# Involution fixed sets of the $M_{24}$ maximal 2-local geometry chamber graph

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**Abstract**

Let $\Gamma$ be the rank three $M_{24}$ maximal 2-local geometry. For the two conjugacy types of involution in $M_{24}$, we describe the fixed point sets of chambers in $\Gamma$.

**keywords:** maximal 2-local geometry, chamber graph, involution, fixed sets

# Introduction

Let $\Gamma$ denote the rank three $M_{24}$ maximal 2-local geometry - the elements of $\Gamma$ being the octads, trios and sextets of the Steiner system $S(24, 8, 5)$. An octad is defined to be incident with a trio if it is one of the octads of the trio, incident with a sextet if it is the union of two of the tetrads of the sextet. A trio and a sextet are incident if the three octads of the trio are unions of the tetrads of the sextet.

In [2] the combinatorial structure of $\mathcal{C}$, the chamber graph of $\Gamma$, is analysed extensively. For background on chamber systems and sporadic group geometries see [3]. The purpose of this note is to investigate and describe in detail the fixed point sets of chambers for involutions in $M_{24}$, using [2] as our starting point. Since for all $\gamma \in \Gamma$, $Stab_{M_{24}}\gamma$ is a 2-local subgroup of $M_{24}$, $\Gamma$ is arguably a characteristic 2 geometry and so such sets are of interest. In fact, these sets have featured in some of the calculations in [4].

Put $G = M_{24}$ and let $\Omega$ be a 24-element set. We assume that the Steiner system $S(24, 8, 5)$ on $\Omega$ which $G$ stabilizes is given by the MOG [1]. So we have

$\Omega =$

| $O_1$ | $O_2$ | $O_3$ |
|---|---|---|
| | | |

where $O_1, O_2$ and $O_3$ are the heavy bricks. We denote the set of all sextets of $\Omega$ by $\mathcal{S}$. A sextet will be described by using $i \in \{1, ..., 6\}$ to identify the 4-element subsets of the MOG which are the tetrads of the sextet. So, for example, $S_0$, the standard sextet is given by

| | | | | | |
|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 |
| 1 | 2 | 3 | 4 | 5 | 6 |
| 1 | 2 | 3 | 4 | 5 | 6 |
| 1 | 2 | 3 | 4 | 5 | 6 |

.

The stabilizer in $G$ of $S_0$ has, apart from $\{S_0\}$,three orbits on $\mathcal{S}$ which we name $\sigma_0, \sigma_1, \sigma_3$. Representatives for these orbits are, respectively,

| | | | | | |
|---|---|---|---|---|---|
| 5 | 1 | 1 | 3 | 3 | 5 |
| 6 | 1 | 2 | 4 | 3 | 6 |
| 2 | 5 | 3 | 2 | 5 | 4 |
| 2 | 6 | 4 | 1 | 6 | 4 |

,

| | | | | | |
|---|---|---|---|---|---|
| 2 | 1 | 3 | 3 | 3 | 3 |
| 1 | 2 | 4 | 4 | 4 | 4 |
| 1 | 2 | 5 | 5 | 5 | 5 |
| 1 | 2 | 6 | 6 | 6 | 6 |

and

| | | | | | |
|---|---|---|---|---|---|
| 1 | 1 | 3 | 3 | 5 | 5 |
| 1 | 1 | 3 | 3 | 5 | 5 |
| 2 | 2 | 4 | 4 | 6 | 6 |
| 2 | 2 | 4 | 4 | 6 | 6 |

.

(See [1] or [2] for more details.)

We shall describe a chamber by first specifying a sextet, then the pairings of the tetrads that give the incident trio and the appropriate octad. For example

$c_0 =$

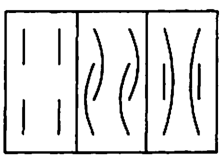
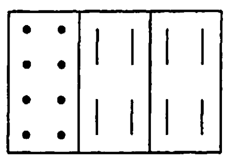
| | | | | | |
|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 |
| 1 | 2 | 3 | 4 | 5 | 6 |
| 1 | 2 | 3 | 4 | 5 | 6 |
| 1 | 2 | 3 | 4 | 5 | 6 |

$\underline{12}|34|56$

is the chamber consisting of the standard sextet, the trio $\{O_1, O_2, O_3\}$ and the octad $O_1$.

