A note on circumferences in k-connected graphs with given independence number

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Abstract

Fouquet and Jolivet conjectured that if G is a k-connected n-vertex graph with independence number $\alpha \geq k \geq 2$, then G has circumference at least $\frac{k(n+\alpha-k)}{\alpha}$. This conjecture was recently proved by O, West and Wu. In this note, we consider the set of k-connected n-vertex graphs with independence number $\alpha > k \geq 2$ and circumference exactly $\frac{k(n+\alpha-k)}{\alpha}$. We show that all of these graphs have a similar structure.

Keywords: Circumference; Connectivity; Independence number
AMS Subject Classification: 05C40, 05C35

1 Introduction

We only consider finite graphs without loops or multiple edges. For a graph G, we use $V(G)$ and $E(G)$ to denote the vertex set and edge set of G, respectively. For any $S \subseteq V(G)$, define $G-S$ to be the subgraph of G with vertex set $V(G)-S$ and edge set $\{e \in E(G) : e$ is not incident with any vertex in $S\}$. We use $G[S]$ to denote the subgraph of G induced by S. A graph G is *k-connected* if $|V(G)| > k$ and every subgraph obtained from G by deleting fewer than k vertices is still connected. The *connectivity* of G, denoted by $\kappa(G)$, is the maximum integer k such that G is k-connected. An *independent*

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set of a graph G is a set of pairwise nonadjacent vertices. The *independence number* of G, denoted by $\alpha(G)$, is the maximum size of such a set. The *circumference* of a graph G, denoted by $c(G)$, is the length of a longest cycle in G. A spanning cycle (respectively, spanning path) is often called a *Hamilton cycle* (respectively, *Hamilton path*).

In 1972, Chvátal and Erdös [2] proved that every k-connected graph with independence number $\alpha \leq k$ contains a Hamilton cycle. There are infinitely many graphs showing that the condition $\alpha \leq k$ is sharp. For $\alpha \geq k$, Fouquet and Jolivet [3] proposed the following conjecture.

Conjecture 1.1 *Let G be a k-connected n-vertex graph with independence number α. If $\alpha \geq k \geq 2$, then $c(G) \geq \frac{k(n+\alpha-k)}{\alpha}$.*

The following two graphs H_1 and H_2 demonstrate the lower bound of $c(G)$ in Conjecture 1.1 is best possible. Let α, k and m be three positive integers such that $\alpha \geq k \geq 2$ and $m \geq 2$. Let G be an arbitrary graph with k vertices, and let K_k and K_m be the complete graphs of order k and m, respectively. Let αK_m be the union of α vertex disjoint copies of K_m. We define H_1 to be the join of the two graphs G and αK_m, and H_2 to be the graph obtained from the join of the two graphs K_k and αK_m by deleting an edge (or even more edges) with one end in K_k and the other end in αK_m. It is easy to check that for each $i = 1, 2$, we have $|V(H_i)| = n = k + \alpha m$, $\kappa(H_i) = k$, $\alpha(H_i) = \alpha$ and $c(H_i) = k(m+1) = \frac{k(n+\alpha-k)}{\alpha}$.

Kouider [6] proved that if G is a k-connected n-vertex graph with independence number α then the vertices of G can be covered by at most $\lceil \frac{\alpha}{k} \rceil$ cycles. This implies that G contains a cycle of length at least $\frac{n}{\lceil \alpha/k \rceil}$, which is close to the conjectured threshold of $\frac{nk}{\alpha} + \frac{k(\alpha-k)}{\alpha}$. Fournier [4] proved Conjecture 1.1 for the case that $\alpha = k+1$ or $\alpha = k+2$ in his doctoral dissertation. In 1984, he [5] further showed that Conjecture 1.1 holds for $k = 2$. The case that $k = 3$ was verified by Manoussakis [7]. Chen, Hu and Wu [1] proved that if $\alpha \geq k \geq 4$ then $c(G) \geq \frac{k(n+\alpha-k)}{\alpha} - \frac{(k-3)(k-4)}{2}$. This shows that Conjecture 1.1 is true for $k = 4$. In the same paper, they also proved this conjecture for $\alpha = k+3$. Recently, O, West and Wu [8] proved Conjecture 1.1 completely.

In this note, we consider the structure of k-connected n-vertex graphs with independence number $\alpha > k \geq 2$ and circumference exactly $\frac{k(n+\alpha-k)}{\alpha}$. We prove the following result.

