(k, α_{n-1})-Fibonacci numbers and P_k-matchings in multigraphs

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ABSTRACT: In this paper we generalize the Fibonacci numbers and the Lucas numbers with respect to n, respectively $n+1$ parameters. Using these definitions we count special subfamilies of the set of n integers. Next we give the graph interpretations of these numbers with respect to the number of P_k-matchings in special graphs and we apply it for proving some identity and also for counting other subfamilies of the set of n integers.

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1 Introduction

We use the standard terminology and notation of the combinatorics and the graph theory, see [1, 2].
The n-th Fibonacci number F_n is defined recursively in the following way $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$, for $n \geq 2$. The n-th Lucas number is defined by $L_0 = 2$, $L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$, for $n \geq 2$.
Many concepts have arisen generalizing the Fibonacci numbers and the Lucas numbers, but a very natural is the concept of the generalized Fibonacci numbers $F(k, n)$ and generalized Lucas numbers $L(k, n)$ introduced by Kwaśnik and Włoch in [7]. This generalization is directly related to studying the concept of k-independent sets in graphs [11, 12, 13]. It is worth mentioning that k-independent sets (and also k-kernels in digraphs) are intensively studied by Galeana-Sánchez and Hernández-Cruz, see for example their last interesting papers [3, 4, 5].
Let $k \geq 2$ be integer and let $X = \{1, ..., n\}$ be the set of n integers, $n \geq 1$. Let $Y \subset X$ such that for each $i, j \in Y$ holds $|i - j| \geq k$. Note that in particular Y can be empty. The number $F(k, n)$ is defined as the number of all subsets Y and it was proved,(see [7]) that $F(k, n) = n + 1$ for $n = 0, 1, ..., k$ and $F(k, n) = F(k, n-1) + F(k, n-k)$ for $n \geq k + 1$. Let $Y^* \subset X$ such that for each $i, j \in Y^*$ holds $k \leq |i - j| \leq n - k$. Then the number $L(k, n)$ is the number of all subsets Y^* including also the empty set, and it was proved that

$L(k,n) = n+1$ for $n = 0, 1, ..., 2k-1$ and
$L(k,n) = (k-1)F(k, n-(2k-1)) + F(k, n-(k-1))$ for $n \geq 2k$. Recently some properties of the generalized Fibonacci numbers $F(k,n)$ and the generalized Lucas numbers $L(k,n)$ were given in [10]. Among other things more comfortable recurrence relation for the generalized Lucas numbers was proved, namely $L(k,n) = L(k,n-1) + L(k,n-k)$, for $n \geq 2k$. Clearly for $k = 2$, $F(2,n) = F_{n+2}$, for $n \geq 0$ and $L(2,n) = L_n$ for $n \geq 2$.

In this paper we give a generalization of the Fibonacci numbers and the Lucas numbers with respect to n, respectively $n+1$ parameters. Next we give a graph interpretation of introduced numbers with respect to the number of all P_k-matchings in special multigraphs.

Let G and H be two graphs. By an H-matching M of G we mean a subgraph of G such that all connected components of M are isomorphic to H. Moreover the empty set also is a H-matching, for every graph H. We can observe that if $H = K_2$, then K_2-matching is a matching in the classical sense. If $H = K_1$, then an induced K_1-matching is an independent set in the classical sense. The definition of H-matching naturally extend the concept of independent sets and matchings.

There are many papers related to the counting problems of induced K_1-matchings and K_2-matchings in graphs, see for example [6].

In 1971 the Japanese chemist Hosoya introduced to the chemical literature the parameter $Z(G)$ of a molecular graph as the number of all K_2-matchings of a graph G. He showed that certain physicochemical properties of alkanes are well correlated with $Z(G)$. In 1989 the American chemists Merrifield and Simmons introduced another graph parameter $\sigma(G)$ as the number of all induced K_1-matchings of a graph G, see [8]. From the formal point of view the definition of the Merrifield-Simmons index is analogous to the definition of the Hosoya index.

In the mathematical literature a real interest in counting of independent sets (i.e induced K_1-matchings) and matchings in graphs was initiated by Prodinger and Tichy in [9]. In this paper among other things they showed the connections between the number of all independent sets $\sigma(G)$, for special graphs and the Fibonacci numbers and the Lucas numbers. In particular for an n-vertex path P_n and an n-vertex cycle C_n they proved that $\sigma(P_n) = F_{n+2}$ and $\sigma(C_n) = L_n$. Consequently $Z(P_n) = F_{n+1}$ and $Z(C_n) = L_n$. This short paper gave impetus for counting independent sets and matchings in graphs.

In recent years a lot of work has been done in counting field and the last survey of Gutmann and Wagner [6] collects and classifies the results concerning these two indices. Most of them have been achieved quite recently, see its references where this type of the problem was studied.

