Graphs with fourth largest signless-Laplacian eigenvalue less than two*

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Abstract

In this paper, all connected graphs with the fourth largest signless-Laplacian eigenvalue less than two are determined.

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1 Introduction

All graphs considered here are simple (so loops and multiple edges are not allowed), undirected and finite. We use standard terminology and notation and refer to [2] for an extensive treatment of digraphs. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G) = \{v_1, \cdots, v_n\}$ and edge set $E(G)$. The order of G is the number $n = |V|$ of its vertices and its size is the number $m = |E|$ of its edges. The adjacency matrix of G is the $0-1$ $n \times n$-matrix indexed by the vertices of G and defined by $a_{ij} = 1$ if and only if $ij \in E$. Then $Q(G) = \Delta(G) + A(G)$, where $\Delta(G)$ is the diagonal matrix whose diagonal entries are the degrees in G, is called the *signless Laplacian* of G. Denote by $(q_1, q_2, \cdots, q_n)$ the Q-spectrum of G, i.e., the spectrum of the signless Laplacian of G. Since $Q(G)$ is real, symmetric and positive semidefinite, assume that the eigenvalues of G are labeled such that $q_1 \geq q_2 \geq \cdots \geq q_n$. Denote $L(G) = \Delta(G) - A(G)$ the *Laplacian matrix* of G. Since $L(G)$ is real, symmetric and positive semidefinite, the eigenvalues of $L(G)$ are denoted by $u_1(G) \geq u_2(G) \geq \cdots \geq u_{n-1}(G) \geq u_n(G) = 0$.

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Cvetković, Rowlinson and Simić [6, 7, 8, 9], discussed the development of a spectral theory of graphs based on the matrix $Q(G)$, and gave several reasons why it is superior to other graph matrices such as the adjacency and the Laplacian matrix. We refer the reader to the above cited papers for basic results on this topic.

Characterizing the structures of graphs by the eigenvalues of some graph matrix is an attractive study field. Some important results have been obtained already. For the largest eigenvalue $\rho \in [0,2]$,$(2, \sqrt{2+\sqrt{5}})$ and $(\sqrt{2+\sqrt{5}}, \frac{3}{2}\sqrt{3}]$ of the graph adjacency matrix, the corresponding graphs are respectively determined by Smith [15], Cvetković et al. [12], Brouwer and Neumaier [3] and Woo and Neumaier [18]. For the second largest eigenvalue $\lambda_2 \in (0, \frac{1}{3})$ and the third eigenvalue $\lambda_3 \in (-\infty, \frac{1-\sqrt{5}}{2})$ of the graph adjacency matrix, the corresponding graphs are found by Cao and Hong [4, 5]. X.D. Zhang [16], characterized the graphs with fourth Laplacian eigenvalue less than two. Recently, J.F. Wang et.al. [17], determined the graphs with signless Laplacian index does not exceed 4.5 and M. Aouchiche et.al. [1], characterized all simple connected graphs with second largest signless-Laplacian eigenvalue at most 3.

The rest paper is organized as follow: In section 2, we give some graphs with $q_4(G) < 2$. In section 3, we characterize all connected graphs with the fourth largest signless-Laplacian eigenvalue less than two.

2 Some graphs with $q_4(G) < 2$

Many researchers studied the relations between the Laplacian matrix and the signless Laplacian matrix of a graph. One of the most important results is the following lemma (see, for example [10, 13])

Lemma 2.1. *In bipartite graphs, the Q-polynomial is equal to the characteristic polynomial of the Laplacian.*

The next two well-known theorems in matrix theory will be used in the proofs of our results.

Theorem 2.1. *(Cauchy-Poincaré Separation Theorem [14]) Let A be an $n \times n$ Hermitian matrix with eigenvalues $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$. For a given integer r, $1 \leq r \leq n$, let A_r, denote any $r\times r$ principal submatrix of A (obtained by deleting $n-r$ rows and the corresponding columns from A). Then for each $1 \leq i \leq r$,*

$$\lambda_i(A) \geq \lambda_i(A_r) \geq \lambda_{n-r+i}(A).$$

Applying this inequality to the adjacency matrix of line graphs leads to the following result.

Theorem 2.2. *(Interlacing theorem [11]) Let G be a graph on n vertices and m edges and let e be an edge of G. Let $q_1 \geq q_2 \geq \cdots \geq q_n$ and $s_1 \geq s_2 \geq \cdots \geq s_n$ be the signless Laplacian eigenvalues of G and $G - e$, respectively. Then,*

$$q_1 \geq s_1 \geq q_2 \geq s_2 \geq \cdots \geq q_n \geq s_n.$$

By the above theorem, the following lemma can be easily obtained.

Lemma 2.2. *Let G be a simple graph of order n. If H is a subgraph of G of order $m \leq n$ (not necessarily an induced subgraph), then for $i = 1, \cdots, m$, we have*

$$q_i(G) \geq q_i(H).$$

In the following, we shall study the set $\mathcal{G}$ of all connected graphs G with the property

$$q_4(G) < 2.$$

As a directed consequence of Lemma 2.2, if $q_4(G) < 2$ and H is a subgraph of G, then $q_4(H) < 2$. The *forbidden subgraphs* of G denoted by the minimal graphs satisfy the fourth largest signless-Laplacian eigenvalue less than two. By a direct calculation, we have the following simple result.