Theorem 1.2 *Let G be a k-connected n-vertex graph with independence number $\alpha > k \geq 2$. Then $c(G) = \frac{k(n+\alpha-k)}{\alpha}$ if and only if there exists a k-cut S in G such that $G - S$ contains exactly α components, each of which is a complete graph of order m ($m \in \mathbb{N}$).*

The two graphs H_1 and H_2 defined above show that the subgraph of G induced by S may not be a complete graph or some vertex in S may not be adjacent to all the vertices of $G - S$. Hence Theorem 1.2 is best possible in some sense. Note that Theorem 1.2 is not true for $\alpha = k$. Let $k, m_1, m_2, \ldots, m_{k-1}$ be k positive integers such that $k \geq 3$ and $m_1 \geq 3$. As before, let K_k and K_{m_i} $(1 \leq i \leq k-1)$ be the complete graphs of order k and m_i, respectively. Let H be the union of the $k-1$ vertex disjoint complete graphs K_{m_i}. We define H_3 to be the graph obtained from the join of the two graphs K_k and H by deleting an edge with both ends in K_{m_1}. Clearly, $\kappa(H_3) = \alpha(H_3) = k$ and H_3 has a Hamilton cycle. But $H_3 - V(K_k)$ contains exactly $k-1$ components and some component of $H_3 - V(K_k)$ is not a complete graph.

We conclude this section with some notation and terminology. Let G be a graph. A *block* in G is a maximal subgraph of G containing no cut vertex. Let F and H be two vertex disjoint subgraphs of G. A path *from F to H* (or an *(F,H)-path*) in G is a path with one endvertex in $V(F)$, the other endvertex in $V(H)$ and no internal vertex in $V(F) \cup V(H)$.

We write $A := B$ to rename B as A. For any graph G and any $v \in V(G)$, we use $N_G(v)$ to denote the neighborhood of v in G. For any subset S of $V(G)$, we let $N_G(S) := (\bigcup_{v \in S} N_G(v)) - S$. For a specified orientation of a cycle C and for distinct vertices x, y of C, we use xCy to denote the subpath of C from x to y in the given orientation.

2 Proof of Theorem 1.2

In this section, we prove the main result of this paper.

The following result was virtually proved by O, West and Wu [8, Lemma 3.2 and Corollary 3.3]. Since we only consider the set of k-connected n-vertex graphs with independence number $\alpha > k \geq 2$ and circumference $\frac{k(n+\alpha-k)}{\alpha}$, we state this result as follows.

Lemma 2.1 *Let G be a k-connected n-vertex graph with independence number $\alpha > k \geq 2$ and circumference $c(G) = \frac{k(n+\alpha-k)}{\alpha}$, and let $l = \alpha - k$. Then there exist cycles $C_0, C_1, \ldots, C_l$ in G such that*

(i) $\alpha(G - \bigcup_{j=0}^{i} V(C_j)) = \alpha - k - i$ *for* $0 \leq i \leq l$,

(ii) $|V(C_i)| = \frac{k(n+\alpha-k)}{\alpha}$ *for* $0 \leq i \leq l$, *and*

(iii) $|V(C_i) - V(C_0)| = |V(C_i) - \bigcup_{j=0}^{i-1} V(C_j)| = \frac{|V(C_0)|}{k} - 1$ *for* $1 \leq i \leq l$.

Proof. By [8, Lemma 3.2], there exist cycles $C_0, C_1, \ldots, C_l$ in G such that $\alpha(G - \bigcup_{j=0}^{i} V(C_j)) \leq \alpha - k - i$ for $0 \leq i \leq l$ and $|V(C_i) - \bigcup_{j=0}^{i-1} V(C_j)| \leq$

$\frac{|V(C_0)|}{k} - 1$ for $1 \le i \le l$. Since $c(G) = \frac{k(n+\alpha-k)}{\alpha}$, it follows from the proof of [8, Corollary 3.3] that $|V(C_0)| = \frac{k(n+\alpha-k)}{\alpha}$ and $|V(C_i) - \bigcup_{j=0}^{i-1} V(C_j)| = \frac{|V(C_0)|}{k} - 1$ for $1 \le i \le l$. Then by the construction of C_i $(0 \le i \le l)$ in [8, Lemma 3.2], we see that $\alpha(G - \bigcup_{j=0}^{i} V(C_j)) = \alpha - k - i$ for $0 \le i \le l$, $|V(C_i) - V(C_0)| = |V(C_i) - \bigcup_{j=0}^{i-1} V(C_j)|$ and $|V(C_i)| = |V(C_0)|$ for $1 \le i \le l$. Hence the assertion of Lemma 2.1 holds. ∎

We can now prove Theorem 1.2. The idea is partially inspired by O, West and Wu [8, Theorem 3.1].