2 Generalization of the Fibonacci numbers

Let $X = \{1, 2, ..., n\}, n \geq 2$ be the set of n integers. Let $\mathcal{X}_n$ be a multi-family of subsets of X such that $\mathcal{X}_n = \{\mathcal{X}_{i,i+1}; i = 1, 2, ..., n-1\}$, where the family $\mathcal{X}_{i,i+1}$ contains $p_i, p_i \geq 1$ subsets $\{i, i+1\}$. In the other words we can write $\mathcal{X}_n = \{\underbrace{\{1,2\},\{1,2\},...,\{1,2\}}_{p_1-\text{times}}, \underbrace{\{2,3\},\{2,3\},...,\{2,3\}}_{p_2-\text{times}},...,$
$\underbrace{\{n-1,n\},\{n-1,n\},...,\{n-1,n\}}_{p_{n-1}-\text{times}}\}, n \geq 2.$

Let $k \geq 2$ be an integer. For fixed $1 \leq t \leq n-k+1$, by $\mathcal{Y}(k,t)$ we denote a subfamily of $\mathcal{X}_n$ such that $\mathcal{Y}(k,t) = \{\{t+j, t+j+1\}, j = 0, ..., k-2\}$

Let $\mathcal{Y} \subset \mathcal{X}_n$ be a subfamily of $\mathcal{X}_n$ such that
(i) $|\mathcal{Y}| = m$, for fixed $m \geq 0$
(ii) for each $\mathcal{Y}(k,t), \mathcal{Y}(k,q) \in \mathcal{Y}$ such that $t \neq q$ holds $|q-t| \geq k$.

Let $\alpha_{n-1} = (p_1, p_2, ..., p_{n-1})$ be the sequence of values p_i, where $i = 1, ..., n-1$ and next let $\alpha_{n-i} = (p_1, p_2, ..., p_{n-i})$ be the subsequence of α_{n-1} obtained by deleting words $p_{n-i+1}, ..., p_{n-1}$, for $1 \leq i \leq n-1$. If $f^{(\alpha_{n-1})}(k,n,m)$ is the number of all m-elements subfamilies $\mathcal{Y}$ then $F^{(\alpha_{n-1})}(k,n) = \sum\limits_{m \geq 0} f^{(\alpha_{n-1})}(k,n,m)$ is the number of all subfamilies $\mathcal{Y}$. The number $F^{(\alpha_{n-1})}(k,n)$ will be named as the (k, α_{n-1})-Fibonacci number. If $p_i = p$ for all $i = 1, ..., n-1$, then the number $F^{(\alpha_{n-1})}(k,n)$ we will denote by $F^p(k,n)$.

Theorem 2.1 *Let $n \geq 2, m \geq 0, 2 \leq k \leq n$ be integers. Then* $f^{(\alpha_{n-1})}(k,n,0) = 1, f^{(\alpha_{n-1})}(k,n,1) = \sum\limits_{i=1}^{n-k+1} \prod\limits_{j=0}^{k-2} p_{i+j}$. *For $m \geq 2$ we have*

$$f^{(\alpha_{n-1})}(k,n,m) = \prod_{i=1}^{k-1} p_{n-i} f^{(\alpha_{n-k-1})}(k, n-k, m-1) + f^{(\alpha_{n-2})}(k, n-1, m).$$

PROOF: For $m = 0, 1$ the initial conditions are obvious. Let $m \geq 2$ and $|\mathcal{Y}| = m$. Let $f_n^{(\alpha_{n-1})}(k,n,m)$ (respectively $f_{-n}^{(\alpha_{n-1})}(k,n,m)$) be the number of all m-elements subfamilies $\mathcal{Y}$ such that $\mathcal{X}_{n-1,n} \cap \mathcal{Y} \neq \emptyset$ (respectively: $\mathcal{X}_{n-1,n} \cap \mathcal{Y} = \emptyset$). Then $f^{(\alpha_{n-1})}(k,n,m) = f_n^{(\alpha_{n-1})}(k,n,m) + f_{-n}^{(\alpha_{n-1})}(k,n,m)$. Two cases occur now:

1. $\mathcal{X}_{n-1,n} \cap \mathcal{Y} \neq \emptyset$.

Then there exists $\mathcal{Y}(k,q) \in \mathcal{Y}$ such that exactly one subset $\{n-1, n\}$ from $\mathcal{X}_{n-1,n}$ belongs to $\mathcal{Y}(k,q)$. Then the definition of the family $\mathcal{Y}$ implies that $\{n-k+i, n-k+i+1\} \in \mathcal{Y}(k,q), i = 1, ..., k-1$ hence $q = n-k+1$. Moreover for each $\mathcal{Y}(k,t) \in \mathcal{Y}, t \neq q$ we have $\{n-k, n-k+1\} \notin \mathcal{Y}(k,t)$ (otherwise the condition (ii) does not hold). This means that each subfamily $\mathcal{Y}(k,t) \in \mathcal{Y}$ is the subfamily of $\mathcal{X}_n - \bigcup\limits_{i=0}^{k-1} \mathcal{X}_{n-k+i,n-k+i+1} = \mathcal{X}_{n-k}$.