Lemma 2.3. *The following graphs as in Fig. 1 are forbidden subgraphs of G in $\mathcal{G}$, i.e. $q_4(H_i) \geq 2$ for $i = 1, \cdots 5$.*

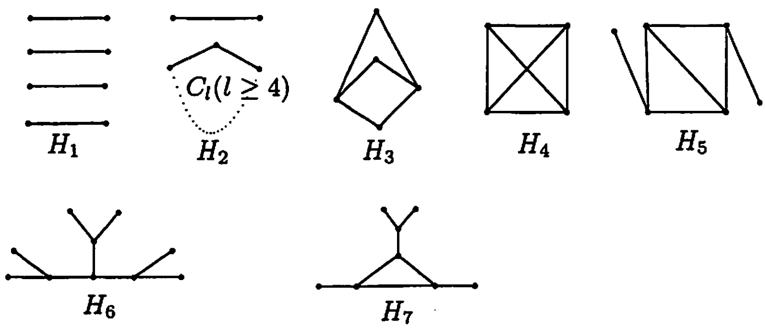
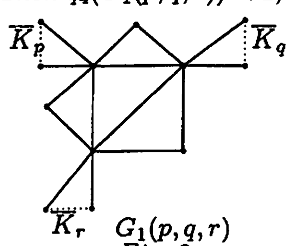


Fig. 1.

Lemma 2.4. *Let $G_1(p, q, r)$ be a graph of order $n = p + q + r + 6$ as in Fig. 2, where $p, q, r \geq 0$. Then $q_4(G_1(p, q, r)) < 2$, i.e. $G_1(p, q, r) \in \mathcal{G}$.*



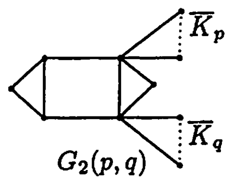
$G_1(p, q, r)$
Fig. 2.

Proof. By Lemma 2.2, we may assume that $p = q = r \geq 1$, since $G_1(p,q,r)$ can be regarded as a connected subgraph of $G_1(p+q+r, p+q+r, p+q+r)$. By a direct calculation, we can show that the characteristic polynomial of $Q(G_1(p,p,p))$ is equal to

$$(\lambda-1)^{n-9}[\lambda^3-(p+9)\lambda^2+(2p+16)\lambda-8][\lambda^3-(6+p)\lambda^2+(10+2p)\lambda-5]^2.$$

Let $f(x) = x^3-(p+9)x^2+(2p+16)x-8$ and $g(x) = x^3-(6+p)x^2+(10+2p)x-5$. Then $f(0) = -8 < 0$, $f(1) = p > 0$, $f(2) = -4 < 0$ and $f(p+11) = 4p^2+82p+410 > 0$. Hence $f(x) = 0$ has exactly one root no less than two. Similarly, we have $g(0) = -5 < 0$, $g(1) = p > 0$, $g(2) = -1 < 0$ and $g(p+6) = 2p^2+22p+55 > 0$. Hence $g(x) = 0$ has exactly one root no less than two. Therefore $Q(G_1(p,p,p))$ has exactly three eigenvalues no less than two. So $q_4(p,q,r) < 2$ and $G_1(p,q,r) \in \mathcal{G}$. □

Lemma 2.5. *Let $G_2(p,q)$ be a graph of order $n = p+q+6$ as in Fig. 3, where $p, q > 0$. Then $q_4(G_2(p,q)) < 2$, i.e. $G_2(p,q) \in \mathcal{G}$.*



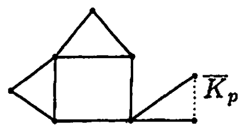
$G_2(p,q)$
Fig. 3.

Proof. By Lemma 2.2, we may assume that $p = q \geq 4$. By a direct calculation, we can show that the characteristic polynomial of $Q(G_2(p,p))$ is equal to

$$(\lambda-1)^{n-8}[\lambda^5-(13+p)\lambda^4+(59+8p)\lambda^3-(115+18p)\lambda^2+(100+12p)\lambda$$
$$-32][\lambda^3-(p+5)\lambda^2+(2p+7)\lambda-3].$$

Let $f(x) = x^5-(13+p)x^4+(59+8p)x^3-(115+18p)x^2+(100+12p)x-32$ and $g(x) = x^3-(p+5)x^2+(2p+7)x-3$. Then $f(0) = -32 < 0$, $f(1) = p > 0$, $f(2-\frac{1}{p}) = -6\frac{1}{p} + 7(\frac{1}{p})^2 + 4(\frac{1}{p})^3 - 3(\frac{1}{p})^4 - (\frac{1}{p})^5 < 0$, $f(2) = 4 > 0$, $f(p+2) = -(3p+2)(p^3-p^2-3p-2) < 0$ and $f(p+6) = p^4+17p^3+95p^2+196p+100 > 0$. Hence $f(x) = 0$ has exactly two roots no less than two. Similarly, we have $g(0) = -3 < 0$, $g(1) = p > 0$, $g(2) = -1 < 0$ and $g(p+4) = p^2+7p+9 > 0$. Hence, $g(x) = 0$ has exactly one root no less than two. Therefore, $Q(G_2(p,p))$ has exactly three eigenvalues no less than two. So $q_4(Q(G_2(p,p)) < 2$ and then $q_4(Q(G_2(p,q)) < 2$, that is, $G_2(p,q) \in \mathcal{G}$. □

Lemma 2.6. *Let $G_3(p)$ be a graph of order $n = p+6$ as in Fig. 4, where $p \geq 4$. Then $q_4(G_3(p)) < 2$, i.e. $G_3(p) \in \mathcal{G}$.*



$G_3(p)$

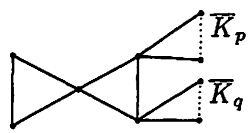
Fig. 4.