Put $B = Stab_G c_0$, and recall that $|B| = 2^{10}.3$. For $k \in \mathbb{N}$, $D_k(c_0)$ denotes the set of chambers in $\mathcal{C}$ whose distance (in the usual graph theoretic sense) from $c_0$ is $k$.

Now $G$ has two conjugacy classes of involutions - as representatives we select

$x =$  and $y =$ .

We use $\mathcal{S}_x$ (respectively $\mathcal{S}_y$) and $\mathcal{C}_x$ (respectively $\mathcal{C}_y$) for the set of sextets and chambers fixed by $x$ (respectively $y$). Before stating our main result on $\mathcal{C}_x$ and $\mathcal{C}_y$, we discuss the $B$-orbits of $\mathcal{S}$. Let $X \in \mathcal{S}$. If $X \in \sigma_0$, then there will be exactly two columns of the MOG and four tetrads of $X$ each of which intersect these two MOG columns in one element. Either of these two columns (of the MOG) will be called *mixed cols* of $X$. When $X \in \sigma_1$, there will be exactly two columns of the MOG for which two of the tetrads of $X$ intersect these columns in 3-elements - these columns we call *3-cols* of $X$. For $X \in \sigma_3$, the six columns of the MOG are partitioned into three pairs by the rule that two tetrads of $X$ each intersect both columns of the pair in two elements. Any three of these pairs of columns we refer to as a *col pair* of $X$. From [(3.1);2], the 12 orbits of $B$ on $\mathcal{S}$ are as follows:-

| $B$-ORBIT | SIZE | DESCRIPTION |
|---|---|---|
| $\{S_0\}$ | 1 | standard sextet |
| $\sigma_0^{(96)}$ | 96 | both mixed cols in $O_1$ |
| $\sigma_0^{(192)}$ | 192 | both mixed cols either in $O_2$ or in $O_3$ |
| $\sigma_0^{(384)}$ | 384 | one mixed col in $O_2$, one mixed col in $O_3$ |
| $\sigma_0^{(768)}$ | 768 | one mixed col in $O_1$, one mixed col either in $O_2$ or in $O_3$ |
| $\sigma_1^{(16)}$ | 16 | both 3-cols in $O_1$ |
| $\sigma_1^{(32)}$ | 32 | both 3-cols either in $O_2$ or in $O_3$ |
| $\sigma_1^{(64)}$ | 64 | one 3-col in $O_2$, one 3-col in $O_3$ |
| $\sigma_1^{(128)}$ | 128 | one 3-col in $O_1$, one 3-col either in $O_2$ or in $O_3$ |
| $\sigma_3^{(6)}$ | 6 | each col pair contained in one of $O_1$, $O_2$, $O_3$ |
| $\sigma_3^{(12)}$ | 12 | one col pair in $O_1$, no col pairs either in $O_2$ or in $O_3$ |
| $\sigma_3^{(24)}$ | 24 | one col pair in $O_2$, no col pairs either in $O_1$ or in $O_3$ or<br>one col pair in $O_3$, no col pair either in $O_1$ or in $O_2$ |
| $\sigma_3^{(48)}$ | 48 | no col pairs in any of $O_1$, $O_2$, $O_3$ |

**Theorem 1** *The chambers in $C_x$ and $C_y$ are described in Tables 1 and 2 respectively. Moreover*

*(i)* $|C_x| = 375, |C_y| = 959$;

*(ii)* $\{c_0\} \cup D_1(c_0) \subseteq C_x$, $|D_2(c_0) \cap C_x| = 36$, $|D_3(c_0) \cap C_x| = 40$, $|D_4(c_0) \cap C_x| = 96$, $|D_5(c_0) \cap C_x| = 160$ *and* $|D_6(c_0) \cap C_x| = 32$; *and*

*(iii)* $\{c_0\} \cup D_1(c_0) \cup D_2(c_0) \subseteq C_y$, $|D_3(c_0) \cap C_y| = 136$, $|D_4(c_0) \cap C_y| = 160$, $|D_5(c_0) \cap C_y| = 224$, $|D_6(c_0) \cap C_y| = 256$ *and* $|D_7(c_0) \cap C_y| = 128$.

