The role of binomial type sequences in determination identities for Bell polynomials

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Abstract

Our paper deals about identities involving Bell polynomials. Some identities on Bell polynomials derived using generating function and successive derivatives of binomial type sequences. We give some relations between Bell polynomials and binomial type sequences in first part, and, we generalize the results obtained in [4] in second part.

Keywords. Partial and complete Bell polynomials. Binomial type sequences. Stirling numbers. Appell polynomials. Generating functions.

1 Introduction

Recall that the (exponential) partial Bell polynomials $B_{n,k}$ are defined by their generating function:

$$\sum_{n=k}^{\infty} B_{n,k}(x_1,x_2,...)\frac{t^n}{n!}=\frac{1}{k!}\left(\sum_{m=1}^{\infty}x_m\frac{t^m}{m!}\right)^k. \tag{1}$$

and the (exponential) complete Bell polynomials A_n are given by:

$$A_n(x_1,x_2,..):=\sum_{k=1}^{n}B_{n,k}(x_1,x_2,...)\quad\text{with}\quad A_0(x_1,x_2,..):=1.$$

Comtet [3] studies the Bell polynomials and gives some basic properties for them. Some applications of Bell polynomials are given by Riordan [5] in combinatorial analysis and by S. Roman [6] in umbral calculus. Recently, by using the Lagrange inversion formula (LIF), Abbas and Bouroubi [1] give some identities for the partial Bell polynomials, and, Mihoubi [4] also gives some extensions involving to the partial and complete Bell polynomials. For any sequence $(x_n; n\geq 1)$ with $x_1=1$ and any nonnegative integers r, s, recall that Proposition 4 in [4] gives:

$$\begin{aligned}B_{n,k}&\left(1,...,ms\frac{B((r+1)(m-1)+s,r(m-1)+s)}{(r(m-1)+s)\binom{(r+1)(m-1)+s}{r(m-1)+s}},...\right)\\&=\binom{n}{k}sk\frac{B((r+1)(n-k)+sk,r(n-k)+sk)}{(r(n-k)+sk)\binom{(r+1)(n-k)+sk}{r(n-k)+sk}},\end{aligned}\tag{2}$$

where $B(n,k) := B_{n,k}(x_1, x_2, x_3, ...)$. Then if we put

(3) $$Y(n,k) := \binom{n}{k} sk \frac{B((r+1)(n-k)+sk, r(n-k)+sk)}{(r(n-k)+sk)\binom{(r+1)(n-k)+sk}{r(n-k)+sk}},$$

we conclude, from (2), that the sequence $(Y(n,k))$ satisfies the equation:

(4) $$B_{n,k}(Y(1,1), Y(2,1), Y(3,1), ...) = Y(n,k).$$

In general, from the definition (1), if we put

$$\psi(t) = \sum_{m=1}^{\infty} x_m \frac{t^m}{m!} \text{ and } Y(n,k) = \frac{1}{k!} D_{t=0}^{n} (\psi(t))^k,$$

then for all integers n, k, with $n \geq k \geq 1$, the sequence $(Y(n,k))$ satisfies (4). Hence, to find identities for the partial Bell polynomials, it suffices to find sequences $(Y(n,k))$ satisfy the equation (4).
Similarly to the partial Bell polynomials, for any sequence $(x_n; n \geq 1)$ with $x_1 = 1$ and any nonnegative integers r, s $(r \geq 1)$, another relation for the complete Bell polynomials is given by Proposition 8 in [4] by:

(5) $$A_n\left(s\frac{x_2}{2}, ..., s\frac{B((r+1)n, nr)}{nr\binom{(r+1)n}{nr}}\right) = s\frac{B((r+1)n+s, nr+s)}{(nr+s)\binom{(r+1)n+s}{nr+s}},$$

where $B(n,k) := B_{n,k}(x_1, x_2, x_3, ...)$, and if we put

(6) $$Z(n,s) := \frac{B((r+1)n+s,\ nr+s)}{(nr+s)\binom{(r+1)n+s}{nr+s}},$$

we conclude, from (5), that the sequence $(Z(n,s))$ satisfies the equation:

(7) $$A_n(sZ(1,0), sZ(2,0), ..., sZ(n,0)) = sZ(n,s),$$

Hence, to find identities for the complete Bell polynomials, it suffices to find sequences $(Z(n,s))$ satisfy the equation (7). Therefore, to determine solutions for (4) and (7), we exploit the strong connection between Bell polynomials and binomial type sequences. For any binomial type sequence $(f_n(x))$, with $f_0(x) := 1$, one of such connections is given in [6, p. 82] by:

(8) $$f_n(x) = \sum_{k=1}^{n} B_{n,k}(x_1, x_2, ...) x^k \text{ with } x_n = \frac{d}{dx} f_n(x)|_{x=0}.$$

On the basis of the results obtained in [4] and the relation (8), we derive in this paper some interesting identities and relations related Bell polynomials and binomial type sequences.
For the next of this paper, we will denote by $D_t f(t)$, $D_t^j f(t)$, $D_{t=x} f(t)$ or $Df(x)$ and $D_{t=x}^j f(t)$, respectively, for the derivative of f, the $j-th$ derivative of f, the derivative of f evaluated at $t = x$, and the $j-th$ derivative of f evaluated at $t = x$.