Thus $\mathcal{Y} = \mathcal{Y}^* \cup \mathcal{Y}(k,q)$, where $\mathcal{Y}^*$ is $(m-1)$-elements subfamily of $\mathcal{X}_{n-k}$ satisfying conditions (i) and (ii).

Since each subset of the subfamily $\mathcal{Y}(k,q) \in \mathcal{Y}$ such that $\mathcal{X}_{n-1,n} \cap \mathcal{Y}(k,q) \neq \emptyset$ could be chosen on $p_{n-1}, p_{n-2}, \dots, p_{n-k+1}$ ways, respectively we have that $f_n^{(\alpha_{n-1})}(k,n,m) = \prod_{i=1}^{k-1} p_{n-i} f^{(\alpha_{n-k-1})}(k,n-k,m-1)$.

2. $\mathcal{X}_{n-1,n} \cap \mathcal{Y} = \emptyset$.

Then for each $\mathcal{Y}(k,q) \in \mathcal{Y}$ we have $\mathcal{X}_{n-1,n} \cap \mathcal{Y}(k,q) = \emptyset$. So $\mathcal{Y}$ is m-elements subfamily of $\mathcal{X}_n \setminus \mathcal{X}_{n-1,n} = \mathcal{X}_{n-1}$. Then $f_{-n}^{(\alpha_n-1)}(k,n,m) = f^{(\alpha_{n-2})}(k,n-1,m)$.

Finally from the above cases $f^{(\alpha_{n-1})}(k,n,m) = \prod_{i=1}^{k-1} p_{n-i} f^{(\alpha_{n-k-1})}(k,n-k,m-1) + f^{(\alpha_{n-2})}(k,n-1,m)$.

Thus the Theorem is proved. □

Theorem 2.2 *Let $k \geq 2$, $n \geq 2$ be integers. Then for $n \geq 2k$*

$$F^{(\alpha_{n-1})}(k,n) = F^{(\alpha_{n-2})}(k,n-1) + \prod_{i=1}^{k-1} p_{n-i} F^{(\alpha_{n-k-1})}(k,n-k)$$

with initial conditions

$$F^{(\alpha_{n-1})}(k,n) = 1 + \sum_{j=1}^{n-(k-1)} \prod_{t=0}^{k-2} p_{j+t} \text{ for } n = 2,\dots,2k-1.$$

PROOF: Let $n \leq 2k-1$. Then

$$F^{(\alpha_{n-1})}(k,n) = \sum_{m \geq 0} f^{(\alpha_{n-1})}(k,n,m) = f^{(\alpha_{n-1})}(k,n,0) + f^{(\alpha_{n-1})}(k,n,1) =$$
$$1 + \sum_{j=1}^{n-(k-1)} \prod_{t=0}^{k-2} p_{j+t} \text{ by Theorem 2.1.}$$

Assume now that $n \geq 2k$. Then

$$F^{(\alpha_{n-1})}(k,n) = \sum_{m \geq 0} f^{(\alpha_{n-1})}(k,n,m) =$$
$$f^{(\alpha_{n-1})}(k,n,0) + f^{(\alpha_{n-1})}(k,n,1) + \sum_{m \geq 2} f^{(\alpha_{n-1})}(k,n,m) =$$
$$1 + \sum_{j=1}^{n-(k-1)} \prod_{t=0}^{k-2} p_{j+t} + \sum_{m \geq 2} f^{(\alpha_{n-2})}(k,n-1,m) +$$
$$\prod_{i=1}^{k-1} p_{n-i} \sum_{m \geq 2} f^{(\alpha_{n-k-1})}(k,n-k,m-1) =$$
$$1 + \sum_{j=1}^{n-(k-1)} \prod_{t=0}^{k-2} p_{j+t} - 1 - \sum_{j=1}^{n-k} \prod_{t=0}^{k-2} p_{j+t} + \sum_{m \geq 0} f^{(\alpha_{n-2})}(k,n-1,m) +$$
$$\prod_{i=1}^{k-1} p_{n-i}(-1 + \sum_{m \geq 0} f^{(\alpha_{n-k-1})}(k,n-k,m)) =$$
$$1 + \sum_{j=1}^{n-k} \prod_{t=0}^{k-2} p_{j+t} + p_{n-k+1} \cdot \ldots \cdot p_{n-1} - 1 - \sum_{j=1}^{n-k} \prod_{t=0}^{k-2} p_{j+t} + \sum_{m \geq 0} f^{(\alpha_{n-2})}(k,n-$$

$$1,m) - p_{n-1}\cdot \ldots \cdot p_{n-k+1} + \prod_{i=1}^{k-1} p_{n-i} \sum_{m\geq 0} f^{(\alpha_{n-k-1})}(k,n-k,m) =$$

$$F^{(\alpha_{n-2})}(k,n-1) + \prod_{i=1}^{k-1} p_{n-i} F^{(\alpha_{n-k-1})}(k,n-k)$$

which ends the proof.