Proof of Theorem 1.2. If there exists a k-cut S in G such that $G - S$ contains exactly α components, each of which is a complete graph of order m $(m \in \mathbb{N})$, then $n = k + \alpha m$. Now it is easy to check that $c(G) = k(m+1) = \frac{k(n+\alpha-k)}{\alpha}$. So we need only to consider the opposite direction.

Suppose $c(G) = \frac{k(n+\alpha-k)}{\alpha}$. Then by Lemma 2.1, there exist cycles $C_0, C_1, \dots, C_l$ in G satisfying (i), (ii) and (iii). By (ii) and (iii), $|V(C_i) - V(C_0)| = \frac{|V(C_0)|}{k} - 1 = \frac{n+\alpha-k}{\alpha} - 1 = \frac{n-k}{\alpha}$ for $1 \le i \le l$. Since $\frac{n-k}{\alpha}$ is a positive integer, we have $n = k + \alpha m$ $(m \in \mathbb{N})$. For each $1 \le i \le l$, let $V_i = V(C_i) - V(C_0)$ and let $G_i = G[V_i]$. Then $|V(C_0)| = \frac{k(n+\alpha-k)}{\alpha} = k(m+1)$ and $|V_i| = \frac{n-k}{\alpha} = m$. By (iii), we see that $V_i \cap V_j = \emptyset$ for $1 \le i < j \le l$ if $l \ge 2$.

We first consider G_l. By (i), we have $\alpha(G_l) = \alpha(G - \bigcup_{j=0}^{l-1} V(C_j)) = 1$. This implies that G_l is a complete graph of order m. Let B be the block of $G - V(C_0)$ containing G_l. Since G is a k-connected graph, by Menger's Theorem, for each $b \in V(B)$, there exist k paths from b to C_0 in G that pairwise share only b (since $|V(C_0)| \ge k$). We call these k paths a (b, C_0)-*fan*. For a fixed orientation of C_0, let $S = \{s_1, s_2, \dots, s_p\}$ be the set of endvertices on C_0 of all the (B, C_0)-paths, and assume that $s_1, s_2, \dots, s_p$ occur on C_0 in the given orientation. Then $p \ge k$. By the maximality of B, any two (B, C_0)-paths with distinct endvertices in B are internally disjoint.

We claim that

(1) $p = k$ and $|V(s_i C_0 s_{i+1})| = m + 2$ for $1 \le i \le k$.

First, suppose that $|V(B)| = 1$. Then $B = G_l$ and $m = 1$. Let b be the only vertex in B. By the maximality of B, we have $N_G(b) = S$. Choose $s_i \in S$ such that $|V(s_i C_0 s_{i+1})|$ is minimum, where $s_{p+1} := s_1$. Then by the pigeonhole principle, we have $|V(s_i C_0 s_{i+1})| \le m+2 = 3$. Let $C := s_{i+1} C_0 s_i \cup \{b s_i, b s_{i+1}\}$. If $|V(s_i C_0 s_{i+1})| = 2$, then $|V(C)| = |V(C_0)| + 1$, which contradicts the assumption that $c(G) = |V(C_0)|$. Hence $|V(s_i C_0 s_{i+1})| = 3$. By the choice of s_i, we deduce that $p = k$ and $|V(s_i C_0 s_{i+1})| = 3$ for $1 \le i \le k$.