2 Main results

For the next of this work, for any sequence $(f_n(x))$ of binomial type with initial term $f_0(x) = 1$ and for a given real number a, we define:

$$f_n(x;a) := \frac{x}{an+x} f_n(an+x) \quad \text{with} \quad f_0(x;a) = 1. \tag{9}$$

The sequence $(f_n(x;a))$ is of binomial type, see [4].
To simplify any expression below we put

$$T = T(n,k) := r(n-k) + sk, \quad R = R(n,s) := nr + s$$
$$\text{and } (f_n(x)) \text{ denotes a sequence of binomial type with } f_0(x) = 1.$$

The two following theorems give some interesting relations between Bell polynomials and binomial type sequences. These relations are used to deduce several identities for partial and complete Bell polynomials as it is illustrated below. To prove these theorems, we use the following Lemma:

Lemma 1 *Let n, k be integers, $n \geq k \geq 1$, and a, α be a real numbers. We have*

$$\begin{aligned} &B_{n,k}(\alpha, ..., mD_{z=0}(e^{\alpha z} f_{m-1}(x+z;a)), ...) \\ &= \binom{n}{k} D^k_{z=0}(e^{\alpha z} f_{n-k}(x+z;a)). \end{aligned} \tag{10}$$

This identity can be replaced when $\alpha = 0$ by:

$$B_{n,k}(D_x f_1(x;a), ..., D_x f_m(x;a), ...) = \frac{1}{k!} D^k_{z=0} f_n(kx+z;a). \tag{11}$$

Theorem 2 *Let a, x, α be real numbers and n, k, r, s be integers with $n \geq k \geq 1$ and $r + s \geq 1$. Then the sequence*

$$Y(n,k) := \binom{n}{k} \frac{sk}{T} D^T_{z=0}(e^{\alpha z} f_{n-k}(Tx+z;a)) \tag{12}$$

satisfies (4). For $\alpha = 0$ the sequence $(Y(n,k))$ can be replaced by:

$$Y(n,k) := \frac{n!}{k!(T+n-k)!} \frac{sk}{T} D^T_{z=0} f_{T+n-k}(Tx+z;a). \tag{13}$$

For $r = s = 0$, we put $Y(n,k) := \binom{n}{k} f_{n-k}(x;a)$.

Theorem 3 *Let a, x, α be real numbers and n, r, s be integers with $n \geq 1$, $r \geq 1$. Then the sequence*

$$Z(n,s) := \begin{cases} \frac{1}{\alpha^s} \frac{1}{R} D^R_{z=0}(e^{\alpha z} f_n(Rx+z;a)) & \text{if } \alpha \neq 0 \\ \frac{1}{(Df_1(0))^s} \frac{1}{R} \frac{n!}{(R+n)!} D^R_{z=0} f_{R+n}(Rx+z;a) & \text{if } \alpha = 0 \end{cases} \tag{14}$$

satisfies (7).

More generally, Theorem 2 can be generalized as follows:

Theorem 4 *Let* $(a_n; n \geq 1)$ *be a real sequence;* n, k, r, s, u, v *be integers with* $n \geq k \geq 1$, $r + s \geq 1$ *and* $x, a, \alpha, \beta, \lambda$ *be real numbers. Then the sequence*

$$(15) \qquad Y(n,k) := \binom{n}{k} \frac{sk}{T} T! \sum_{j \geq T} B_{j,T}(a_1, a_2, ...) D^{ju+vT}_{z=\beta j+\lambda T} \left\{ e^{\alpha z} f_{n-k}(z; a) \right\} \frac{x^j}{j!}$$

satisfies (4). For $\alpha = 0$ *the above sequence can be replaced by:*

$$(16) \qquad Y(n,k) := \frac{n!}{k!} \frac{sk}{T} T! \sum_{j=T}^{h+T} B_{j,T}(a_1, a_2, ...) \frac{D^{ju+vT}_{z=\beta j+\lambda T} f_{(u+v)T+n-k}(z; a)}{((u+v)T + n - k)!} \frac{x^j}{j!}$$

where $h = \left[\frac{n-k}{u}\right]$ *for* $u \geq 1$, $h = \infty$ *for* $u = 0$ *and* $[x]$ *is the largest integer* $\leq x$.

More generally, Theorem 3 can be generalized as follows:

Theorem 5 *Let* $(a_n; n \geq 1)$ *be a real sequence;* n, r, s, u, v *be integers with* $r \geq 1$ *and* $a, \alpha, \beta, \lambda$ *be real numbers. Then for* $\alpha \neq 0$, *the sequence*

$$(17) \qquad Z(n,s) := \begin{cases} \frac{R!}{\gamma_1^s R} \sum_{j \geq R} B_{j,R}(a_1, a_2, ...) D^{ju+vR}_{z=\beta j+\lambda R} \left\{ e^{\alpha z} f_n(z; a) \right\} \frac{x^j}{j!} & \text{if } \alpha \neq 0 \\ \frac{n! R!}{\gamma_2^s R} \sum_{j=R}^{g+R} \frac{B_{j,R}(a_1, a_2, ...)}{(n+(u+v)R)!} D^{ju+vR}_{z=\beta j+\lambda R} f_{n+(u+v)R}(z; a) \frac{x^j}{j!} & \text{if } \alpha = 0 \end{cases}$$

satisfies (7), where $g = \left[\frac{n}{u}\right]$ *for* $u \geq 1$, *and,* $g = \infty$ *for* $u = 0$; $\gamma_1 = \alpha^v \varphi(x\alpha^u)$;
$\gamma_2 = \begin{cases} a_1 x (Df_1(0))^{u+v} & \text{if } u \geq 1 \\ (Df_1(0))^v \varphi(x) & \text{if } u = 0 \end{cases}$ *and* $\varphi(x) = \sum_{i=1}^{\infty} a_i \frac{x^i}{i!}$.

Remark 6 *For* $a_n = 0$ $(n \geq 2)$ *in Theorem 4 we obtain Theorem 2. For* $a_n = 0$ $(n \geq 2)$ *in Theorem 5 we obtain Theorem 3. For* $x = \alpha = 0$ *in Theorem 2 we obtain Corollary 5 in [4]. For* $x = \alpha = 0$ *in Theorem 3 we get Corollary 9 in [4]. From (12) the sequence* $Y_1(n,k) := \lim_{\alpha \to 0} \alpha^{-T} Y(n,k)$ *satisfies (4) and gives Proposition 1 in [4]. From (14) the sequence* $Z_1(n,s) := \lim_{\alpha \to 0} \alpha^{-nr} Z(n,s)$ *satisfies (7) and gives Proposition 3 in [4]. By using (8) and (9) we can construct several binomial type polynomials as* $p_n(t) := t \sum_{k=1}^{n} Y(n,k)(bn+t)^{k-1}$ *with* $p_0(t) := 1$, *where* $(Y(n,k))$ *is given by (13) or (12).*

3 Applications

We give in this section another versions of Theorems 2, 3 and we present some particular cases of the above results.