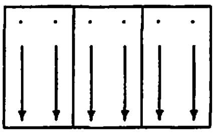
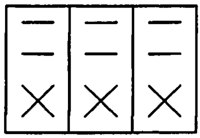
<table>
<tr><th>$S(c)$</th><th>CHAMBERS, $c$</th><th>SIZE</th><th>ROW</th><th>DISC</th></tr>
<tr><td>$S_0$</td><td colspan="4">all chambers incident with $S_0$(see [(3.2);2])</td></tr>
<tr><td>$\underset{6}{\begin{array}{|cc|cc|cc|}\hline 1&1&3&3&5&5\\ 1&1&3&3&5&5\\ 2&2&4&4&6&6\\ 2&2&4&4&6&6\\ \hline\end{array}} \in \sigma_3^{(6)}$</td><td>$\underline{12}|34|56$<br>$12|\underline{34}|56,\ 12|34|\underline{56}$<br>$\underline{12}|35|46,\ \underline{12}|36|45$</td><td>1×6<br>2×6<br>2×6</td><td>1<br>2<br>3</td><td>1<br>2<br>2</td></tr>
<tr><td>$\underset{4}{\begin{array}{|cc|cc|cc|}\hline 2&1&3&5&4&6\\ 1&2&3&5&4&6\\ 1&1&4&6&3&5\\ 2&2&4&6&3&5\\ \hline\end{array}} \in \sigma_3^{(12)*}$</td><td>$\underline{12}|34|56$<br>$12|\underline{34}|56,\ 12|34|\underline{56}$<br>$\underline{12}|35|46,\ \underline{12}|36|45$<br>$\mathcal{P}_1$</td><td>1×4<br>2×4<br>2×4<br>4×4</td><td>1<br>3<br>2<br>5</td><td>2<br>3<br>3<br>4</td></tr>
<tr><td>$\underset{8}{\begin{array}{|cc|cc|cc|}\hline 3&5&1&2&4&6\\ 4&6&1&1&3&5\\ 3&5&2&1&4&6\\ 4&6&2&2&3&5\\ \hline\end{array}} \in \sigma_3^{(24)*}$</td><td>$\underline{12}|34|56$<br>$12|\underline{34}|56,\ 12|34|\underline{56}$<br>$\underline{12}|35|46,\ \underline{12}|36|45$<br>$\mathcal{P}_1$</td><td>1×8<br>2×8<br>2×8<br>4×8</td><td>1<br>3<br>2<br>5</td><td>3<br>4<br>4<br>5</td></tr>
<tr><td>$\underset{3\times 8}{\begin{array}{|cc|cc|cc|}\hline 1&4&2&6&3&5\\ 2&3&2&5&3&6\\ 2&4&1&6&4&6\\ 1&3&1&5&4&5\\ \hline\end{array}} \in \sigma_3^{(48)}$</td><td>$12|34|\underline{56}$<br>$\underline{12}|34|56,\ 12|\underline{34}|56$<br>$13|24|\underline{56},\ 14|23|\underline{56}$</td><td>1×(3 × 8)<br>2×(3 × 8)<br>2×(3 × 8)</td><td>1<br>3<br>2</td><td>4<br>5<br>5</td></tr>
<tr><td>$\underset{8}{\begin{array}{|cc|cc|cc|}\hline 4&5&2&5&1&3\\ 3&6&1&5&2&3\\ 3&5&1&6&1&4\\ 4&6&2&6&2&4\\ \hline\end{array}} \in \sigma_3^{(48)}$</td><td>$\underline{12}|34|56$<br>$12|\underline{34}|56,\ 12|34|\underline{56}$<br>$\underline{12}|35|46,\ \underline{12}|36|45$<br>$\mathcal{P}_1$</td><td>1×8<br>2×8<br>2×8<br>4×8</td><td>1<br>3<br>2<br>5</td><td>4<br>5<br>5<br>6</td></tr>
</table>