Thus the Theorem is proved. □

If $p_i = 1$, for $i = 1,...,n-1$ then for $k \geq 2$ the number $F^{(\alpha_{n-1})}(k,n)$ gives the generalized Fibonacci number $F(k, n-(k-1))$. Moreover if additionally $k = 2$ then $F(2, n-1)$ gives the Fibonacci number F_{n+1} defined in the classical sense.

Let $X = \{1, 2, ..., n\}$, $n \geq 3$ be the set of integers. For $i, j \in X \cup \{0\}$ let $i \oplus j = i + j$ when $i + j \leq n$ or $i \oplus j = i + j - n$ when $i + j \geq n + 1$.

Let $\mathcal{X}_n^*$ be a multifamily of subsets of X such that $\mathcal{X}_n^* = \{\mathcal{X}_{i,i+1}^*;\ i = 1, 2, ..., n-1\} \cup \{\mathcal{X}_{n,1}^*\}$, where the family $\mathcal{X}_{i,i+1}^*$ contains p_i subsets $\{i, i+1\}$, $i = 1, ..., n-1$ and $\mathcal{X}_{n,1}^*$ contains p_n subsets $\{n, 1\}$, $p_i \geq 1$, for $i = 1, ..., n$. In the other words we can write $\mathcal{X}_n^* = \{\underbrace{\{1,2\},...,\{1,2\}}_{p_1-\text{times}},$ $\underbrace{\{2,3\},...,\{2,3\}}_{p_2-\text{times}},...,\underbrace{\{n-1,n\},...,\{n-1,n\}}_{p_{n-1}-\text{times}},\underbrace{\{n,1\},...,\{n,1\}}_{p_n-\text{times}}\}$.

Let $k \geq 2$ be an integer. For fixed $1 \leq t \leq n$ by $\mathcal{F}(k,t)$ we denote a subfamily of $\mathcal{X}_n^*$ such that $\mathcal{F}(k,t) = \{\{t \oplus j, t \oplus (j+1)\},\ j = 0, ..., k-2\}$.

Let $\mathcal{F} \subset \mathcal{X}_n^*$ be a subfamily of $\mathcal{X}_n^*$ such that

(iii) $|\mathcal{F}| = m$, for fixed $m \geq 0$,

(iv) for each $\mathcal{F}(k,t)$, $\mathcal{F}(k,q) \in \mathcal{F}$ such that $t \neq q$ holds $k \leq |q-t| \leq n-k$.

Let $\alpha_n = (p_1, p_2, ..., p_n)$ be the sequence of values p_i, $i = 1, ..., n$. For the future considerations we define the following subsequence of α_n.

Let $\alpha_{r,s}$, $1 \leq r \leq s \leq n$, be the subsequence of α_n obtained from α_n by deleting words $p_r, p_{r+1}, ..., p_s$. Let α_n^q, $1 \leq q \leq n$, be the subsequence of α_n obtained from α_n by deleting words $p_1, p_2, ..., p_{q-1}$. Let $\alpha_{k,q}^*$ be the subsequence of α_n obtained from α_n by deleting words $p_q, p_{q\oplus 1}, ..., p_{q\oplus(k-2)}$.

If $l^{(\alpha_n)}(k,n,m)$ is the number of all m-elements subfamilies $\mathcal{F}$, then $L^{(\alpha_n)}(k,n) = \sum_{m\geq 0} l^{(\alpha_n)}(k,n,m)$ is the number of all subfamilies $\mathcal{F}$. The number $L^{(\alpha_n)}(k,n)$ will be named as (k,α_n)-Lucas number.

Theorem 2.3 *Let $n \geq 3$, $m \geq 0$, $2 \leq k \leq n$ be integers. Then* $l^{(\alpha_n)}(k,n,0) = 1$, $l^{(\alpha_n)}(k,n,1) = \sum_{i=1}^{n} \prod_{j=0}^{k-2} p_{i\oplus j}$ *and for $m \geq 2$ we have*

$$l^{(\alpha_n)}(k,n,m) =$$

$$\sum_{q=n-k+1}^{n} \left[\left(\prod_{i=q}^{q\oplus(k-2)} p_i \right) f^{(\alpha_{k,q}^*)}(k, n-k, m-1) \right] + f^{(\alpha_{n-2})}(k, n-1, m).$$

PROOF: For $m=0$ and $m=1$ the initial conditions are obvious. Assume that $m \geq 2$. Let $l_n^{(\alpha_n)}(k,n,m)$ (respectively $l_{-n}^{(\alpha_n)}(k,n,m)$) be the number of all m-elements subfamilies $\mathcal{F}$ such that $\mathcal{X}_{n-1,n}^* \cap \mathcal{F} \neq \emptyset$ or $\mathcal{X}_{n,1}^* \cap \mathcal{F} \neq \emptyset$ (respectively: $\mathcal{X}_{n-1,n}^* \cap \mathcal{F} = \emptyset$ and $X_{n,1}^* \cap \mathcal{F} = \emptyset$). Then $l^{(\alpha_n)}(k,n,m) = l_n^{(\alpha_n)}(k,n,m) + l_{-n}^{(\alpha_n)}(k,n,m)$. Two cases occur now:

1. $\mathcal{X}_{n-1,n}^* \cap \mathcal{F} \neq \emptyset$ or $X_{n,1}^* \cap \mathcal{F} \neq \emptyset$.