Proof. By Lemma 2.2, we may assume that $p \geq 4$. By a direct calculation, we can show that the characteristic polynomial of $Q(G_3(p))$ is equal to

$$(\lambda-1)^{n-7}(\lambda^2-5\lambda+5)[\lambda^5-(12+p)\lambda^4+(48+9p)\lambda^3-(21p+81)\lambda^2$$
$$+(60+14p)\lambda-16]$$

Let $f(x)=x^2-5x+5$ and $g(x)=x^5-(12+p)x^4+(48+9p)x^3-(21p+81)x^2+(60+14p)x-16$. Then $g(0)=-16<0$, $g(1)=p>0$, $g(2-\frac{1}{p})=-2+\frac{1}{p}-2(\frac{1}{p})^2+7(\frac{1}{p})^3-2(\frac{1}{p})^4-(\frac{1}{p})^5<0$, $g(2)=4>0$, $g(p+3)=2-6p-4p^2<0$ and $g(p+6)=20+360p+3p^4+45p^3+221p^2>0$. Hence $g(x)=0$ has exactly two roots no less than two. It is easy to see that $f(x)=0$ has exactly one root no less than two. Therefore, $Q(G_3(p))$ has exactly three eigenvalues no less than two. So $q_4(Q(G_3(p))<2$ and $G_3(p)\in\mathcal{G}$. □

Lemma 2.7. *Let $G_4(p,q)$ be a graph of order $n=p+q+5$ as in Fig. 5, where $p\geq 4$. Then $q_4(G_4(p,q))<2$, i.e. $G_4(p,q)\in\mathcal{G}$.*



$G_4(p,q)$

Fig. 5.

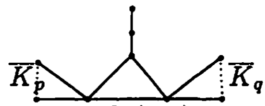
Proof. By Lemma 2.2, we may assume that $p=q\geq 4$. By a direct calculation, we can show that the characteristic polynomial of $Q(G_4(p,q))$ is equal to

$$(\lambda-1)^{n-6}[\lambda^4-(p+11)\lambda^3+(7p+39)\lambda^2-(10p+53)\lambda$$
$$+24]\lambda^2-(p+2)\lambda+1]$$

Let $f(x)=x^4-(p+11)x^3+(7p+39)x^2-(10p+53)x+24$ and $g(x)=x^2-px-2x+1$. Then $f(0)=24>0$, $f(1)=-4p<0$, $f(2)=2>0$, $f(p+1)=-p(3p^2-13p+8)<0$ and $f(p+6)=2p^3+23p^2+67p-18>0$. Hence $f(x)=0$ has exactly two roots no less than two. It is easy to see that $g(x)=0$ has exactly one root no less than two. Therefore, $Q(G_4(p,q))$ has exactly three eigenvalues no less than two. So $q_4(Q(G_4(p,q))<2$ and $G_4(p,q)\in\mathcal{G}$.

□

Lemma 2.8. *Let $G_5(p,q)$ be a graph of order $n = p+q+5$ as in Fig. 6, where $p, q \geq 4$. Then $q_4(G_5(p,q)) < 2$, i.e. $G_5(p,q) \in \mathcal{G}$.*



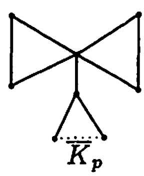
$G_5(p,q)$
Fig. 6.

Proof. By Lemma 2.2, we may assume that $p = q \geq 4$. By a direct calculation, we can show that the characteristic polynomial of $Q(G_5(p,q))$ is equal to

$$(\lambda-1)^{n-7}[\lambda^5 - (p+10)\lambda^4 + (34+6p)\lambda^3 - (48+9p)\lambda^2 + (2p+27)\lambda$$
$$-4][\lambda^2 - (p+2)\lambda + 1]$$

Let $f(x) = x^5 - (p+10)x^4 + (34+6p)x^3 - (48+9p)x^2 + (2p+27)x - 4$ and $g(x) = x^2 - px - 2x + 1$. Then $f(0) = -4 < 0$, $f(0.2) = -824/3125 + 54/625p > 0$, $f(1) = -2p < 0$, $f(2) = 2 > 0$, $f(p+1) = -p(4 - 2p + 3p^3 - 7p^2) < 0$ and $f(p+6) = 590 + 651p + 242p^2 + 2p^4 + 37p^3 > 0$. Hence $f(x) = 0$ has exactly two roots no less than two. It is easy to see that $g(x) = 0$ has exactly one root no less than two. Therefore, $Q(G_5(p,q))$ has exactly three eigenvalues no less than two. So $q_4(Q(G_5(p,q)) < 2$ and $G_5(p,q) \in \mathcal{G}$. □

Lemma 2.9. *Let $G_6(p)$ be a graph of order $n = p+6$ as in Fig. 7, where $p \geq 6$. Then $q_4(G_6(p)) < 2$, i.e. $G_6(p) \in \mathcal{G}$.*



$G_6(p)$
Fig. 7.

Proof. By Lemma 2.2, we may assume that $p \geq 6$. By a direct calculation, we can show that the characteristic polynomial of $Q(G_6(p))$ is equal to

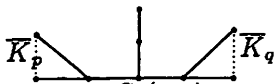
$$(\lambda-1)^{n-5}(\lambda-3)[\lambda^4 - (p+10)\lambda^3 + (27+8p)\lambda^2 - (26+11p)\lambda + 8].$$

Let $f(x) = x^4 - (p+10)x^3 + (27+8p)x^2 - (26+11p)x + 8$. Then $f(0) = 8 > 0$, $f(1) = -4p < 0$, $f(2) = 2p > 0$, $f(p) = -2p^3 + 16p^2 - 26p + 8 < 0$ and $f(p+4) = 2p^3 + 8p^2 + 30p - 92 > 0$. Hence $f(x) = 0$ has exactly two roots no less than two. Hence $q_4(Q(G_6(p))) < 2$, that is, $G_6(p) \in \mathcal{G}$.