Table 1 $C_x$ $(\mathcal{P}_1 = \{15|\underline{34}|26,\ 16|\underline{34}|25,\ 13|24|\underline{56},\ 14|23|\underline{56}\})$

| $S(c)$ | CHAMBERS, $c$ | SIZE | ROW | DISC |
|---|---|---|---|---|
| $S_0$ | all chambers incident with $S_0$(see (3.2) of [2]) | | | |
| $\underset{32}{\begin{array}{cccccc} 1 & 3 & 5 & 1 & 3 & 5 \\ 2 & 4 & 6 & 1 & 3 & 6 \\ 3 & 2 & 2 & 5 & 5 & 4 \\ 4 & 1 & 2 & 6 & 6 & 4 \end{array}} \in \sigma_0^{(96)}$ | $12\vert 34\vert \underline{56}$<br>$13\vert 24\vert \underline{56}$, $14\vert 23\vert \underline{56}$<br>$12\vert \underline{34}\vert 56$, $\underline{12}\vert 34\vert 56$<br>$\mathcal{P}_2$ | 1×32<br>2×32<br>2×32<br>4×32 | 1<br>2<br>3<br>4 | 5<br>6<br>6<br>7 |
| $\underset{16}{\begin{array}{cccccc} 2 & 1 & 3 & 3 & 3 & 3 \\ 1 & 2 & 4 & 4 & 4 & 4 \\ 1 & 2 & 5 & 5 & 5 & 5 \\ 1 & 2 & 6 & 6 & 6 & 6 \end{array}} \in \sigma_1^{(16)}$ | $\left.\begin{array}{c}\underline{12}\vert 34\vert 56, \underline{12}\vert 35\vert 46, \\ \underline{12}\vert 36\vert 45\end{array}\right\}$<br>$12\vert 34\vert \underline{56}$, $12\vert \underline{34}\vert 56$ | 3×16<br>2×16 | 1<br>2 | 3<br>4 |
| $S' = \underset{2}{\begin{array}{cccccc} 1 & 1 & 3 & 3 & 5 & 5 \\ 1 & 1 & 3 & 3 & 5 & 5 \\ 2 & 2 & 4 & 4 & 6 & 6 \\ 2 & 2 & 4 & 4 & 6 & 6 \end{array}} \in \sigma_3^{(6)}$ | all chambers incident with $S'$(see(3.2) of [2]) | | | |
| $\underset{4}{\begin{array}{cccccc} 1 & 1 & 3 & 3 & 5 & 5 \\ 2 & 2 & 4 & 4 & 6 & 6 \\ 2 & 2 & 4 & 4 & 6 & 6 \\ 1 & 1 & 3 & 3 & 5 & 5 \end{array}} \in \sigma_3^{(6)}$ | $\underline{12}\vert 34\vert 56$<br>$12\vert \underline{34}\vert 56$, $12\vert 34\vert \underline{56}$<br>$\underline{12}\vert 35\vert 46$, $\underline{12}\vert 36\vert 45$ | 1×4<br>2×4<br>2×4 | 1<br>2<br>3 | 1<br>2<br>2 |
| $S'' = \underset{4}{\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 4 & 5 & 6 \\ 2 & 2 & 5 & 6 & 3 & 4 \\ 1 & 1 & 5 & 6 & 3 & 4 \end{array}} \in \sigma_3^{(12)}$ | all chambers incident with $S''$(see(3.2) of [2]) | | | |
| $\underset{8}{\begin{array}{cccccc} 1 & 2 & 3 & 6 & 4 & 5 \\ 1 & 1 & 4 & 5 & 3 & 6 \\ 2 & 1 & 3 & 6 & 4 & 5 \\ 2 & 2 & 4 & 5 & 3 & 6 \end{array}} \in \sigma_3^{(12)}$ | $\underline{12}\vert 34\vert 56$<br>$\underline{12}\vert 35\vert 46$, $\underline{12}\vert 36\vert 45$<br>$12\vert \underline{34}\vert 56$, $12\vert 34\vert \underline{56}$ | 1×8<br>2×8<br>2×8 | 1<br>2<br>3 | 2<br>3<br>3 |
| $\underset{8}{\begin{array}{cccccc} 3 & 1 & 5 & 6 & 4 & 2 \\ 3 & 1 & 6 & 5 & 4 & 2 \\ 4 & 2 & 6 & 6 & 3 & 1 \\ 4 & 2 & 5 & 5 & 3 & 1 \end{array}} \in \sigma_3^{(24)\dagger}$ | $12\vert 34\vert \underline{56}$<br>$13\vert 24\vert \underline{56}$, $14\vert 23\vert \underline{56}$<br>$12\vert \underline{34}\vert 56$, $\underline{12}\vert 34\vert 56$<br>$\mathcal{P}_2$ | 1×8<br>2×8<br>2×8<br>4×8 | 1<br>2<br>3<br>4 | 3<br>4<br>4<br>5 |
| $\underset{16}{\begin{array}{cccccc} 1 & 4 & 2 & 6 & 3 & 5 \\ 2 & 3 & 2 & 5 & 3 & 6 \\ 2 & 4 & 1 & 6 & 4 & 6 \\ 1 & 3 & 1 & 5 & 4 & 5 \end{array}} \in \sigma_3^{(48)}$ | $12\vert 34\vert \underline{56}$<br>$13\vert 24\vert \underline{56}$, $14\vert 23\vert \underline{56}$<br>$\underline{12}\vert 34\vert 56$, $12\vert \underline{34}\vert 56$<br>$\mathcal{P}_2$ | 1×16<br>2×16<br>2×16<br>4×16 | 1<br>2<br>3<br>4 | 4<br>5<br>5<br>6 |

