_p$. Without loss of generality, $a_{i-1} \neq 0$ in $\mathbb{Z}_p$. Doing computations in $\mathbb{Z}_p$, we get $m_i = \frac{a_{i-2}}{a_{i-1}}$, and so $b_{i-2} = \frac{a_{i-2}b_{i-1}}{a_{i-1}}$. Thus, $a_{i-1}b_{i-2} \equiv a_{i-2}b_{i-1} \pmod{p}$. For the induction step, suppose $a_{j+1}b_j \equiv a_jb_{j+1} \pmod{p}$. We then have, in $\mathbb{Z}_p$,

$$a_{j+1}b_j = a_jb_{j+1} = a_j(m_{j+1}b_j - b_{j-1})$$

A little algebra gives us $a_jb_{j-1} = b_j(m_{j+1}a_j - a_{j+1}) = a_{j-1}b_j$, which proves the claim.

In particular, we have $a_1b_2 \equiv a_2b_1 \pmod{p}$, and so $-m_1m_2 \equiv 1 - m_1m_2 \pmod{p}$. This implies that $1 \equiv 0 \pmod{p}$, a contradiction, which completes the proof. □

As a consequence, we get the following.

Theorem 5.9. Let G_1 and G_2 be non-AW caterpillar graphs over $\mathbb{Z}_k$, where $k = p^e$ with p a prime. Then $G_1.w.G_2(r)$ is non-AW over $\mathbb{Z}_k$ for all $r \ge 0$.

Proof. Let the spine of G_i be P_{n_i} for $i = 1, 2$. Since $d_{n_1}^{G_1}(x) \equiv 0 \pmod{k}$ has no solution, we cannot have $\gcd(a_{n_1}, k) = 1$. Thus, $p | a_{n_1}$. By Lemma 5.8, p does not divide b_{n_1}, and so p does not divide d_{n_1}. Therefore $\gcd(d_{n_1}, k) = 1$.

After $v_{n_1}(G_1)$ is toggled, we toggle w $-d_{n_1}$ times, and we toggle each leaf of w d_{n_1} times. This leaves $v_1(G_2)$ with label $-d_{n_1}$. Now the only vertices that remain to be toggled are $v_1(G_2), \ldots, v_{n_2}(G_2)$. If it were possible to toggle these vertices so that $v_1(G_2), \ldots, v_{n_2}(G_2)$ have label 0 (even if we ignore the label of w), then $\pi_{-d_{n_1}}$ would be a winnable labeling for G_2. This would imply, by Lemma 5.2(2), that π_1 is a winnable labeling, which implies that G_2 is AW by Lemma 5.2(3). This is a contradiction and completes the proof. □

We now derive a set of non-AW caterpillar graphs that we paste together to generate all non-AW caterpillars. We call a non-AW caterpillar graph *irreducibly non-AW* if it cannot be written $G_1.w.G_2(r)$ for any non-AW G_1 and G_2, and any $r \ge 0$. The following is a useful characterization of irreducibly non-AW caterpillar graphs.

Lemma 5.10. Let $k = p^e$ with p prime, and G be a caterpillar graph over with spine P_n. Then G is irreducibly non-AW over $\mathbb{Z}_k$ if and only if $\gcd(a_i, k) = 1$ for all $i \le n-1$ and $\gcd(a_n, k) \neq 1$.

Proof. Suppose that $\gcd(a_i, k) = 1$ for all $i \le n-1$ and $\gcd(a_n, k) \neq 1$. Since $\gcd(a_n, k) \neq 1$, G is not AW by Lemma 5.8. Furthermore, if $G =$

$G_1 v_i G_2$, then $\gcd(a_{i-1}, k) = 1$ implies that G_1 is AW. Thus, G is irreducibly non-AW.

Conversely, suppose that G is irreducibly non-AW. For contradiction, assume $\gcd(a_i, k) \neq 1$ for some $i \leq n-1$. Let j be minimal such that $\gcd(a_j, k) \neq 1$. We claim that $\gcd(a_{j+1}, k) = 1$. If $j = 1$, then $a_2 = 1 - m_1 m_2$. Since p divides $a_1 = -m_1$, then $a_2 \equiv 1 \pmod{p}$. If $j \geq 2$, then $a_{j+1} = m_{j+1} a_j - a_{j-1}$ and $p | a_j$ imply that $a_{j+1} \equiv a_{j-1} \pmod{p}$. In either case, $\gcd(a_{j+1}, k) = 1$. Since G is not AW, we must have $j \leq n-2$. We have $G = G_1.v_{j+1}.G_2$, where $\ell_i(G_1) = \ell_i(G)$ for $1 \leq i \leq j$, and $\ell_i(G_2) = \ell_{i+j+1}(G)$ for $1 \leq i \leq n-j-1$. We claim that G_1 and G_2 are non-AW, which would imply that G is not irreducibly AW, completing the proof.