3.1 Some applications of Theorem 2

The following corollaries gives a practical version of Theorem 2.

Corollary 7 *Under the hypothesis of Theorem 2 the sequence*

$$Y(n,k) := \binom{n}{k}\frac{sk}{T}\sum_{j=0}^{n-k}\binom{T}{j}D^{j}_{z=Tx}f_{n-k}(z;a)\,\alpha^{j} \tag{18}$$

satisfies (4).

Proof. From (12), we have

$$Y(n,k) = \binom{n}{k}\frac{sk}{T}\sum_{j=0}^{n-k}\binom{T}{j}D^{T-j}_{z=0}\left(e^{z/\alpha}\right)D^{j}_{z=0}f_{n-k}(Tx+z;a)$$
$$= \binom{n}{k}\frac{sk}{T}\sum_{j=0}^{n-k}\binom{T}{j}D^{j}_{z=0}f_{n-k}(Tx+z;a)\left(\frac{1}{\alpha}\right)^{T-j}.$$

To terminate, it suffices to remark that $\alpha^{T}Y(n,k)$ satisfies (4). ∎

Example 8 *For $f_n(x) = x^n$ the sequence $(Y(n,k))$ in (18) becomes:*

$$Y(n,k) :=$$
$$\binom{n}{k}\frac{sk}{T}\sum_{j=0}^{n-k}\binom{T}{j}\frac{(n-k)!}{(n-k-j)!}(Tx+a(n-k))^{n-k-j-1}(Tx+aj)\,\alpha^{j}$$

and for $a=0$, $\alpha=1$ the last sequence becomes:

$$Y(n,k) := T!\binom{n}{k}\frac{sk}{T}\sum_{j=0}^{n-k}\binom{n-k}{j}\frac{(Tx)^{n-k-j}}{(T-j)!}.$$

The following corollary gives a practical version of Theorem 2 when $\alpha = 0$.

Corollary 9 *Under the hypothesis of Theorem 2 the sequence*

$$Y(n,k) := \frac{sk}{T}\binom{n}{k}\binom{T+n-k}{n-k}^{-1}\sum_{j=0}^{n-k}\binom{T+j-1}{T-1}\times$$
$$B(T+n-k,T+j)\left(((r+1)c-b)j+b(n-k)+csk\right)\times$$
$$(b(n-k)+csk)^{T+j-1} \tag{19}$$

satisfies (4), and in particular the sequence

$$\text{(20)} \qquad Y(n,k) := \frac{sk}{T}\binom{n}{k}\binom{T+n-k}{n-k}^{-1} \times$$
$$\sum_{j=0}^{n-k}\binom{T+j-1}{T-1} B(T+n-k, T+j)(T+n-k)^j x^j$$

satisfies (4), where $B(n,k) := B_{n,k}(x_1, x_2, x_3, ..)$.

Proof. Let $(x_n; n \geq 1)$ be a sequence of real numbers. From (13), it suffices to express $Y(n,k)$ by considering the binomial type sequence:

$$\text{(21)} \qquad f_n(x;a) := x\sum_{k=1}^{n} B_{n,k}(x_1, x_2, ...)(an+x)^{k-1} \text{ with } f_0(x) = 1,$$

put after $b := ra + rx + a, \quad c := x + a$. To obtain (20), it suffices to choice $b = (r+1)c$, $c = x$ in (19). ■

Example 10 *By using the well-known identity* $B_{n,k}(1,2,...) = \binom{n}{k}k^{n-k}$, *if* $x_n = n$, *the sequence given by (20) becomes:*

$$Y(n,k) = \frac{n!}{k!}\frac{sk}{(T+n-k)!}\sum_{j=0}^{n-k}\binom{T+n-k}{T+j}(T+j)^{n-k-j}(T+n-k)^j\frac{x^j}{j!},$$

and by using the well-known identity $B_{n,k}(1!, 2!, ...) = \binom{n}{k}\frac{(n-1)!}{(k-1)!}$ *(the Lah numbers), if* $x_n = n!$, *the sequence given by (20) becomes:*

$$Y(n,k) = \frac{n!}{k!}\frac{sk}{T+n-k}\sum_{j=0}^{n-k}\binom{T+n-k}{T+j}(T+n-k)^j\frac{x^j}{j!}.$$

More examples can be obtained by choosing x_n *in (19) or (20) as* $1, nf_{n-1}(x;a)$, $D_x f_n(x;a)$.

Example 11 *For* $f_n(x) = B_n(x) = \sum_{j=1}^{n}\binom{n}{j}S(n,j)$, $r = 0$, $s = 1$ *and* $\alpha = 0$ *in (13), with* $S(n,j)$ *are the Stirling numbers of the second kind, we obtain:*

$$B_{n,k}\left(1, ..., \frac{xB_{m+1}(am+x) - (x^2 + amx - am)B_m(am+x)}{(am+x)^2}, ...\right)$$
$$= kx\sum_{l=0}^{n-k}\binom{l+k}{k}S(n, l+k)(an+kx)^{l-1}.$$

For $r = 0$, $s = 1$, $\alpha = 0$ *and* $f_n(x) := [x]_n = \sum_{j=1}^{n} s(n,j)\, x^j$ *in (13), where* $s(n,j)$ *are the Stirling numbers of the first kind, we obtain:*

$$B_{n,k}\left(1, \ldots, [am + x - 1]_{m-1}\left(m - \sum_{i=1}^{m-1} \frac{am - i}{am + x - i}\right), \ldots\right)$$
$$= kx \sum_{l=0}^{n-k} \binom{l+k}{k} s(n, l+k)\,(an + kx)^{l-1}.$$

For $r = 0$, $s = 1$, $\alpha = 0$ *and* $f_n(x) := [x]^n = \sum_{j=1}^{n} |s(n,j)|\, x^j$ *in (13), where* $|s(n,j)|$ *are the absolute Stirling numbers of the first kind, we obtain:*

$$B_{n,k}\left(1, \ldots, [am + x - 1]^{m-1}\left(m - \sum_{i=1}^{m-1} \frac{am + i}{am + i + x}\right), \ldots\right)$$
$$= kx \sum_{l=0}^{n-k} \binom{l+k}{k} |s(n, l+k)|\,(an + kx)^{l-1}.$$

3.2 Some applications of Theorem 3

The following corollaries gives a practical version of Theorem 3.