Then there exists $\mathcal{F}(k,q) \in \mathcal{F}$ such that exactly one subset $\{n-1,n\}$ from $\mathcal{X}_{n-1,n}^*$ or exactly one subset $\{n,1\}$ from $\mathcal{X}_{n,1}^*$ belongs to $F(k,q)$. Then the definition of the family $\mathcal{F}$ implies that $q \in \{n-k+1, n-k+2, ..., n\}$. This means that each subfamily $\mathcal{F}(k,t) \in \mathcal{F}$, $t \neq q$ is the subfamily of one of the following subfamilies $\mathcal{X}_n^* - \bigcup_{i=0}^{k-1} \{q \oplus i, q \oplus (i+1)\} = \mathcal{X}_{n-k}$. Using Theorem 2.1 we obtain that

$$l_n^{(\alpha_n)}(k,n,m) = \sum_{q=n-k+1}^{n} \left(\prod_{i=q}^{q\oplus(k-2)} p_i \right) f^{(\alpha_{k,q}^*)}(k,n-k,m-1).$$

2. $\mathcal{X}_{n-1,n}^* \cap \mathcal{F} = \emptyset$ and $X_{n,1}^* \cap \mathcal{F} = \emptyset$.

Then for each $\mathcal{F}(k,q) \in \mathcal{F}$ we have $\{n-1,n\} \notin \mathcal{F}(k,q)$ and $\{n,1\} \notin \mathcal{F}(k,q)$ for all p_i subsets, $i = n-1, n$. So $\mathcal{F}$ is m-elements subfamily of $\mathcal{X}_n^* - (\mathcal{X}_{n-1,n} \cup \mathcal{X}_{n,1}) = \mathcal{X}_{n-2}$. Then by Theorem 2.1 we have that $l_{-n}^{(\alpha_n)}(k,n,m) = f^{(\alpha_{n-2})}(k,n-1,m)$.

Finally from the above cases $l^{(\alpha_n)}(k,n,m) =$

$$\sum_{q=n-k+1}^{n} \left[\left(\prod_{i=q}^{q\oplus(k-2)} p_i \right) f^{(\alpha_{k,q}^*)}(k,n-k,m-1) \right] + f^{(\alpha_{n-2})}(k,n-1,m).$$

Thus the Theorem is proved. □

Theorem 2.4 *Let $n \geq 3$, $2 \leq k \leq n$. Then for $n \geq 2k$*

$$L^{(\alpha_n)}(k,n) = \sum_{q=n-k+1}^{n} \left(\prod_{i=q}^{q\oplus(k-2)} p_i \right) F^{(\alpha_{k,q}^*)}(k,n-k) + F^{(\alpha_{n-2})}(k,n-1)$$

with initial conditions $L^{(\alpha_n)}(k,n) = 1 + \sum_{i=1}^{n} \prod_{j=0}^{k-2} p_{i\oplus j}$ *for* $n = 1, ..., 2k-1$.

PROOF: Let $n \leq 2k-1$. Then $L^{(\alpha_n)}(k,n) = \sum_{m\geq 0} l^{(\alpha_n)}(k,n,m) =$

$l^{(\alpha_n)}(k,n,0) + l^{(\alpha_n)}(k,n,1) = 1 + \sum_{i=1}^{n} \prod_{j=0}^{k-2} p_{i\oplus j}$, by Theorem 2.3.

Assume now that $n \geq 2k$. Then

$L^{(\alpha_n)}(k,n) = \sum_{m\geq 0} l^{(\alpha_n)}(k,n,m) =$

$l^{(\alpha_n)}(k,n,0) + l^{(\alpha_n)}(k,n,1) + \sum_{m\geq 2} l^{(\alpha_n)}(k,n,m) = 1 + \sum_{i=1}^{n} \prod_{j=0}^{k-2} p_{i\oplus j} +$

$$\sum_{m\geq 2}\left(\sum_{q=n-k+1}^{n}\left(\prod_{i=q}^{q\oplus(k-2)} p_i\right) f^{(\alpha^*_{k,q})}(k,n-k,m-1)+f^{(\alpha_{n-2})}(k,n-1,m)\right)=$$