□

Combining Lemma 2.1 with the result of X.D. Zhang in [16], we have the following lemma.

Lemma 2.10. *Let $G_8(p,q)$ be a tree of order $n = p+q+5$ as in Fig. 8, where $p, q \geq 0$. Then $q_4(Q(G_7(p,q)) < 2$.*



$G_7(p,q)$

Fig. 8.

3 All connected graphs with $q_4(G) < 2$

In this section, we characterize all connected graphs whose fourth largest signless-Laplacian eigenvalue is less than two. Denote by Γ_n the set of all connected graphs of order n that do not have any subgraphs isomorphic to one of H_1-H_7 in Fig. 1.

Lemma 3.1. *Let $G \in \Gamma_n$. If G contains a cycle of order 4, then G must be a connected subgraph of one of the graphs $G_1(p,q,r)$, $G_2(p,q)$, $G_3(p)$ and G_8 as in Fig. 9.*

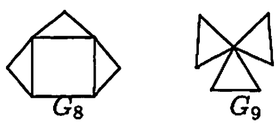


Fig. 9.

Proof. Let $G \in \Gamma_n$ be a connected graph of order n on vertex set $V = \{v_1, \cdots, v_n\}$. Since G contains a cycle of order 4 and does not contain K_4, we only need to consider the following two cases.

Case 1. G contains a $K_4 - e$ (the graph obtained from a complete graph of order 4 by deleting an edge) as an induced subgraph, say $v_1 \sim v_i$ for $i = 2,3,4$, $v_2 \sim v_3$ and $v_3 \sim v_4$, where $\sim$ stands for the adjacency relationship, in the sequel. Since H_2 is a forbidden subgraph, the subgraph of G induced by vertex set $U = \{v_5, \cdots, v_n\}$ has no edges. Moreover, since H_3 is a forbidden subgraph, no vertex in U is adjacent to three vertices in $\{v_1, v_2, v_3, v_4\}$. Therefore, at most two vertices of U are adjacent to two vertices in $\{v_1, v_2, v_3, v_4\}$. Hence we consider the following three subcases.

Subcase 1.1. There exist two vertices, say v_5, v_6, in U such that they are adjacent to two vertices in $\{v_1, v_2, v_3, v_4\}$, respectively. Since H_3 is a forbidden subgraph, we may assume that $v_5 \sim v_1$ and $v_5 \sim v_2$. Then $v_6 \nsim v_4$, since H_5 is a forbidden subgraph. So v_6 is adjacent to two vertices in

$\{v_1, v_2, v_3\}$. Note that H_2 and H_3 are forbidden subgraphs. Then $v_6 \sim v_2$, $v_6 \sim v_3$ and $v_i \nsim v_4$ for $i = 7, \cdots, n$. Hence G is a subgraph of $G_1(p, q, r)$.

Subcase 1.2. Precisely one vertex of U is adjacent to two vertices in $\{v_1, v_2, v_3, v_4\}$. Since H_3 is a forbidden subgraph, we may assume that $v_5 \sim v_1$ and $v_5 \sim v_2$. Since H_5 is a forbidden subgraph, $v_i \nsim v_4$ for $i = 6, \cdots, n$. So G must be a subgraph of $G_1(p, q, r)$.

Subcase 1.3. No vertex in U is adjacent to two vertices in $\{v_1, v_2, v_3, v_4\}$. Since H_1 is a forbidden subgraph, there are no four disjoint edges having one of their end vertices in $\{v_1, v_2, v_3, v_4\}$ and the other one in U. Moreover, since H_5 is a forbidden subgraph of G, there are no two disjoint edges having one of their end vertices in $\{v_2, v_4\}$ and the other one in U. Hence G must be a connected subgraph of $G_1(p, q, r)$.

Case 2. G contains a cycle of order 4 as an induced subgraph, say $v_1v_2v_3v_4$. Since H_2 is a forbidden subgraph, the subgraph of G induced by vertex set $U = \{v_5, \cdots, v_n\}$ has no edges. Moreover, since H_1 is a forbidden subgraph, there are no four vertices of U adjacent to two vertices of $\{v_1, v_2, v_3, v_4\}$. Hence we consider the following four subcases.

Subcase 2.1. There exist three vertices, say v_5, v_6, v_7, in U such that they are adjacent to two vertices in $\{v_1, v_2, v_3, v_4\}$, respectively. Since H_3 is a forbidden subgraph, we may assume that $v_5 \sim v_1$ and $v_5 \sim v_2$. Similarly, we assume that $v_6 \sim v_1$ and $v_6 \sim v_4$ and $v_7 \sim v_4$ and $v_7 \sim v_3$. Since H_1 is a forbidden subgraph, there are no vertices in U adjacent to $\{v_1, v_2, v_3, v_4\}$. Therefore G must be G_8. □

Subcase 2.2. There exist two vertices, say v_5, v_6, in U such that they are adjacent to two vertices in $\{v_1, v_2, v_3, v_4\}$, respectively. Since H_3 is a forbidden subgraph, without loss of generality, we assume that $v_5 \sim v_1$ and $v_5 \sim v_2$. If $v_6 \sim v_3$ and $v_6 \sim v_4$, there exists a vertex, say $v_7 \in U$ is adjacent to one vertex of $\{v_1, v_2, v_3, v_4\}$, by symmetric, we may assume that $v_7 \sim v_1$. Hence $v_i \nsim v_3$, v_4 for $i = 8, \cdots, n$ since H_1 is a forbidden subgraph. Therefore, G must be a connected subgraph of $G_2(p, q)$. If $v_6 \sim v_1$ and $v_6 \sim v_4$ (or $v_6 \sim v_2$ and $v_6 \sim v_3$), without loss of generality, we assume that $v_6 \sim v_1$ and $v_6 \sim v_4$. If there exists a vertex of U, say v_7 is adjacent to v_1, v_2 or v_4, we have $v_i \nsim v_3$ $(i = 8, \cdots, n)$ since H_1 is a forbidden subgraph. Hence G must be a connected subgraph of $G_1(p, q, r)$. If there exists a vertex of U, say v_7 is adjacent to v_3, we have $v_i \nsim v_1$, v_2 and v_4 $(i = 8, \cdots, n)$ since H_1 is a forbidden subgraph. Therefore G must be a connected subgraph of $G_3(p)$.