Table 2 $C_y$ ($\mathcal{P}_2 = \{\ \underline{13}|24|56,\ 13|\underline{24}|56,\ \underline{14}|23|56,\ 14|\underline{23}|56\ \}$)

The first column of Table 1 gives a representative sextet (not necessarily a $C_B(x)$–orbit representative) which is fixed by $x$. Under the sextet we record the number of sextets it represents. The second column describes all the chambers which are fixed by $x$ and are incident with the sextets in the first column. In columns three and four we record the number of the chambers (in column two) and the row the chambers are to be found in (3.2) of [2]. The final column gives $i$ for which the chambers belong to $D_i(c_0)$ - this data follows from the fourth column and (3.2) of [2]. Table 2 gives the analogous information for $y$. The sets $\sigma_3^{(12)*}, \sigma_3^{(24)*}$ and $\sigma_3^{(24)\dagger}$ will be defined later.

Put $\rho =$  and $\tau =$ .

Observe that $\rho, \tau \in B$ and that $\{1, \rho, \rho^2, \tau, \rho\tau, \rho^2\tau\}$ is a complete set of right coset representatives for $C_B(x)$ in $B$.

Set $S_1 =$

| | | | | | |
|---|---|---|---|---|---|
| 1 | 4 | 2 | 6 | 3 | 5 |
| 2 | 3 | 2 | 5 | 3 | 6 |
| 2 | 4 | 1 | 6 | 4 | 6 |
| 1 | 3 | 1 | 5 | 4 | 5 |

, $S_2 = S_1^{\rho}$, $S_3 = S_1^{\rho^2}$ and $S_4 = S_1^{\tau} =$

| | | | | | |
|---|---|---|---|---|---|
| 4 | 1 | 6 | 2 | 5 | 3 |
| 3 | 2 | 5 | 2 | 6 | 3 |
| 3 | 1 | 5 | 1 | 5 | 4 |
| 4 | 2 | 6 | 1 | 6 | 4 |

. Note that $S_1 \in \sigma_3^{(48)}$. Let $\sigma_3^{(12)*}$ (respectively $\sigma_3^{(24)*}$) denote the sextets in $\sigma_3^{(12)}$ (respectively $\sigma_3^{(24)}$) all of whose tetrads intersect the partition of $\Omega$ given by the $<x>$*–orbits* in $1^4$.