Corollary 12 *Under the hypothesis of Theorem 3 the sequence*

$$Z(n,s) := \frac{1}{R} \sum_{j=0}^{n} \binom{R}{j} D^j_{z=Rx} f_n(z;a)\, \alpha^j \tag{22}$$

satisfies (7).

Proof. From (14) we have
$Z(n,s) = \frac{\alpha^s}{R} D^R_{z=0}\left(e^{z/\alpha} f_n(Rx + z; a)\right)$
$= \frac{\alpha^s}{R} \sum_{j=0}^{n} \binom{R}{j} D^j_{z=0}\left(f_n(Rx + z; a)\right) \frac{1}{\alpha^{R-j}}$
$= \frac{1}{R} \sum_{j=0}^{n} \binom{R}{j} D^j_{z=0}\left(f_n(Rx + z; a)\right) \frac{\alpha^j}{\alpha^{nr}}.$
To terminate, it suffices to remark that $\alpha^{nr} Z(n,s)$ satisfies (4). ■

Example 13 *For* $f_n(x) = x^n$ *and* $r \geq 1$ *the sequence given by (22) becomes:*

$$Z(n,s) = \frac{1}{R} \sum_{j=0}^{n} \binom{R}{j} \frac{n!}{(n-j)!} (Rx + an)^{n-j-1} (Rx + aj)\, \alpha^j$$

and for $a = 0, x = 1$, the last sequence becomes:

$$Z(n,s) = \frac{1}{R}\sum_{j=0}^{n}\binom{R}{j}\frac{n!}{(n-j)!}R^{n-j}\alpha^{j}.$$

Corollary 14 *Under the hypothesis of Theorem 3 the sequence*

$$Z(n,s) := \frac{1}{(Df_1(0))^s}\frac{1}{R}\binom{R+n}{n}^{-1}\sum_{j=0}^{n}\binom{R+j-1}{R-1}\times \tag{23}$$

$$B(R+n,R+j)(cn+bs)^{j-1}\{((r+1)b-c)j+cn+bs\}$$

satisfies (7), and in particular the sequence

$$Z(n,s) := \frac{1}{(Df_1(0))^s}\frac{1}{R}\binom{R+n}{n}^{-1}\times \tag{24}$$

$$\sum_{j=0}^{n}\binom{R+j-1}{R-1}B(R+n,R+j)(R+n)^j x^j$$

satisfies (7), where $B(n,k) := B_{n,k}(x_1, x_2, x_3, ..)$.

Proof. Let $(x_n; n \geq 1)$ be a sequence of real numbers. From (14), it suffices to express $Z(n,s)$ by considering the binomial type sequence defined in (21), put after $b = a + x$, $c = a + ar + xr$. To obtain (24), it suffices to choice $b = (r+1)c$, $c = x$ in (23). ■

Example 15 *If $x_n = n$, the sequence given by (24) becomes:*

$$Z(n,s) = \frac{1}{(Df_1(0))^s}\sum_{j=0}^{n}\binom{n}{j}(R+j)^{n-j-1}(R+n)^j x^j$$

and if $x_n = n!$, the sequence given by (24) becomes:

$$Z(n,s) = \frac{(R+n-1)!}{(Df_1(0))^s}\sum_{j=0}^{n}\binom{n}{j}\frac{(R+n)^j}{(R+j)!}x^j.$$

3.3 Some applications of Theorem 4

Some particular cases of Theorem 4 are given by the following corollaries:

Corollary 16 *Under the hypothesis of Theorem 4 and $v \geq u$ the sequences*

$$Y_1(n,k) := \binom{n}{k}\frac{sk}{T}\sum_{j=0}^{T}\binom{T}{j}D_{z=\beta j+\lambda T}^{ju+vT}\{e^{\alpha z}f_{n-k}(z;a)\}x^j y^{T-j}, \tag{25}$$

$$Y_2(n,k) := \frac{n!}{k!}\frac{sk}{T}\sum_{j=0}^{T}\binom{T}{j}\frac{D_{z=\beta j+\lambda T}^{ju+vT}f_{vT+n-k}(z;a)}{(vT+n-k)!}x^j y^{T-j}$$

satisfy (4).

Proof. Let $a_1 = p,\ a_2 = 2q$ and $a_m = 0$ for $m \geq 3$. To obtain (25) it suffices to use the identity $B_{j,T}(p, 2q, 0, 0, ...) = \binom{T}{j-T} p^{2T-j} q^{j-T}$ in (16) and in (15). ■