$$1+\sum_{i=1}^{n}\prod_{j=0}^{k-2} p_{i\oplus j}-\sum_{q=n-k+1}^{n}\prod_{i=q}^{q\oplus(k-2)} p_i+$$

$$\sum_{m\geq 0}\sum_{q=n-k+1}^{n}\left(\prod_{i=q}^{q\oplus(k-2)} p_i\right) f^{(\alpha^*_{k,q})}(k,n-k,m-1)-$$

$$1-\sum_{i=1}^{n-k}\prod_{j=0}^{k-2} p_{i+j}+\sum_{m\geq 0} f^{(\alpha_{n-2})}(k,n-1,m)=$$

$$\sum_{q=n-k+1}^{n}\left(\prod_{i=q}^{q\oplus(k-2)} p_i\right) F^{(\alpha^*_{k,q})}(k,n-k)+F^{(\alpha_{n-2})}(k,n-1)+$$

$$\sum_{i=1}^{n}\prod_{j=0}^{k-2} p_{i\oplus j}-\sum_{i=1}^{n-k}\prod_{j=0}^{k-2} p_{i+j}-\sum_{q=n-k+1}^{n}\prod_{i=q}^{q\oplus(k-2)} p_i.$$

Claim 1. $\sum_{i=1}^{n}\prod_{j=0}^{k-2} p_{i\oplus j}-\sum_{i=1}^{n-k}\prod_{j=0}^{k-2} p_{i+j}-\sum_{q=n-k+1}^{n}\prod_{i=q}^{q\oplus(k-2)} p_i=0$

Proof: $\sum_{i=1}^{n}\prod_{j=0}^{k-2} p_{i\oplus j}-\sum_{i=1}^{n-k}\prod_{j=0}^{k-2} p_{i+j}-\sum_{q=n-k+1}^{n}\prod_{i=q}^{q\oplus(k-2)} p_i=$

$$\sum_{i=1}^{n-k}\prod_{j=0}^{k-2} p_{i\oplus j}+\sum_{i=n-k+1}^{n}\prod_{j=0}^{k-2} p_{i\oplus j}-\sum_{i=1}^{n-k}\prod_{j=0}^{k-2} p_{i+j}-\sum_{q=n-k+1}^{n}\prod_{i=q}^{q\oplus(k-2)} p_i=$$

$$\sum_{i=n-k+1}^{n}\prod_{j=0}^{k-2} p_{i\oplus j}-\sum_{q=n-k+1}^{n}\prod_{i=q}^{q\oplus(k-2)} p_i=$$

$p_{n-k+1}p_{(n-k+1)\oplus 1}\cdots p_{(n-k+1)\oplus(k-2)}+$
$p_{n-k+2}p_{(n-k+2)\oplus 1}\cdots p_{(n-k+2)\oplus(k-2)}+\ldots+$
$p_n p_{n\oplus 1}\cdots p_{n\oplus(k-2)}-p_{n-k+1}p_{n-k+2}\cdots p_{(n-k+1)\oplus(k-2)}-$
$p_{n-k+2}p_{n-k+3}\cdots p_{(n-k+2)\oplus(k-2)}-\cdots-p_n p_{n\oplus 1}\cdots p_{n\oplus(k-2)}=0.$

Using Claim 1 we have

$$L^{(\alpha_n)}(k,n)=\sum_{q=n-k+1}^{n}\left(\prod_{i=q}^{q\oplus(k-2)} p_i\right) F^{(\alpha^*_{k,q})}(k,n-k)+F^{(\alpha_{n-2})}(k,n-1).$$

Thus the Theorem is proved. □

If $p_i=1$ for $i=1,2,\ldots,n$, $n\geq 3$ then $L^{(\alpha_n)}(k,n)$ gives the generalized Lucas number $L(k,n)$. Additionally if $k=2$ and $n\geq 3$, then $L(2,n)$ gives the Lucas number L_n.

3 Graph interpretations and their applications

In this section we give some graph interpretations of the (k,α_{n-1})-Fibonacci numbers and the (k,α_n)-Lucas numbers with respect to the number of H-

matchings of special graphs. It is interesting that the set X can be represented as the vertex set of the multipath $P_n^{(\alpha_{n-1})}$, where vertices from $V(P_n^{(\alpha_{n-1})}) = \{x_1, ..., x_n\}$, $n \geq 2$ are numbered in the natural fashion. Moreover the family $\mathcal{X}_n$ corresponds to $E(P_n^{(\alpha_{n-1})}) = \{x_i x_{i+1}; x_i x_{i+1}$ repeats p_i times, $p_i \geq 1$, $i = 1, ..., n-1\}$. Then $\mathcal{Y}$ corresponds to a P_k-matching of a multigraph $P_n^{\alpha_{n-1}}$. Thus in the graph terminology the number $F^{(\alpha_{n-1})}(k,n)$, for $n \geq 2$, $k \geq 2$ is equal to the number of all P_k-matchings of the graph $P_n^{(\alpha_{n-1})}$.