So we may assume that $|V(B)| \geq 2$. Let $S_1 = \{s_{i_1}, s_{i_2}, \ldots, s_{i_q}\} \subseteq S$ be the set of endvertices on C_0 such that for each $s_{i_j} \in S_1$, there exist two disjoint (B, C_0)-paths P_{i_j} and P_{i_j+1} in G with $V(P_{i_j}) \cap V(C_0) = \{s_{i_j}\}$ and $V(P_{i_j+1}) \cap V(C_0) = \{s_{i_j+1}\}$. Clearly, $S_1 \neq \emptyset$. Let $b_{i_j} \in V(B)$ and $b_{i_j+1} \in V(B)$ be the other endvertices of P_{i_j} and P_{i_j+1}, respectively. Since B is a block of $G - V(C_0)$ and G_l is a complete graph, by Menger's Theorem, we can always find a path P in B from b_{i_j} to b_{i_j+1} containing V_l (by considering $|\{b_{i_j}, b_{i_j+1}\} \cap V_l| = 0, 1$ or 2). Now let $C := s_{i_j+1}C_0s_{i_j} \cup P_{i_j} \cup P \cup P_{i_j+1}$. If $|V(s_{i_j}C_0s_{i_j+1})| \leq m+1$, then we have $|V(C)| \geq |V(C_0)| + 1$, which contradicts the assumption that $c(G) = |V(C_0)|$. So we know that $|V(s_{i_j}C_0s_{i_j+1})| \geq m + 2$ for $1 \leq j \leq q$, and hence $q \leq k$.

Let $S_2 = S - S_1$. By the definition of S_1, for each $s_j \in S_2$, there exist two internally disjoint paths P_j and P_{j+1} in G from the same vertex of $G - V(C_0)$, say v_j, to C_0 such that $V(P_j) \cap V(C_0) = \{s_j\}$ and $V(P_{j+1}) \cap V(C_0) = \{s_{j+1}\}$. It is possible that $v_j \notin V(B)$. Let $C := s_{j+1}C_0s_j \cup P_j \cup P_{j+1}$. If $|V(s_jC_0s_{j+1})| = 2$, then $|V(C)| \geq |V(C_0)| + 1$, contradicting the assumption that $c(G) = |V(C_0)|$. Hence we have $|V(s_jC_0s_{j+1})| \geq 3$ for each $s_j \in S_2$. For each $b \in V(B)$, a (b, C_0)-fan has k endvertices on C_0. Since $q \leq k$, at least $k - q$ of them are contained in S_2. Moreover, again by the definition of S_1, for distinct vertices of B, these endvertices contained in S_2 are pairwise distinct. Therefore, we see that $|S_2| = p - q \geq |V(B)|(k - q)$.

Then we have $k(m+1) = |V(C_0)| \geq q(m+1) + 2(p-q) \geq q(m+1) + 2|V(B)|(k-q) \geq q(m+1) + 2m(k-q) = (2k-q)m + q$. Since $|V(B)| \geq 2$, it is easy to check the above inequality holds if and only if $p = q = k$. So we have $|V(s_iC_0s_i + 1)| = m + 2$ for $1 \leq i \leq k$. This proves (1).

We also claim that

(2) $B = G_l$.

Suppose for a contradiction that $B \neq G_l$. Then $|V(B)| \geq 2$. By (iii), $G - V_l$ is 2-connected. Then by Menger's Theorem, there exist two internally disjoint $(B - V_l, C_0)$-paths P_1 and P_2 in $G - V_l$ with distinct endvertices on C_0. Without loss of generality, let $b_1 \in V(B) - V_l$ and s_1 be the two endvertices of P_1, and let $b_2 \in V(B) - V_l$ and $s_i \neq s_1$ be the two endvertices of P_2.

First, assume that $k = 2$. Then $s_i = s_2$. Suppose $b_1 \neq b_2$. Since B is a block of $G - V(C_0)$ and G_l is a complete graph, by Menger's theorem, we can always find a path P in B from b_1 to b_2 containing V_l. Let $C := s_2C_0s_1 \cup P_1 \cup P \cup P_2$. Since $b_1, b_2 \in V(B) - V_l$, we have $|V(C)| \geq |V(C_0)| + 2$, which contradicts the assumption that $c(G) = |V(C_0)|$. So we may assume that $b_1 = b_2$. Then there must exist a path P_3 (disjoint with P_1) in G from $B - \{b_1\}$ to C_0 with $V(P_3) \cap V(C_0) = \{s_2\}$ or a path P_3' (disjoint with P_2) in G from $B - \{b_1\}$ to C_0 with $V(P_3') \cap V(C_0) = \{s_1\}$; for otherwise, b_1 would be a cut vertex in G, a contradiction. By symmetry between s_1 and s_2, let P_3 be a path (disjoint with P_1) in G from $B - \{b_1\}$ to C_0 with endvertices $b_3 \in V(B) - \{b_1\}$ and s_2. As before, let P' be a path in B from b_1 to b_3 containing V_l, and let $C' := s_2C_0s_1 \cup P_1 \cup P' \cup P_3$. Since $b_1 \in V(B) - V_l$, $|V(C')| \geq |V(C_0)| + 1$, a contradiction.