Some particular cases of (26) are given by:
For $r = u = 0,\ v = 1$ or $u = v = r = 0$ in (25), the sequences

$$
\begin{aligned}
(26)\ Y_1(n,k) &:= \binom{n}{k}\sum_{j=0}^{sk}\binom{sk}{j} D^{sk}_{z=\beta j+\lambda k}\left\{e^{\alpha z} f_{n-k}(z;a)\right\} x^j y^{sk-j}, \\
Y_2(n,k) &:= \frac{n!}{k!}\sum_{j=0}^{sk}\binom{sk}{j}\frac{D^{sk}_{z=\beta j+\lambda k} f_{n+(s-1)k}(z;a)}{(n+(s-1)k)!} x^j y^{sk-j}, \\
Y_3(n,k) &:= \binom{n}{k}\sum_{j=0}^{sk}\binom{sk}{j} f_{n-k}(\beta j+\lambda k;a)\, x^j y^{sk-j},
\end{aligned}
$$

satisfy (4), and if $s = 1$ the last sequences give

$$
(27)\quad B_{n,k}\left(\alpha y, ..., m\left(\alpha y f_{m-1}(\lambda;a) + x D_{z=\beta} f_{m-1}(z;a)\right), ...\right) = \binom{n}{k}\sum_{j=0}^{k}\binom{k}{j} D^k_{z=0}\left\{e^{\alpha z} f_{n-k}((\beta-\lambda) j + \lambda k + z; a)\right\} x^j y^{k-j},
$$

$$
(28)\quad B_{n,k}\left((x+y) D f_1(0), ..., y D_{z=\lambda} f_n(z;a) + x D_{z=\beta} f_n(z;a), ...\right) = \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j} D^k_{z=0} f_n((\beta-\lambda) j + \lambda k + z; a)\, x^j y^{k-j}
$$

$$
(29)\quad B_{n,k}\left(y + x, ..., m\left(y f_{m-1}(\lambda;a) + x f_{m-1}(\beta;a)\right), ...\right) = \binom{n}{k}\sum_{j=0}^{k}\binom{k}{j} f_{n-k}((\beta-\lambda) j + \lambda k; a)\, x^j y^{k-j}
$$

Example 17 *For $f_n(x) = x^n,\ a = 0$ and $u = 0$ or 1 in (25) the sequences*

$$
\begin{aligned}
Y_1(n,k) &= \frac{sk}{T}\binom{n}{k}\sum_{j=0}^{T}\binom{T}{j}(\alpha T+\beta j)^{n-k} x^{T-j} y^j \text{ and} \\
Y_2(n,k) &= \frac{n!}{k!}\frac{sk}{T}\sum_{j=0}^{n-k}\binom{T}{j}\frac{(\alpha T+\beta j)^{n-k-j}}{(n-k-j)!} x^{T-j} y^j
\end{aligned}
$$

satisfy (4), and for $f_n(x) = x^n$, $a = 0$, $u = 0$ *or* 1, $a_n = n$ *in Theorem 4 the sequences*

$$Y_3(n,k) = \frac{sk}{T}\binom{n}{k}\sum_{j=0}^{\infty}(\alpha T + \beta j)^{n-k}\frac{(Tx)^j}{j!} \text{ and}$$

$$Y_4(n,k) = \frac{sk}{T}\binom{n}{k}\sum_{j=0}^{n-k}\binom{n-k}{j}(\alpha T + \beta j)^{n-k-j}(Tx)^j$$

satisfy (4).

An interesting relation between Bell polynomials and Appell polynomials can be viewed as a special case of Theorem 4 and it is given by:

Corollary 18 *For the sequence of polynomials* $(A_n(x,y,z))$ *defined by:*

$$A_n(x,y,z) := \sum_{j=0}^{n}\binom{n}{j}a_j((x+y)j + x + y + z)(xj + yn + x + y + z)^{n-1-j}$$

we have

$$B_{n,k}(A_0(x,y,z),..,mA_{m-1}(x,y,z),...) = \sum_{j=k}^{n}\binom{n}{j}B_{j,k}(a_0, 2a_1, 3a_2, ...)(jx + ny + kz)^{n-j-1}((x+y)j + kz),$$

and in particular when $x = y = 0$ *we get:*

(30) $$B_{n,k}(A_0(z), 2A_1(z), ...) = \sum_{j=k}^{n}\binom{n}{j}B_{j,k}(a_0, 2a_1, ...)(kz)^{n-j},$$

where $A_n(z) := \sum_{j=0}^{n}\binom{n}{j}a_j z^{n-j}$ *is an Appell polynomial.*

Proof. If suffices to use (16) with $f_n(x;a) = a(an+x)^{n-1}$, $r = v = 0$, $x = 1$, $s = u = 1$ and replace a_n by na_{n-1}. ∎

Example 19 *For any sequence* $(\varphi_n; n \geq 1)$ *for real numbers, let* I_n *be the identity matrix of order* n *and* (A_n) *be the sequence of matrices defined by:* $A_0 := 1$, $A_n := (a_{ij})$ *for* $1 \leq i,j \leq n$ *with* $a_{ij} = \varphi_{j-i+1}$ *if* $j \geq i$, $a_{i,i-1} = i-1$ *and* $a_{ij} = 0$ *otherwise. Then from [7, p. 110] and from (30) we get:*

$$B_{n,k}(1, ..., m\det(A_{m-1} + xI_{m-1}), ...) = \sum_{j=k}^{n}\binom{n}{j}B_{j,k}(1, ..., m\det A_{m-1}, ...)(kx)^{n-j}.$$

3.4 Some applications of Theorem 5

A particular case of Theorem 5 is given by the following corollary:

Corollary 20 *Under the hypothesis of Theorem 5 and $v \geq u$ the sequences*

$$(31)\quad Z_1(n,s) := \frac{1}{\gamma_1^s R}\sum_{j=0}^{R}\binom{R}{j} D^{ju+vR}_{z=\beta j+\lambda R}\left\{e^{\alpha z} f_n(z;a)\right\} x^j y^{R-j} \text{ and}$$

$$Z_2(n,s) := \frac{n!}{\gamma_2^s R}\sum_{j=0}^{R}\binom{R}{j}\frac{D^{ju+vR}_{z=\beta j+\lambda R} f_{n+(u+v)R}(z;a)}{(n+(u+v)R)!} x^j y^{R-j}$$

satisfy (7), where $\gamma_1 = \alpha^v(x\alpha^u + y)$ *and* $\gamma_2 = \begin{cases} (x+y)(Df_1(0))^v & \text{if } u \geq 1 \\ y(Df_1(0))^v & \text{if } u = 0. \end{cases}$