Let $\#_H(G)$ be the number of all H-matchings of a graph G. From the above it immediately follows:

Theorem 3.1 *Let $n \geq 2, k \geq 2, p_i \geq 1$, $i = 1, ..., n-1$ be integers. Then* $\#_{P_k}(P_n^{(\alpha_{n-1})}) = F^{(\alpha_{n-1})}(k, n-(k-1))$.

The graph interpretation of the number $F^{(\alpha_{n-1})}(k,n)$ can be used for proving some identities.

Theorem 3.2 *Let $n \geq 2, k \geq 2$ be integers. Then for $2 \leq m \leq n-k+1$*
$$F^{(\alpha_{n-1})}(k,n) = F^{(\alpha_{m-2})}(k,m-1)F^{(\alpha_{n-1}^{m+1})}(k,n-m)+$$
$$\sum_{i=0}^{k-1}\prod_{j=1}^{k-1} p_{m-k+i+j}F^{(\alpha_{m-k+i-1})}(k,m-k+i)F^{(\alpha_{n-1}^{m+1+i})}(k,n-m-i).$$

PROOF: To prove this identity we use the graph interpretation of the number $F^{(\alpha_{n-1})}(k,n)$. Consider the multipath $P_n^{(\alpha_{n-1})}$ with $V(P_n^{(\alpha_{n-1})}) = \{x_1, ..., x_n\}$, $n \geq 2$ and with the numbering of its vertices in the natural fashion. Let $x_m \in V(P_n^{(\alpha_{n-1})})$ and $2 \leq m \leq n-k+1$ and let M be an arbitrary P_k-matching of a multipath $P_n^{(\alpha_{n-1})}$. Let $\#_{P_k}^{-m}(P_n^{(\alpha_{n-1})})$ (respectively $\#_{P_k}^{m}(P_n^{(\alpha_{n-1})})$) be the number of P_k-matchings of $P_n^{(\alpha_{n-1})}$ and there is an element $P_k^* \in M$ such that $x_m \notin P_k^*$ (respectively; $x_m \in P_k^*$).
We consider two possibilities:
1. $x_m \notin M$.
Then it is clear that $M = M_1 \cup M_2$, where M_1 is a P_k-matching of a graph $P_n^{(\alpha_{n-1})} \setminus \bigcup_{i=0}^{n-m-1}\{x_{n-i}\}$ which is isomorphic to the graph $P_{m-1}^{(\alpha_{m-2})}$. Moreover M_2 is a P_k-matching of a graph $P_n^{(\alpha_{n-1})} \setminus \bigcup_{i=1}^{m}\{x_i\}$ which is isomorphic to the graph $P_{n-m}^{(\alpha_{n-1}^{m+1})}$. Hence $\#_{P_k}^{-m}(P_n^{(\alpha_{n-1})}) = F^{(\alpha_{m-2})}(k,m-1)F^{(\alpha_{n-1}^{m+1})}(k,n-m)$.
2. $x_m \in M$.
Since x_m is a vertex of P_k-element of a matching M, then it is clear that there are exactly k different subsets of $V(P_n^{(\alpha_{n-1})})$ which give a P_k-element

of M which can belong to M. Proving analogously as in Case 1 and considering these k possibilities we obtain that $\#^{m}_{P_k}(P_n^{(\alpha_{n-1})}) =$
$\sum_{i=0}^{k-1}\prod_{j=1}^{k-1} p_{m-k+i+j}F^{(\alpha_{m-k+i-1})}(k, m-k+i)F^{(\alpha_{n-1}^{m+1+i})}(k, n-m-i)$.
Finally from the above cases we have that
$F^{(\alpha_{n-1})}(k,n) = F^{(\alpha_{m-2})}(k, m-1)F^{(\alpha_{n-1}^{m+1})}(k, n-m) +$
$\sum_{i=0}^{k-1}\prod_{j=1}^{k-1} p_{m-k+i+j}F^{(\alpha_{m-k+i-1})}(k, m-k+i)F^{(\alpha_{n-1}^{m+1+i})}(k, n-m-i)$.
Thus the Theorem is proved. □

From the above Theorem it immediately follows:

Corollary 1 *If $p_i = p$ for all $i = 1, ..., n-1$, then for $k \geq 2$ we have*

$$F^{(p)}(k,n) = F^{(p)}(k, m-1)F^{(p)}(k, n-m) + p^{k-1}\sum_{i=0}^{k-1} F^{(p)}(k, m-k+i)F^{(p)}(k, n-m-i).$$

If $p = 1$ and $k = 2$, then $F_{n+1} = F_m F_{n-m+1} + F_{m-1}F_{n-m+1} + F_m F_{n-m}$.