Subcase 2.3. Precisely one vertex of U is adjacent to two vertices in $\{v_1, v_2, v_3, v_4\}$. Since H_3 is a forbidden subgraph, we may assume that $v_5 \sim v_1$ and $v_5 \sim v_2$. If there exists no vertex of U is adjacent to v_3 and v_4, then G must be a connected subgraph of $G_1(p, q, r)$. If there exists a vertex of U, say v_6, is adjacent to v_3 (or v_4), we may assume that $v_3 \sim v_6$. If there exists a vertex $v_7 \sim v_1$ or v_2, $v_i \nsim v_3$, $i = 8, \cdots, n$ since H_1 is a forbidden subgraph. Hence G must be a connected subgraph of $G_1(p, q, r)$. If $v_7 \sim v_3$, $v_i \nsim v_1$ and v_2, for $i = 8, \cdots, n$, then G must be a connected subgraph of $G_2(p, q)$.

Subcase 2.4. No vertex in U is adjacent to two vertices in $\{v_1, v_2, v_3, v_4\}$. Since H_1 is a forbidden subgraph, there are no four disjoint edges having one of their end vertices in $\{v_1, v_2, v_3, v_4\}$ and the other one in U. Therefore, G must be a subgraph of $G_1(p, q, r)$.

Lemma 3.2. *Let $G \in \Gamma_n$. If G does not contain a cycle of order 4 and contains a cycle of order 3, then G must be a subgraph of one of the graphs $G_1(p, q, r)$, $G_2(p, q)$, $G_3(p)$, $G_4(p, q)$, $G_5(p, q)$, $G_6(p)$, G_8 and G_9 as in Fig. 9.*

Proof. Let G contain a cycle of order 3, say v_1, v_2, v_3. Since H_1 is a forbidden subgraph, there exist at most two disjoint edges in the induced subgraph of $U = \{v_4, \cdots, v_n\}$. If the induced subgraph $G[U]$ has no edges, then G must be a subgraph of $G_1(p, q, r)$. We first consider when $G[U]$ contains exactly one edge. Hence we may assume that the induced subgraph $G[U]$ consists of a star graph $K_{1,s}$, say $v_4, \cdots, v_{s+4}$, $s \geq 1$ and some isolated vertices, where $v_4 \sim v_i$ for $i = 5, \cdots, s+4$. We only need to consider the following two cases.

Case 1. v_4 is adjacent to one vertex in $\{v_1, v_2, v_3\}$, say $v_4 \sim v_1$. Then $v_i \nsim v_j$ for $i = 4, \cdots, s+4$ and $j = 2, 3$, since G does not contain a cycle of order 4. We consider the following two subcases.

Subcase 1.1. There exist two vertices, say v_{s+5}, v_{s+6}, in $\{v_{s+5}, \cdots, v_n\}$ such that $v_{s+5} \sim v_2$ and $v_{s+6} \sim v_3$. Since H_1 is a forbidden subgraph, $v_i \nsim v_1$ for $i = s+7, \cdots, n$. Moreover, since H_7 is a forbidden subgraph, we have $s = 1$. Hence G must be a subgraph of $G_5(p, q)$.

Subcase 1.2. There exists at most one vertex in $\{v_{s+5}, \cdots, v_n\}$ that is adjacent to one of v_2 and v_3, say $v_i \nsim v_3$ for $i = s+5, \cdots, n$. Since G does not contain a cycle of order 4, there exists at most one vertex in $\{v_5, \cdots, v_{s+4}\}$ that is adjacent to v_1. Hence G must be a subgraph of $G_1(p, q, r)$.

Case 2. $v_4 \nsim v_i$ for $i = 1,2,3$. Then there exists one vertex, say v_5, in $\{v_5, \cdots, v_{s+4}\}$ that is adjacent to one vertex in $\{v_1, v_2, v_3\}$, say, $v_5 \sim v_1$. Then $v_i \nsim v_1$ for $i = 6, \cdots, s+4$, since G does not contain a cycle of order 4. Moreover, since H_2 is a forbidden subgraph, there exists at most one vertex of $\{v_6, \cdots, v_{s+4}\}$ is adjacent to one of v_2 and v_3. Hence we consider the following two subcases.

Subcase 2.1. There exists only one vertex, say v_6, in $\{v_6, \cdots, v_{s+4}\}$ such that v_6 is adjacent to one vertex in $\{v_2, v_3\}$, say $v_6 \sim v_2$. Since H_2 is a forbidden subgraph, $v_i \nsim v_3$ for $i = s+5, \cdots, n$, then v_i is adjacent to v_1 or v_2 for $i = s+5, \cdots, n$. Hence G must be a subgraph of $G_1(p,q,r)$.