**Lemma 2** *(i)* $\mathcal{S}_x = \{S_0\} \cup \sigma_3^{(6)} \cup \sigma_3^{(12)*} \cup \sigma_3^{(24)*} \cup S_1^{C_B(x)} \cup S_2^{C_B(x)} \cup S_3^{C_B(x)} \cup S_4^{C_B(x)}$ *where* $\left|\sigma_3^{(6)}\right| = 6, \left|\sigma_3^{(12)*}\right| = 4, \left|\sigma_3^{(24)*}\right| = 8$ *and* $\left|S_i^{C_B(x)}\right| = 8$ *for* $i = 1, 2, 3, 4$; *and*

*(ii)* $x$ *induces a permutation of cycle type* $1^2 2^2$ *on the tetrads of sextets in* $\sigma_3^{(6)} \cup S_1^{C_B(x)} \cup S_2^{C_B(x)} \cup S_3^{C_B(x)}$ *and of type* $2^3$ *on the tetrads of the sextets in* $\sigma_3^{(12)*} \cup \sigma_3^{(24)*} \cup S_4^{C_B(x)}$.

**Proof.** Let $S \in \mathcal{S}_x$. Suppose $S \in \sigma_0$. Then $S$ will have a mixed col and hence, as $x$ is fixed-point-free on the MOG columns, $x$ must interchange (say) tetrads 1 and 2 and tetrads 3 and 4 of $S$. Now there will be a MOG column intersecting tetrad 1 of $S$ in two elements and intersecting tetrads 5 and 6 of $S$ each in one element. This is incompatible with $x$ interchanging tetrads 1 and 2 of $S$, and thus $S \notin \sigma_0$. If $S \in \sigma_1$, then $S$ would have a 3-col, whence $x$ cannot fix $S$. Therefore we must have $S \in \sigma_3$. Clearly $S_0 \in \mathcal{S}_x$ and, by inspection, $\sigma_3^{(6)} \subset \mathcal{S}_x$ ($\sigma_3^{(6)}$ consists of the sextets in the third column of the MOG). The fifth and sixth columns of the MOG comprise the set $\sigma_3^{(12)}$ and $\sigma_3^{(24)}$is obtained from $\sigma_3^{(12)}$ by moving (bodily) the left-most brick to either the $O_2$ or $O_3$ position. Checking reveals that $\mathcal{S}_x \cap \sigma_3^{(12)} = \sigma_3^{(12)*}$ and $\mathcal{S}_x \cap \sigma_3^{(24)} = \sigma_3^{(24)*}$. Turning to $\sigma_3^{(48)}$ we observe that $Stab_B S_1 \leqslant C_B(x)$ and hence $\sigma_3^{(48)}$ is the union of six $C_B(x)-$orbits each of size 8. For representatives of these orbits we may take $S_1, S_2, S_3, S_4, S_1^{\rho\tau}, S_1^{\rho^2\tau}$. It is readily seen that the latter two sextets are not fixed by $x$ and therefore $\mathcal{S}_x$ is as stated. The action of $x$ on the tetrads of the sextets in $\mathcal{S}_x$ is clearly observed, so proving Lemma 1. ∎

Let $S_5 = \begin{array}{|cc|cc|cc|} \hline 1 & 3 & 5 & 1 & 3 & 5 \\ 2 & 4 & 6 & 1 & 3 & 6 \\ 3 & 2 & 2 & 5 & 5 & 4 \\ 4 & 1 & 2 & 6 & 6 & 4 \\ \hline \end{array}$ ($\in \sigma_0^{(96)}$) and $S_6 = \begin{array}{|cc|cc|cc|} \hline 1 & 4 & 2 & 6 & 3 & 5 \\ 2 & 3 & 2 & 5 & 3 & 6 \\ 2 & 4 & 1 & 6 & 4 & 6 \\ 1 & 3 & 1 & 5 & 4 & 5 \\ \hline \end{array}$ ($\in \sigma_3^{(48)}$).

Let $\sigma_3^{(24)\dagger}$ be the set of sextets $X$ in $\sigma_3^{(24)}$ with the property that if $T$ is a tetrad of $X$ with $T \cap O_1 \neq \emptyset$, then $T \cap (O_1 \cup O_2)$ is a $< x >$-orbit.