Hence we may assume that $k \geq 3$. Then there must exist a path P_4 (disjoint with P_1) in G from $B - \{b_1\}$ to C_0 with $V(P_4) \cap V(C_0) = \{s_2\}$ or $V(P_4) \cap V(C_0) = \{s_k\}$; for otherwise, $S' := (S - \{s_2, s_k\}) \cup \{b_1\}$ would be a $(k-1)$-cut in G, a contradiction. By symmetry between s_2 and s_k, let $b_4 \in V(B) - \{b_1\}$ and s_2 be the two endvertices of P_4. By the same argument as before, we can find a path in B from b_1 to b_4 containing V_l and construct a cycle of length at least $|V(C_0)|+1$ in G (since $b_1 \in V(B) - V_l$), again a contradiction. So (2) holds.

We further claim that

(3) $N_G(V_l) = S$.

Suppose to the contrary that $N_G(V_l) \neq S$. Then by the argument in (1) and by (2), we have $|V_l| = |V(B)| \geq 2$. Since G is a k-connected graph, $|N_G(V_l)| \geq k = |S|$, and hence $N_G(V_l) - S \neq \emptyset$. Let uv be an edge in G with $v \in V_l$ and $u \in N_G(V_l) - S$. By (iii), $G - V_l$ is 2-connected. Then by Menger's Theorem, there exist two internally disjoint (u, C_0)-paths Q_1' and Q_2' in $G - V_l$ with distinct endvertices on C_0. Then $Q_1 := Q_1' \cup \{uv\}$ and $Q_2 := Q_2' \cup \{uv\}$ are two paths in G from G_l to C_0 with $|V(Q_1)| \geq 3$ and $|V(Q_2)| \geq 3$. Without loss of generality, let s_1 and $s_i \neq s_1$ be the other endvertices of Q_1 and Q_2, respectively.

Suppose that $k = 2$. Therefore, $s_i = s_2$. Then there must exist a path Q_3 (disjoint with Q_1) in G from $G_l - \{v\}$ to C_0 with $V(Q_3) \cap V(C_0) = \{s_2\}$ or a path Q_3' (disjoint with Q_2) in G from $G_l - \{v\}$ to C_0 with $V(Q_3') \cap V(C_0) = \{s_1\}$; for otherwise, v would be a cut vertex in G, a contradiction. By symmetry between s_1 and s_2, let Q_3 be a path (disjoint with Q_1) in G from $G_l - \{v\}$ to C_0 with endvertices $w \in V_l - \{v\}$ and s_2. Let Q be a Hamilton path in G_l from v to w, and let $C := s_2C_0s_1 \cup Q_1 \cup Q \cup Q_3$. Since $|V(Q_1)| \geq 3$, we have $|V(C)| \geq |V(C_0)| + 1$, a contradiction.

So we may assume that $k \geq 3$. Then there must exist a path Q_4 (disjoint with Q_1) in G from $G_l - \{v\}$ to C_0 with $V(Q_4) \cap V(C_0) = \{s_2\}$ or $V(Q_4) \cap V(C_0) = \{s_k\}$; for otherwise, $S' := (S - \{s_2, s_k\}) \cup \{v\}$ would be a $(k-1)$-cut in G, a contradiction. By symmetry between s_2 and s_k, let $w' \in V(B) - \{b_1\}$ and s_2 be the two endvertices of Q_4. By the same argument as before, since $|V(Q_1)| \geq 3$, we can find a cycle of length at least $|V(C_0)| + 1$ in G, again a contradiction. So we have (3).

We then consider G_{l-1} if $l \geq 2$. By (i), $\alpha(G[V_{l-1} \cup V_l]) = \alpha(G - \bigcup_{j=0}^{l-2} V(C_j)) = 2$. Since $N_G(V_l) \cap V_{l-1} = \emptyset$ (by (3)), we know that G_{l-1}

is also a complete graph of order m. Let B' be the block of $G - V(C_0)$ containing G_{l-1}. Let $T = \{t_1, t_2, \ldots, t_{p'}\}$ be the set of endvertices on C_0 of all the (B', C_0)-paths, and assume that $t_1, t_2, \ldots, t_{p'}$ occur on C_0 in the given orientation. Then by the same argument as for G_l, we can also conclude that $p' = k$, $|V(t_i C_0 t_{i+1})| = m + 2$ for $1 \le i \le k$ and $N_G(V_{l-1}) = T$.