Proof. It suffices to put in Theorem 5 $x = 1$; $a_1 = p$, $a_2 = 2q$ and $a_m = 0$ for $m \geq 3$ and use the identity $B_{j,R}(p, 2q, 0, 0, \ldots) = \binom{R}{j-R} p^{2R-j} q^{j-R}$. ■

Example 21 *For $f_n(x) = x^n$, $a = 0$ and $u = 0$ or 1, the sequence given by (31) becomes:* $Z_1(n,s) = (x+y)^{-s}\frac{1}{R}\sum_{j=0}^{R}\binom{R}{j}(\beta j + \alpha R)^n x^j y^{R-j}$ *and*

$$Z_2(n,s) = \frac{n!}{R}\sum_{j=0}^{n}\frac{1}{(n-j)!}\binom{R}{j}(\beta j+\alpha R)^{n-j} x^j,$$

and for $f_n(x) = x^n$, $a = 0$, $x = 1$, $u = 0$ or 1, $a_n = nt^{n-1}$ the sequence given by (17) becomes: $Z_3(n,s) = \frac{1}{R}\sum_{j=0}^{n}\binom{n}{j}(\beta j+\alpha R)^{n-j} R^j t^j$ *and*

$$Z_4(n,s) = \frac{\exp(-sx)}{R!}\sum_{j=0}^{\infty}(\beta j+\alpha R)^n \frac{(Rx)^j}{j!}.$$

4 Proof of the main results

Proof of Lemma 1. Let $F(t;a)^x := 1 + \sum_{n=1}^{\infty} f_n(x;a)\frac{t^n}{n!}$. Now, because $f_n(x;a)$ is a polynomial of degree n, then the proof follows from the following expansions of $g_k(t) := \sum_{n=0}^{\infty} D^k_{z=0}\left(e^{\alpha z} f_n(kx+z;a)\right)\frac{t^{n+k}}{n!}$:

$$g_k(t) = \sum_{n=k}^{\infty}\frac{n!}{(n-k)!} D^k_{z=0}\left(e^{\alpha z} f_{n-k}(kx+z;a)\right)\frac{t^n}{n!}, \text{ and}$$

$$\begin{aligned} g_k(t) &= D^k_{z=0}\left(e^{\alpha z}\sum_{n=0}^{\infty} f_n(kx+z;a)\frac{t^{n+k}}{n!}\right) \\ &= D^k_{z=0}\left(e^{\alpha z} F(t;a)^{kx+z} t^k\right) \\ &= t^k F(t;a)^{kx} D^k_{z=0}\left(e^{(\alpha+\ln F(t;a))z}\right) \end{aligned}$$

$$
\begin{aligned}
&= t^k F(t;a)^{kx} (\alpha + \ln F(t;a))^k \\
&= t^k e^{-\alpha k x} \left((e^{\alpha} F(t;a))^x \ln (e^{\alpha} F(t;a)) \right)^k \\
&= t^k e^{-\alpha k x} \left(D_x (e^{\alpha} F(t;a))^x \right)^k \\
&= t^k \left(\sum_{m=0}^{\infty} e^{-\alpha x} D_x \left(e^{\alpha x} f_m(x;a) \right) \tfrac{t^m}{m!} \right)^k \\
&= \left(\sum_{m=1}^{\infty} m e^{-\alpha x} D_x \left(e^{\alpha x} f_{m-1}(x;a) \right) \tfrac{t^m}{m!} \right)^k \\
&= k! \sum_{n=k}^{\infty} B_{n,k} \left(\alpha, ..., m e^{-\alpha x} D_x \left(e^{\alpha x} f_{m-1}(x;a), ... \right) \right) \tfrac{t^n}{n!}.
\end{aligned}
$$

Then by identification we get:

$$
B_{n,k} \left(\alpha, ..., m D_x \left(e^{\alpha x} f_{m-1}(x;a) \right), ... \right) = \binom{n}{k} D_{z=0}^k \left(e^{\alpha z} f_{n-k}(kx+z;a) \right).
$$

To obtain (10), it suffices to remark that for $m \geq 1$ we have:

$$
e^{-\alpha x} D_x \left(e^{\alpha x} f_m(x;a) \right) = D_{z=0} \left(e^{\alpha z} f_m(x+z;a) \right).
$$

When $\alpha = 0$, it's easy to verify that the identity (10) is equivalent to (11). ■

Proof of Theorem 2. When we replace x_n by αx_n in (2) and we use the well-known identities

$$
\begin{aligned}
&B_{n,k} (\alpha x_1, \alpha x_2, \alpha x_3 ...) = \alpha^k B_{n,k} (x_1, x_2, x_3, ...) \text{ and} \\
&B_{n,k} \left(\alpha x_1, \alpha^2 x_2, \alpha^3 x_3 ... \right) = \alpha^n B_{n,k} (x_1, x_2, x_3, ...),
\end{aligned}
\tag{32}
$$

it results that the identity (2) remains true for $x_1 \neq 1$. Then for $r + s \geq 1$ and for the choice $x_n = n D_{z=0} \left(e^{\alpha z} f_{n-1}(x+z;a) \right)$ in (2), the identity (10) proves that the sequence $(Y(n,k))$ given by (3) becomes:

$$
\begin{aligned}
Y(n,k) &= \binom{n}{k} \tfrac{sk}{T} \frac{B_{T+n-k,\ T} \left(\alpha, ..., m D_{z=0} \left(e^{\alpha z} f_{m-1}(x+z;a) \right), ... \right)}{\binom{T+n-k}{T}} \\
&= \binom{n}{k} \tfrac{sk}{T} D_{z=0}^T \left\{ e^{\alpha z} f_{n-k}(Tx+z;a) \right\}.
\end{aligned}
$$

For the particular case $\alpha = 0$, if we take $x_n = D_{z=0} \left(f_n(x+y;a) \right)$ in (2) and we use the identity (11), the sequence $(Y(n,k))$ given by (3) becomes:

$$
\begin{aligned}
Y(n,k) &= \binom{n}{k} \tfrac{sk}{T} \binom{T+n-k}{T}^{-1} B_{T+n-k,\ T} \left(D_{z=0} \left(e^{\alpha z} f_1(x+z;a) \right), ... \right) \\
&= \tfrac{n!}{k!} \tfrac{sk}{T} \tfrac{1}{(T+n-k)!} D_{z=0}^T f_{T+n-k}(Tx+z;a).
\end{aligned}
$$