**Lemma 3** *(i)* $\mathcal{S}_y = \{S_0\} \cup S_5^{C_B(y)} \cup \sigma_1^{(16)} \cup \sigma_3^{(6)} \cup \sigma_3^{(12)} \cup \sigma_3^{(24)\dagger} \cup S_6^{C_B(y)}$ *where* $\left|S_5^{C_B(y)}\right| = 32$, $\left|\sigma_1^{(16)}\right| = 16$, $\left|\sigma_3^{(6)}\right| = 6$, $\left|\sigma_3^{(12)}\right| = 12$, $\left|\sigma_3^{(24)\dagger}\right| = 8$ *and* $\left|S_6^{C_B(y)}\right| = 16$*; and*

*(ii) $y$ induces a permutation of cycle type $1^6$ on the tetrads of two of the sextets in $\sigma_3^{(6)}$ and four of the sextets in $\sigma_3^{(12)}$ and of cycle type $1^2 2^2$ on the remaining four sextets of $\sigma_3^{(6)}$ and eight sextets in $\sigma_3^{(12)}$. On the tetrads of the sextets in $S_5^{C_B(y)} \cup S_4^{C_B(y)}$ $y$ induces type $1^4 2$ and on those in $\sigma_1^{(16)}$ $y$ induces $1^2 2^2$.*

**Proof.** Let $S \in \mathcal{S}_y$.We first consider the case $S \in \sigma_0$. If $S \in \sigma_0^{(192)} \cup \sigma_0^{(384)} \cup \sigma_0^{(768)}$, then $S$ has a mixed col in $O_2 \cup O_3$ whence $y$ acts fixed-point-free on this column. But $S$ has a tetrad which has a non-empty intersection with this mixed col and $O_1$, a contradiction. Therefore $S \in \sigma_0^{(96)}$. Noting that $Stab_B S_5 \leqslant C_B(x) \in Syl_2 B$ we see that $\sigma_0^{(96)}$ is the union of three $C_B(y)-$orbits each of size 32. As representatives for these three orbits we may take $S_5, S_5^{\rho}$ and $S_5^{\rho^2}$ and readily we see that $y$ fixes $S_5$ but fixes neither of $S_5^{\rho}$ and $S_5^{\rho^2}$. Thus $\mathcal{S}_x \cap \sigma_0 = S_5^{C_B(y)}$.

Because the sizes of the $B-$orbits of $\sigma_1$ are all coprime to 3, they are also $C_B(y)-$orbits. Checking $B-$orbit representatives (see (3.2) of [2]) reveals that $\mathcal{S}_y \cap \sigma_1 = \sigma_1^{(16)}$. That $\sigma_3^{(6)} \cup \sigma_3^{(12)} \cup \sigma_3^{(24)\dagger} \subseteq \mathcal{S}_y$ is also readily checked. Finally, looking at $\sigma_3^{(48)}$ we note that $Stab_B S_6 \leqslant C_B(y)$ and so $\sigma_3^{(48)}$ is the union of three $C_B(y)$−orbits each of size 16 with representatives $S_6, S_6^{\rho}, S_6^{\rho^2}$. Thus, as $y$ fixes $S_6$ but not $S_6^{\rho}$ and $S_6^{\rho^2}$, we obtain $\mathcal{S}_x \cap \sigma_3 = \sigma_3^{(6)} \cup \sigma_3^{(12)} \cup \sigma_3^{(24)\dagger} \cup S_6^{C_B(y)}$, so giving (i), and (ii) follows by inspection. ∎

**Proof of Theorem 1**

From Lemma 2 we have $\mathcal{S}_x$. Let $X \in \mathcal{S}_x$. Now if $x$ induces $1^6$ on the tetrads of $X$, then all 45 chambers containing $X$ will be fixed by $x$. While $x$ inducing $1^2 2^2$ on the tetrads of $X$ (suppose $x$ induces (34)(56)) means that $x$ fixes the chambers determined by $\{\underline{12}|34|56, 12|\underline{34}|56, 12|34|\underline{56}, \underline{12}|35|46, \underline{12}|36|45\}$. And if $x$ induces $2^3$ on the tetrads of $X$ (say $x$ induces (12)(34)(56)) then $x$ fixes the chambers given by $\{\underline{12}|34|56, 12|\underline{34}|56, 12|34|\underline{56}, \underline{12}|35|46, \underline{12}|36|45, 15|\underline{34}|26, 16|\underline{34}|25, 13|24|\underline{56}, 14|23|\underline{56}\}$. These observations together with the information supplied by Lemma 2 yields $\mathcal{C}_x$ as given in Table 1. Similarly, Lemma 3 gives us $\mathcal{C}_y$.

# References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