Subcase 2.2. $v_i \nsim v_j$ for $i = 6, \cdots, s+4$ and $j = 2,3$. If there exist at least two vertices, say v_{s+5} and v_{s+6}, in $\{v_{s+5}, \cdots, v_n\}$ such that $v_{s+5} \sim v_2$ and $v_{s+6} \sim v_3$, then $s = 1$ and $v_i \nsim v_1$ for $i = s+5, \cdots, n$, since G does not contain H_1 as a subgraph. Hence G must be a subgraph of $G_5(p,q)$. Therefore, we may assume that $v_i \nsim v_3$ for $i = 4, \cdots, n$. Then G must be a subgraph of $G_1(p,q,r)$.

In the following, we will consider when $G[U]$ contains two disjoint edges. Hence we may assume that the induced subgraph $G[U]$ consists of two star graphs $K_{1,s}$ and $K'_{1,t}$ $s,t \geq 1$. Since H_1 is a forbidden subgraph, there are no isolated vertices. We suppose $V(K_{1,s}) = \{v_4, \cdots, v_{s+4}\}$ and $V(K'_{1,t}) = \{v_{s+5}, \cdots, v_n\}$, where $v_4 \sim v_i$ for $i = 5, \cdots, s+4$; $v_{s+5} \sim v_j$ for $j = s+6, \cdots, n$. In the following, we consider the following two cases.

Case 1. $K_{1,s}$ and $K'_{1,t}$ are not connected.

Subcase 1.1. Both v_4 and v_{s+5} are adjacent to vertices of $\{v_1, v_2, v_3\}$. We shall discuss by the following two subcases.

Subcase 1.1.1 v_4 and v_{s+5} are adjacent to the same vertex of $\{v_1, v_2, v_3\}$, say $v_1 \sim v_4$ and $v_1 \sim v_{s+5}$. Since G does not contain a cycle of order 4, then $v_i \nsim v_2, v_3$ $(i = 5, \cdots, s+4, s+6, \cdots, v_n)$. If $v_i \nsim v_1$ $(i = 5, \cdots, s+4)$ and $v_j \nsim v_1$ $(j = s+6, \cdots, n)$, then G must be a subgraph of $G_4(p,q)$. If there exists exactly a vertex of $\{v_5, \cdots, v_{s+4}\}$ or a vertex of $\{v_{s+6}, \cdots, v_n\}$ is adjacent to v_1, by symmetry, suppose that $v_5 \sim v_1$, then $s = 1$. Hence G is the subgraph of $G_6(p)$. If there exist both a vertex of $\{v_5, \cdots, v_{s+4}\}$ and a vertex of $\{v_{s+6}, \cdots, v_n\}$ are adjacent to v_1, then $s = 1$ and $n = 7$. Hence G is the graph G_9.

Subcase 1.1.2. v_4 and v_{s+5} are adjacent to the different vertices of $\{v_1, v_2, v_3\}$, say $v_4 \sim v_1$ and $v_{s+5} \sim v_2$. Then $v_i \nsim v_2, v_3$ for $i = 5, \cdots, s+4$ and $v_j \nsim v_1, v_3$ for $j = s+6, \cdots, n$ since G does not contain a cycle of order 4. If $v_i \nsim v_1$ $(i = 5, \cdots, s+4)$ and $v_j \nsim v_2$ $(j = s+6, \cdots, n)$, then G must be a subgraph of $G_4(p,q)$. If there exists exactly a vertex of $\{v_5, \cdots, v_{s+4}\}$

or a vertex of $\{v_{s+6}, \cdots, v_n\}$ is adjacent to v_1, by symmetry, suppose that $v_5 \sim v_1$, then $s = 1$. Hence G is the subgraph of $G_3(p)$. If there exist both a vertex of $\{v_5, \cdots, v_{s+4}\}$ and a vertex of $\{v_{s+6}, \cdots, v_n\}$, then $s = 1$ and $n = 7$. Hence G is a subgraph of G_8.

Subcase 1.2. At most one of v_4 and v_{s+5} is adjacent to the a vertex of $\{v_1, v_2, v_3\}$, say, $v_i \nsim v_{s+5}$ for $i = 1, 2, 3$. Then there exists a vertex say v_{s+6} in $\{v_{s+6}, \cdots, v_n\}$ is adjacent to a vertex of $\{v_1, v_2, v_3\}$ such that $v_{s+6} \sim v_1$. Then $n = s + 6$. Hence G is a subgraph of $G_3(p)$.

Subcase 1.3. None of v_4 and v_{s+5} is adjacent to the a vertex of $\{v_1, v_2, v_3\}$. Then there exists a vertex of $\{v_5, \cdots, v_n\}$, say v_5, and a vertex of $\{v_{s+6}, \cdots, v_n\}$, say v_{s+6}, adjacent to a vertex of $\{v_1, v_2, v_3\}$. Then we have $s = 1$ and $n = 7$. If v_5 and v_7 are adjacent to a same vertex of $\{v_1, v_2, v_3\}$, say $v_5 \sim v_1$ and $v_7 \sim v_1$, then G must be a subgraph of $G_3(p)$. If v_5 and v_7 are adjacent to the different vertices of $\{v_1, v_2, v_3\}$, say $v_5 \sim v_1$ and $v_7 \sim v_2$, then G must be a subgraph of G_8.

Case 2. $K_{1,s}$ and $K'_{1,t}$ are connected.

Subcase 2.1. v_4 is adjacent to v_{s+5}. Since G does not contain a cycle of order 4, we have v_4 and v_{s+5} do not adjacent to different vertices of $\{v_1, v_2, v_3\}$. Then we follow the following three subcases.