Note that for the case $r = s = 0$, the sequence $(Y(n,k))$ given by (12) is not defined. We put in this case $Y(n,k) = \binom{n}{k} f_{n-k}(x;a)$. Proposition 1 in [4] proves that this sequence satisfies (4). ■

Proof of Theorem 3. Case $\alpha \neq 0$:
Let $x_n = \frac{n}{\alpha} D_{z=0} \left\{ e^{\alpha z} f_{n-1}(x+z;a) \right\}$. We have $x_1 = 1$ and then the identity (10) proves that the sequence $(Z(n,s))$ given by (6) becomes:

$$Z(n,s) = \frac{1}{\alpha^R}\frac{B_{R+n,\ R}(\alpha,\dots,mD_{z=0}(e^{\alpha z}f_{m-1}(x+z;a)),\dots)}{R\binom{R+n}{R}}$$
$$= \frac{1}{\alpha^R}\frac{1}{R}D_{z=0}^R\left(e^{\alpha z}f_n(Rx+z;a)\right).$$

Case $\alpha = 0$: Because $D_x f_1(x) = xDf_1(0)$, then for $x_n = \frac{D_x f_n(x;a)}{D_x f_1(x;a)}$ we get $D_x f_1(x;a) = D_x\left(\frac{x}{x+a}f_1(x+a)\right) = Df_1(0) \neq 0$. We have $x_1 = 1$ and then by using the identity (11), the sequence $(Z(n,s))$ given by (6) becomes:

$$Z(n,s) = \frac{B_{R+n,\ R}(D_{z=0}(f_1(x+z;a)),\dots,D_{z=0}(f_m(x+z;a)),\dots)}{R\binom{R+n}{R}(D_x f_1(x;a))^R}$$
$$= \frac{1}{(Df_1(0))^R}\frac{1}{R}\frac{n!}{(R+n)!}D_{z=0}^R f_{R+n}(kx+z;a).$$

Note that if $Z(n,s)$ satisfies (7) then $\lambda^n Z(n,s)$ satisfies (7). ■

Proof of Theorem 4. Let $(a_n; n \geq 1)$ be a real sequence; u, v be a nonnegative integers, $F(t)^x := \sum_{n=0}^{\infty} f_n(x)\frac{t^n}{n!}$, $(f_0(x) = 1)$, and $G_k(t) := \sum_{n\geq 0} F(n,k)\frac{t^{n+k}}{n!}$ with $F(n,k) := \sum_{j\geq k} B_{j,k}(a_1,a_2,\dots)D_{z=0}^{ju+vk}\left(e^{\alpha z}f_n(\beta j+\lambda k+z;a)\right)\frac{x^j}{j!}$.

Then we have

$$G_k(t) = t^k\sum_{j\geq k}B_{j,k}(a_1,a_2,\dots)\frac{x^j}{j!}\left(\sum_{n\geq 0}D_{z=0}^{ju+vk}\{e^{\alpha z}f_n(\beta j+\lambda k+z;a)\}\frac{t^n}{n!}\right)$$
$$= t^k\sum_{j\geq k}B_{j,k}(a_1,a_2,\dots)\frac{x^j}{j!}D_{z=0}^{ju+vk}\left(e^{\alpha z}\sum_{n\geq 0}f_n(\beta j+\lambda k+z;a)\frac{t^n}{n!}\right)$$
$$= t^k\sum_{j\geq k}B_{j,k}(a_1,a_2,\dots)\frac{x^j}{j!}D_{z=0}^{ju+vk}\left(e^{\alpha z}(F(t;a))^{\beta j+\lambda k+z}\right)$$
$$= t^k\sum_{j\geq k}B_{j,k}(a_1,a_2,\dots)\frac{x^j}{j!}(F(t;a))^{\beta j+\lambda k}D_{z=0}^{ju+vk}\left(e^{\alpha z}F(t;a)^z\right)$$
$$= t^k\sum_{j\geq k}B_{j,k}(a_1,a_2,\dots)\frac{x^j}{j!}(F(t;a))^{\beta j+\lambda k}(\alpha+\ln F(t;a))^{ju+vk}$$
$$= t^k\sum_{j\geq k}B_{j,k}(a_1,a_2,\dots)\frac{x^j}{j!}F(t;a)^{\beta j+\lambda k}\left(D_{z=0}\left(e^{\alpha z}(F(t;a))^z\right)\right)^{ju+vk}$$
$$= t^k\sum_{j\geq k}B_{j,k}(u_1,u_2,\dots)\frac{x^j}{j!}$$
$$= \frac{t^k}{k!}\left(\sum_{m\geq 1}u_m\frac{x^m}{m!}\right)^k$$
$$= \frac{t^k}{k!}\left(\sum_{m\geq 1}a_m D_{z=0}^{mu+v}\left(e^{\alpha z}F(t;a)^{\beta m+\lambda+z}\right)\frac{x^m}{m!}\right)^k$$
$$= \frac{t^k}{k!}\left(\sum_{m\geq 1}a_m\sum_{j\geq 0}D_{z=0}^{mu+v}\left(e^{\alpha z}f_j(\beta m+\lambda+z;a)\frac{t^j}{j!}\right)\frac{x^m}{m!}\right)^k$$
$$= \frac{t^k}{k!}\left(\sum_{j\geq 0}\frac{t^j}{j!}\sum_{m\geq 1}a_m D_{z=0}^{mu+v}\left(e^{\alpha z}f_j(\beta m+\lambda+z;a)\right)\frac{x^m}{m!}\right)^k$$