Since $\gcd(a_j, k) \neq 1$, G_1 is non-AW. It suffices to prove that G_2 is non-AW. Suppose, for contradiction, that G_2 is AW, and let y be the number of times v_{j+2} is toggled to win π_1. As we toggle the vertices of G in an attempt to win π_1, consider the situation after v_{j+1} and its adjacent leaves are toggled. Then v_{j+1} has label $d_{j+1}(x)$, v_{j+2} has label $t_{j+1} = -d_j(x)$, and the remaining vertices have label 0. Note that only vertices in G_2 are toggled from here on out, and that v_{j+2} will be toggled $-d_{j+1}(x)$ times. But Lemma 5.4 implies that if v_{j+2} is toggled $-yd_j(x)$ times, then the game can be won. Thus, the game can be won if $-yd_j(x) \equiv -d_{j+1}(x) \pmod{k}$ can be solved for x. By substituting $d_j(x) = a_j x + b_j$ and $d_{j+1}(x) = a_{j+1}x + b_{j+1}$, this equivalence becomes

$$(a_{j+1} - a_j y)x \equiv b_j y - b_{j+1} \pmod{k}$$

We know p divides a_j but not a_{j+1}, so $\gcd(a_{j+1} - a_j y, k) = 1$. Therefore, the equivalence can be solved, which makes G an AW graph. This is a contradiction, and so G_2 is non-AW. □

This characterization of irreducibly non-AW caterpillar graphs makes them relatively straightforward to construct. Let $k = p^e$, let G be irreducibly non-AW over $\mathbb{Z}_k$, and let P_n be the spine of G. If $n = 1$, then p divides $a_1 = -m_1$, giving us p^{e-1} choices for m_1 (and thus ℓ_1) modulo k. If $n = 2$, then since p does not divide a_1, we have $p^{e-1}(p-1)$ choices for m_1 modulo k. We then need $p | a_2$, and so $m_1 m_2 \equiv 1 \pmod{p}$. If a is the inverse of m_1 modulo p, then $a + rp$ are incongruent solutions for $0 \leq r \leq p^{e-1} - 1$, giving us p^{e-1} choices for m_2, and $p^{2(e-1)}(p-1)$ possible irreducibly non-AW caterpillar graphs. For $n \geq 3$, we proceed similarly. For $j \geq 3$, suppose that we have arranged that p does not divide a_i for $1 \leq i \leq j-1$. We have $a_j = m_j a_{j-1} - a_{j-2}$, and so $p | a_j$ precisely when $m_j a_{j-1} \equiv a_{j-2} \pmod{p}$. This equivalence has a unique solution $m_j = a$, and, as in the $n = 2$ case, we get inequivalent solutions $a + rp$ modulo k, where $0 \leq r \leq p^{e-1} - 1$. If $j = n$, we can choose any of the p^{e-1} solutions modulo k, and if $j < n$, we choose any of the other $p^{e-1}(p-1)$ equivalence classes to make G irreducibly non-AW. Note that applying this process to the case $k = 2$ gives us the graphs of types T_1 and $T_{2,j}$ in Theorem 5.6(2). Thus, the task of generating all non-AW caterpillars can be reduced to inverting elements of $\mathbb{Z}_p$. We also get the following.

Corollary 5.11. Let $k = p^e$, where p is prime. If we consider each ℓ_i modulo k, then the number of irreducibly non-AW caterpillar graphs whose spine is P_n is $p^{n(e-1)}(p-1)^{n-1}$.

The following generalization of Amin and Slater's result follows directly from Lemma 5.10.

Theorem 5.12. Let $k = p^e$, where p is prime. Then all non-AW caterpillar graphs over $\mathbb{Z}_k$ can be generated by repeated applications of pasting irreducibly non-AW caterpillar graphs.

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