Subcase 2.1.1. v_4 and v_{s+5} are adjacent to a vertex of $\{v_1, v_2, v_3\}$, say $v_1 \sim v_4$ and $v_1 \sim v_{s+5}$. Since G does not contain a cycle of order 4, we have $v_i \nsim \{v_1, v_2, v_3\}$ for $i = 5, \cdots, s+4, s+6, \cdots, n$. Then G be a subgraph of $G_4(p, q)$.

Subcase 2.1.2. Exactly one of v_4 and v_{s+5} is adjacent to a vertex of $\{v_1, v_2, v_3\}$, say $v_4 \sim v_1$.

If there exists a vertex of $\{v_{s+6}, \cdots, v_n\}$ say, v_{s+6} is adjacent to v_2 or v_3, we have $n = s + 6$ since H_2 is a forbidden subgraph. If $v_i \nsim v_1$, $(i = 5, \cdots, s+4)$, then G is a subgraph of $G_2(p, q)$. If there exists some vertex, say v_6 is adjacent to v_1, then $s = 1$. Hence G is a subgraph of G_8.

If $v_i \nsim v_2, v_3$, for $i = s+6, \cdots, n$. If $v_i \nsim v_1$, $(i = 5, \cdots, s+4)$, then G is a subgraph of $G_2(p, q)$. If there exists some vertex, say v_6 is adjacent to v_1, then $n = s + 6$. Hence G is the graph $G_3(p)$.

Subcase 2.1.3. Both of v_4 and v_{s+5} are non-adjacent to $\{v_1, v_2, v_3\}$. Then there exists a vertex of $\{v_5, \cdots, v_{s+4}, v_{s+6}, \cdots, v_n\}$ is adjacent to a vertex of $\{v_1, v_2, v_3\}$, say $v_5 \sim v_1$. Since H_1 is a forbidden subgraph, we have $s = 1$. Since there are no cycle of order 4, we have $v_i \nsim v_1$ for $i = s+6, \cdots, n$. If there exists a vertex of $v_{s+6}, \cdots, v_n$ is adjacent to v_2 or

v_3, say $v_{s+6} \sim v_2$, then $n = 7$. Hence G is a subgraph of G_8. If $v_i \nsim v_2, v_3$ for $i = s+6, \cdots, n$, then G is the subgraph of $G_2(p,q)$.

Subcase 2.2. v_4 is adjacent to one vertex of $\{v_{s+6}, \cdots, v_n\}$, say $v_4 \sim v_{s+6}$. Since H_1 is a forbidden subgraph, we have v_4 and v_{s+5} do not adjacent to the same vertex of $\{v_1, v_2, v_3\}$. Then we follow the following three subcases.

Subcase 2.2.1. v_4 and v_{s+5} are adjacent to vertices of $\{v_1, v_2, v_3\}$, say $v_1 \sim v_4$ and $v_2 \sim v_{s+5}$. If $v_i \nsim v_1$ for $i = 5, \cdots, s+5$ and $v_j \nsim v_2$ for $i = s+6, \cdots, n$, then G is a subgraph of $G_2(p,q)$. If there exists a vertex of $\{v_5, \cdots, v_{s+5}\}$, say $v_5 \sim v_1$ or there exists a vertex of $\{v_{s+6}, \cdots, v_n\}$, say $v_{s+6} \sim v_2$, then $s = 2$ and $n = 7$. Hence G is the subgraph of G_8.

Subcase 2.2.2. Exactly one of v_4 and v_{s+5} is adjacent to a vertex of $\{v_1, v_2, v_3\}$, say $v_4 \sim v_1$. Since G does not contain a cycle of order 4, therefore $v_i \nsim v_2, v_3$ $(i = 5, \cdots, s+4, s+6)$ and $v_j \nsim v_1$ $(j = s+5, s+7, \cdots, n)$. If $v_i \nsim v_1$, v_2, v_3 for $i = 5, \cdots, s+4, s+7, \cdots, n$, then G is a subgraph of $G_2(p,q)$. If there exists a vertex of $\{v_5, \cdots, v_{s+5}\}$, say $v_5 \sim v_1$ or a vertex of $\{v_{s+7}, \cdots, v_n\}$ say $v_{s+7} \sim v_2$ (or v_3), then $s = 2$ and $n = 7$. Hence G is a subgraph of G_8. If $v_{s+6} \sim v_1$, then G is the subgraph of $G_3(p)$ or $G_4(p,q)$.

Subcase 2.2.3. Both of v_4 and v_{s+5} are non-adjacent to $\{v_1, v_2, v_3\}$. If v_{s+6} is adjacent to a vertex of $\{v_1, v_2, v_3\}$, then similarly, we have G is the subgraph of $G_3(p)$ or $G_4(p,q)$. If there exists a vertex of $\{v_5, \cdots, v_{s+4}, v_{s+7}, \cdots,$ then we have $s = 2$ and $n = 7$. Hence G is a subgraph of G_8.

Subcase 2.3. There exists a vertex of $\{v_5, \cdots, v_{s+4}\}$ is adjacent to a vertex of $\{v_{s+6}, \cdots, v_n\}$, say $v_5 \sim v_{s+6}$. Since H_1 is a forbidden subgraph, we have $n = s+6$, or $s = 1$. By symmetry, we suppose that $n = s+6$. Since H_2 is a forbidden subgraph, we have there are no two vertices of $\{v_4, \cdots, v_n\}$ are adjacent to a vertex of $\{v_1, v_2, v_3\}$. Then we follow the following three subcases.

Subcase 2.3.1. v_4 and v_{s+5} are adjacent to vertices of $\{v_1, v_2, v_3\}$, since H_2 is a forbidden subgraph, v_4 and v_{s+5} are not adjacent to the same vertex of $\{v_1, v_2, v_3\}$, we may assume that $v_1 \sim v_4$ and $v_2 \sim v_{s+5}$. Then $s = 1$ and $n = 7$ and $v_i \nsim v_j$ for $i = 1, 2, 3$ and $j = 5, 7$, therefore G is the subgraph of G_8.