We claim that

(4) $S = T$.

Suppose for a contradiction that $S \ne T$. Then $S \cap T = \emptyset$. By symmetry between S and T, we may assume that $s_1, t_1, s_2, t_2, \ldots, s_k, t_k$ occur on C_0 in the given orientation. Let $s_1u_1, s_2u_2 \in E(G)$ with $u_1, u_2 \in V_l$, and let $t_1v_1, t_2v_2 \in E(G)$ with $v_1, v_2 \in V_{l-1}$. If $m \ge 2$, we can choose u_1, u_2, v_1, v_2 so that $u_1 \ne u_2$ and $v_1 \ne v_2$. Let P be a Hamilton path in G_l from u_1 to u_2, and let Q be a Hamilton path in G_{l-1} from v_1 to v_2. (If $m = 1$, then let $P := \emptyset$ and $Q := \emptyset$.) Now let $C := t_2 C_0 s_1 \cup P \cup t_1 C_0 s_2 \cup Q \cup \{s_1u_1, s_2u_2, t_1v_1, t_2v_2\}$. Since $|V(s_1 C_0 t_1)| = |V(s_2 C_0 t_2)| \le m + 1$, we have $|V(C)| \ge |V(C_0)| + 2$, contradicting the assumption that $c(G) = |V(C_0)|$. This proves (4).

By (i) and by the same arguments as for G_l and G_{l-1}, we know that

(5) G_i is a complete graph of order m and $N_G(V_i) = S$ for each $1 \le i \le l$.

We now consider C_0. For each $1 \le i \le k$, let $U_i = V(s_i C_0 s_{i+1}) - \{s_i, s_{i+1}\}$ and let $G_{l+i} = G[U_i]$. Then $|U_i| = m$ (by (1)) and $U_i \cap U_j = \emptyset$ for $1 \le i < j \le k$.

We claim that

(6) there is no edge in G with one end in U_i and the other end in U_j for $1 \le i < j \le k$.

For otherwise, let $r_i r_j$ be an edge in G with $r_i \in U_i$ and $r_j \in U_j$. Let $s_iu_1, s_ju_2, s_{i+1}v_1, s_{j+1}v_2 \in E(G)$ such that $u_1, u_2, v_1, v_2 \in V_1$. If $m \ge 2$, we can further choose u_1, u_2, v_1, v_2 so that $u_1 \ne u_2$ and $v_1 \ne v_2$. Let P be a Hamilton path in G_1 from u_1 to u_2, and let Q be a Hamilton path in G_1 from v_1 to v_2. (If $m = 1$, then let $P := \emptyset$ and $Q := \emptyset$.) Now let $C := r_i C_0 s_j \cup r_j C_0 s_i \cup P \cup \{s_iu_1, s_ju_2, r_ir_j\}$ and let $C' := s_{i+1} C_0 r_j \cup s_{j+1} C_0 r_i \cup Q \cup \{s_{i+1}v_1, s_{j+1}v_2, r_ir_j\}$. Since $|U_i| = |U_j| = m$, it is easy to check that $|V(C)| + |V(C')| = 2|V(C_0)| + 2$. But this implies either $|V(C)| \ge |V(C_0)| + 1$ or $|V(C')| \ge |V(C_0)| + 1$, which contradicts the assumption that $c(G) = |V(C_0)|$. So (6) holds.

By (5) and (6), we have $G - S$ contains exactly $l + k = \alpha$ components. Since $\alpha(G) = \alpha$, we see that G_{l+i} is also a complete graph of order m for each $1 \le i \le k$. This completes the proof of Theorem 1.2. ∎

Acknowledgements. The authors would like to thank the referees for their valuable comments and suggestions, which have led to considerable improvement of the presentation of this work. This work was supported by the National Natural Science Foundation of China (Nos. 11001129 and 11226289) and by the Fundamental Research Funds for Nanjing University of Aeronautics and Astronautics (No. NN2012103).

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