$$= \frac{t^k}{k!}\left(\sum_{j\geq 0} F(j,1)\frac{t^j}{j!}\right)^k$$
$$= \frac{1}{k!}\left(\sum_{j\geq 1} jF(j-1,1)\frac{t^j}{j!}\right)^k$$
$$= \sum_{n\geq k} B_{n,k}\left(F(j,1),...,jF(j-1,1),...\right)\frac{t^n}{n!},$$

where $u_m = a_m F(t;a)^{\beta m+\lambda}\left(D_{z=0}\left(e^{\alpha z}\left(F(t;a)\right)^z\right)\right)^{mu+v}$, $m\geq 1$.
Then we obtain $B_{n,k}\left(F(0,1),...,jF(j-1,1),...\right) = k!\binom{n}{k}F(n-k,k)$, and this means that the sequence

$$(33)\qquad \begin{array}{c} X(n,k) := \\ k!\binom{n}{k}\sum_{j\geq k} B_{j,k}(a_1,a_2,..)\, D_{z=0}^{ju+vk}\left\{e^{\alpha z} f_{n-k}(\beta j+\lambda k+z;a)\right\}\frac{x^j}{j!}\end{array}$$

satisfies (4). To obtain (15), it suffices to take $x_n = X(n,1)$ in (2), where $(X(n,k))$ is given by (33). Indeed, the sequence given by (3) becomes:

$$\begin{aligned} Y(n,k) &= \tbinom{n}{k}\tfrac{sk}{T}\tbinom{T+n-k}{T}^{-1} B_{T+n-k,\,T}\left(X(1,1),X(2,1),...\right)\\ &= \tbinom{n}{k}\tfrac{sk}{T}\tbinom{T+n-k}{T}^{-1} X(T+n-k,\,T)\\ &= \tbinom{n}{k}\tfrac{sk}{T}T!\sum_{j\geq T} B_{j,T}(a_1,a_2,...)\, D_{z=0}^{ju+vT}\left\{e^{\alpha z} f_{n-k}(\beta j+\lambda T+z;a)\right\}\tfrac{x^j}{j!}.\end{aligned}$$

For the particular case $\alpha = 0$ and $u \geq 1$ we remark that $X(n,1) = 0$ for $u+v > n-1$ and then the identity (33) becomes:

$$(34)\qquad B_{n,k}\left(\underbrace{0,...,0}_{u+v}, X(u+v+1,1), X(u+v+2,1),...\right) = X(n,k),$$

or, equivalently, by using the well-known identity

$$B_{n,k}(0,...,0,a_{r+1},...) = \frac{n!}{(n-rk)!}B_{n-rk,k}\left(\frac{a_{1+r}}{(1+r)!},...,\frac{m!a_{m+r}}{(m+r)!},...\right),$$

the identity (34) becomes:

$$(35)\quad B_{n,\,k}\left(X^*(1,1),X^*(2,1),...\right) = X^*(n,k) := \frac{n!X(n+(u+v)k,k)}{(n+(u+v)k)!}.$$

To obtain (16) it suffices to take $x_n = X^*(n,1)$ in (2). Indeed,

$$\begin{aligned} Y(n,k) &= \tbinom{n}{k}\tfrac{sk}{T}\tbinom{T+n-k}{T}^{-1} B_{T+n-k,\,T}\left(X^*(1,1),X^*(2,1),...\right)\\ &= \tbinom{n}{k}\tfrac{sk}{T}\tbinom{T+n-k}{T}^{-1} X^*(T+n-k,\,T)\\ &= \tfrac{n!}{k!}\tfrac{sk}{T}T!\sum_{j\geq T} B_{j,T}(a_1,a_2,...)\times\\ &\qquad D_{z=0}^{ju+vT}\left\{e^{\alpha z}\tfrac{f_{(u+v)T+n-k}(\beta j+\lambda T+z;a)}{((u+v)T+n-k)!}\right\}\tfrac{x^j}{j!}.\ \blacksquare\end{aligned}$$

Proof of Theorem 5. For $\alpha \neq 0$ it suffices to put $x_n = X(n,1)/X(1,1)$ in (5), where $(X(n,k))$ is the sequence given by (33). Indeed, we have:
$x_1 = X(1,1) = \sum_{j\geq 1} a_j \alpha^{ju+v} \frac{x^j}{j!} = \alpha^v \varphi(x\alpha^u)$ with $\varphi(x) := \sum_{j\geq 1} a_j \frac{x^j}{j!}$.
The sequence $(Z(n,s))$ given by (6) becomes:

$$Z(n,s) = \frac{1}{(X(1,1))^R} \frac{1}{R} \binom{R+n}{R}^{-1} X(R+n,R)$$
$$= \frac{1}{(\alpha^v \varphi(x\alpha^u))^R} \frac{R!}{R} \sum_{j\geq R} B_{j,R}(a_1,a_2,...) \times$$
$$D_{z=0}^{ju+vR} \{e^{\alpha z} f_n(\beta j + \lambda R + z; a)\} \frac{x^j}{j!}.$$

For $\alpha = 0$ it suffices to put in (5) $x_n = X^*(n,1)/X^*(1,1)$, where $(X^*(n,k))$ is the sequence given by (35). We have

$$X^*(1,1) = \frac{X(u+v+1,1)}{(u+v+1)!} = \begin{cases} a_1 x (Df_1(0))^{u+v} & \text{if } u \geq 1 \\ (Df_1(0))^v \varphi(x) & \text{if } u = 0. \end{cases}$$

The sequence $(Z(n,s))$ given by (6) becomes:

$$Z(n,s) = \frac{1}{(X^*(1,1))^R} \frac{1}{R} \binom{R+n}{R}^{-1} X^*(R+n,R)$$
$$= \frac{n!}{(X^*(1,1))^R} \frac{R!}{R} \sum_{j\geq R} B_{j,R}(a_1,a_2,...) \times$$
$$D_{z=0}^{ju+vR} \left\{ \frac{f_{n+(u+v)R}(\beta j + \lambda R + z; a)}{(n+(u+v)R)!} \right\} \frac{x^j}{j!}. \blacksquare$$

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