Subcase 2.3.2. Exactly one of v_4 and v_{s+5} is adjacent to a vertex of $\{v_1, v_2, v_3\}$, say v_1. If $v_4 \sim v_1$, then $v_i \nsim v_1$ for $i = 5, \cdots, s+3$ and $v_{s+4} \nsim v_2, v_3$ since H_1 is a forbidden subgraph. Moreover, since H_2 is a forbidden subgraph, $v_{s+6} \nsim v_1$. If $v_{s+4} \sim v_1$, then $s = 1$, $n = 7$ and

$v_{s+6}(v_7) \nsim v_2, v_3$. Hence G is the subgraph of G_8. Similarly, if $v_{s+4} \nsim v_1$ and $v_{s+6} \nsim v_2, v_3$, then G must be the subgraph of $G_2(p,q)$. If $v_{s+4} \nsim v_1$ and $v_{s+6} \sim v_2$ or v_3, then $s = 1$ and $n = 7$, hence G is a subgraph of G_8. If $v_{s+5} \sim v_1$, then $s = 1$ and $n = 7$, therefore G is the subgraph of G_8.

Subcase 2.2.3. Both of v_4 and v_{s+5} are non-adjacent to $\{v_1, v_2, v_3\}$. Since H_1 is the forbidden subgraph, there is no vertex of $\{v_5, \cdots, v_{s+3}\}$ is adjacent to $\{v_1, v_2, v_3\}$. Then either v_{s+4} or s_{s+5} is adjacent to a vertex of $\{v_1, v_2, v_3\}$, hence we have $s = 1$ and $v_7 \nsim v_1$ since G does not contain a cycle of order 4. Then G is the subgraph of G_8. □

Lemma 3.3. *Let $G \in \Gamma_n$. If G does not contain a cycle of order 3, 4 and contains a cycle of order 5, then G is a subgraph of $G_1(p,q,r)$.*

Proof. Since G does not contain a cycle of order 3, 4, and contains a cycle of order 5, G contains a cycle of order 5 as an induced subgraph of G, say $v_i \sim v_{i+1}$ for $i = 1, 2, 3, 4$ and $v_1 \sim v_5$. Moreover, v_i is not adjacent to two vertices in $\{v_1, \cdots, v_5\}$ for $i = 6, \cdots, n$ since G does not contain a cycle of order 3. On the other hand, since H_2 is a forbidden subgraph, the subgraph of G induced by the vertex set $U = \{v_6, \cdots, v_n\}$ contains no edges. We may assume that $v_6 \sim v_1$. If there exists a vertex, say v_7, in U such that $v_7 \sim v_2$ (or $v_7 \sim v_5$), then $v_i \nsim v_j$ for $i = 8, \cdots, n$ and $j = 3, 5$ (or 2), since G does not contain H_1 as a subgraph. Hence G must be a subgraph of $G_1(p,q,r)$. If $v_i \nsim v_j$, for $i = 7, \cdots, n$, $j = 2, 5$, then G must be a subgraph of $G_1(p,q,r)$. □

Lemma 3.4. *Let $G \in \Gamma_n$. If G does not contain a cycle of order 3, 4, 5 and contains a cycle of order $s \geq 6$, then G is a subgraph of one of $G_1(p,q,r)$ and the cycle of order 7.*

Proof. Since H_1 is a forbidden subgraph of G, G does not contain a cycle of order $s \geq 8$. Moreover, if G contains a cycle of order 7, then G must be the graph of the cycle of order 7. Hence G contains a cycle of order 6 as an induced subgraph of G, say $v_i \sim v_{i+1}$ for $i = 1, \cdots, 5$ and $v_1 \sim v_6$. On the other hand, since G does not contain H_2 as a subgraph, the subgraph of G induced by the vertex set $U = \{v_7, \cdots, v_n\}$ has no edges. We may assume that $v_7 \sim v_1$. Then $v_7 \nsim v_j$ for $j = 2, \cdots, 6$. Further, $v_i \nsim v_j$ for $i = 8, \cdots, n$ and $j = 2, 4, 6$. Hence G must be a subgraph of $G_1(p,q,r)$. □

Combining Lemma 2.1 with the result of X.D. Zhang in [16], we get the following lemma immediately.

Lemma 3.5. *Let $G \in \Gamma_n$. If G is a tree, then G must be a subgraph of $G_1(p,q,r)$ or $G_7(p,q)$.*

We sum up the results of Lemma 3.1-3.5 as follows.

Theorem 3.1. *A connected graph G of order n belongs to Γ_n, i.e. a connected graph G does not contain any subgraph isomorphic to any one of the graphs H_1-H_7 if and only if G is a subgraph of one of the following graphs: $G_1(p,q,r)$, $G_2(p,q)$, $G_3(p)$, $G_4(p,q)$, $G_5(p,q)$, $G_6(p)$, $G_7(p,q)$, G_8, G_9 and the cycle of order 7.*

We now present our main result.

Theorem 3.2. *A connected graph G of order $n \geq 4$ satisfies $q_4(G) < 2$ if and only if G is a subgraph of one of the following graphs: $G_1(p,q,r)$, $G_2(p,q)$, $G_3(p)$, $G_4(p,q)$, $G_5(p,q)$, $G_6(p)$, $G_7(p,q)$, G_8, G_9 and the cycle of order 7.*

Proof. The assertions follow from Theorem 3.1 and Lemmas 2.3-2.9. □

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