Now we show an application of the graph representation for counting of another family of subsets of the set of n integers. Let $k \geq 2$ be an integer. Let $X = \{1, 2, ..., n\}$, $n \geq 3$, be the set of n integers and let $\mathcal{K} = \{\mathcal{Y}^*(i,k); i = 1, 2, ..., n-k+1\}$, where $\mathcal{Y}^*(i,k)$ is a family of all necessarily different two elements subsets of the set $\{i, i+1, ..., i+k-1\}$ providing that subsets $\{i, i+1\}$ repeats p_i-times, $i = 1, ..., n-k-2$, respectively and remaining subsets appear exactly once.
Let $\mathcal{Y}^*$ be a subfamily of $\mathcal{K}$ such that
(v) for each $\mathcal{Y}^*(i,k), \mathcal{Y}^*(j,k), i \neq j$ holds $|j-i| \geq k$.
Denote by $\eta(k,n)$ the number of all subfamilies $\mathcal{Y}^*$ of the multifamily $\mathcal{K}$. To count the number $\eta(k,n)$ we will use given earlier the graph interpretation of the number $F^{(\alpha_{n-1})}(k, n-(k-1))$. Firstly, auxiliary, we need the concept of the d-power of a graph G. Let $d \geq 1$ be an integer. For a given graph G by d-th power of a graph G we mean the graph denoted by G^d such that $V(G^d) = V(G)$ and $xy \in E(G)$ if and only if $d_G(x,y) \leq d$, where $d_G(x,y)$ denotes the distance between x and y in a graph G.

Theorem 3.3 *Let $n \geq 3, k \geq 2$, $p_i \geq 1$, $i = 1, ..., n-1$ be integers. Then* $\eta(k,n) = F^{(\alpha_{n-1})}(k, n-(k-1))$.

PROOF: In the graph interpretation the set X corresponds to the vertex set of the graph $(P_n^{(\alpha_{n-1})})^{k-1}$ and the family $\mathcal{K}$ corresponds to the set $E((P_n^{(\alpha_{n-1})})^{k-1})$. Using the graph interpretation of the number $F^{(\alpha_{n-1})}(k, n-$

$(k-1)$) and the operation of $(k-1)$-power of a graph $P_n^{(\alpha_{n-1})}$ we have that every P_k-matching of a graph $P_n^{(\alpha_{n-1})}$ induces to a K_k-matching in a graph $(P_n^{(\alpha_{n-1})})^{k-1}$, hence the result immediately follows.
Thus the Theorem is proved. □

Now we give the graph representation of the (k, α_n)-Lucas number.

The set $X = \{1, 2, \ldots, n\}$ can be regarded as the vertex set of the multicycle $C_n^{(\alpha_n)}$ with $V(C_n^{(\alpha_n)}) = \{x_1, x_2, \ldots, x_n\}$, $n \geq 3$, where vertices from $V(C_n^{(\alpha_n)})$ are numbered in the natural fashion. Then the multifamily $\mathcal{X}_n^*$ corresponds to $E(C_n^{(\alpha_n)}) = E(P_n^{(\alpha_{n-1})}) \cup \{x_n x_1;\ x_n x_1$ repeats p_n times$\}$. Thus, we have the following

Theorem 3.4 *Let $n \geq 3$, $k \geq 2$, $p_i \geq 1$, $i = 1, 2, \ldots, n$ be integers. Then $\#_{P_k}(C_n^{(\alpha_n)}) = L^{(\alpha_n)}(k, n)$.*

Now we show another application of this graph interpretation. We will use it for counting of another subfamily of the set of n integers.

Let $X = \{1, 2, \ldots, n\}$, $n \geq 3$, $k \geq 2$ be integers. Let $\mathcal{L} = \mathcal{K} \cup \mathcal{L}_1$, where $\mathcal{L}_1 = \{\mathcal{Y}^{**}(i, k); i = n-(k-2), n-(k-3), \ldots, n\}$, where $\mathcal{Y}^{**}(i, k)$ is a family of all necessarily different two elements subsets of the set $\{n-s, (n-s) \oplus 1, (n-s) \oplus 2, \ldots, (n-s) \oplus (k-1), s = 0, 1, \ldots, k-2\}$ providing that subsets $\{i, i \oplus 1\}, i = n-(k-2), n-(k-3), \ldots, n$ appear p_i times and remaining subsets appear exactly once.
Let $\mathcal{Y}^{**}$ be a subfamily of different elements of the multifamily $\mathcal{L}$ such that
(vi) for each $\mathcal{Y}^{**}(i, k), \mathcal{Y}^{**}(j, k), i \neq j, k \leq |j-i| \leq n-k$.
Let $R(k, n)$ be the number of all subfamilies $\mathcal{Y}^{**}$ of the multifamily $\mathcal{L}$.

Proving analogously as in Theorem 3.3 we obtain

Theorem 3.5 *Let $n \geq 3$, $2 \leq k \leq n$, p_i be integers, $i = 1, 2, \ldots, n$. Then $R(k, n) = L^{(\alpha_n)}(k, n)$.*

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