A note on 2-subset-regular self-complementary 3-uniform hypergraphs

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Abstract

We show that a 2-subset-regular self-complementary 3-uniform hypergraph with n vertices exists if and only if $n \geq 6$ and n is congruent to 2 modulo 4.

1 Introduction

A *k-uniform hypergraph* of order n is an ordered pair $\Gamma = (V, E)$, where $V = V(\Gamma)$ is an arbitrary set of size n, and $E = E(\Gamma)$ is a subset of $V^{(k)} = \{e \subseteq V : |e| = k\}$. Note that the notion of a 2-uniform hypergraph coincides with the usual notion of a simple graph. We shall call a k-uniform hypergraph simply a *k-hypergraph.*

A k-hypergraph Γ is *self-complementary* if it is isomorphic to its complement Γ^C, defined by $V(\Gamma^C) = V(\Gamma)$ and $E(\Gamma^C) = V(\Gamma)^{(k)} \setminus E(\Gamma)$. Equivalently, $\Gamma = (V, E)$ is self-complementary whenever there exists a permutation $\tau \in \mathrm{Sym(V)}$, called the *antimorphism of* Γ, such that for all $e \in V^{(k)}$ the equivalence $e \in E \Leftrightarrow e^\tau \notin E(\Gamma)$ holds. Antimorphisms of uniform hypergraphs were characterized in terms of their cyclic decompositions by Wojda in [7].

A k-hypergraph Γ is *t-subset-regular* if there exists an integer λ, also called the *t-valence* of Γ, such that each t-element subset of $V(\Gamma)$ is a subset of exactly λ elements of $E(\Gamma)$. Clearly t-subset-regular k-hypergraphs generalize the notion of regular graphs, and can also be viewed as a bridge between graph theory and design theory. Namely, a t-subset-regular k-hypergraph of t-valence λ and order n is simply a t-(n, k, λ) design. Moreover, such a k-hypergraph Γ is self-complementary if and only if the pair

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$\{\Gamma, \Gamma^C\}$ forms a large set of t-designs LS[2](t, n, k) with the additional property that the two designs constituting the large set are isomorphic (see [1] for the definition of a large set of designs).

Here a question of existence of a self-complementary t-subset-regular hypergraph with prescribed parameters n, k, and t arises naturally. An easy counting argument shows that whenever a self-complementary t-subset-regular k-hypergraphs on n vertices exists, then $\binom{n-i}{k-i}$ is even for all $i = 0, \ldots, t$.

It can be seen that the above divisibility conditions can in fact be expressed in terms of certain congruence conditions on n modulo an appropriate power of 2 (see [4]). For example if $k = 2^\ell$ or $k = 2^\ell + 1$ for some positive integer ℓ, then n is congruent to one of $t, \ldots, k-1$ modulo $2^{\ell+1}$. In particular, if $k = 2$ and $t = 1$, then $n \equiv 1 \pmod 4$; if $k = 3$ and $t = 1$, then $n \equiv 1$ or $2 \pmod 4$; if $k = 3$ and $t = 2$, then $n \equiv 2 \pmod 4$.

In [5] the following question, strengthening Hartman's conjecture [3] about existence of large sets of (not necessarily isomorphic) designs, was raised:

Question. [5] *Is it true that for every triple of integers $t < k < n$ such that $\binom{n-i}{k-i}$ is even for all $i = 0, \ldots, t$, there exists a self-complementary t-subset-regular k-hypergraph of order n?*

It is well known (see [6]) that a regular self-complementary graph on n vertices exists if and only if n is congruent to 1 modulo 4, showing that the answer to the above question is affirmative for $k = 2$ and $t = 1$. Recently, the answer was proved to be affirmative also for the case $k = 3$ and $t = 1$ (see [5]). The aim of this note is to show that the answer to the question above is affirmative also for the remaining case of 3-hypergraphs, namely for the case $k = 3$, $t = 2$. More precisely, in Section 2 we present a construction which proves the following:

Theorem 1 *If $n \geq 6$ and n is congruent to 2 modulo 4, then there exists a 2-subset-regular self-complementary 3-hypergraph on n vertices.*

2 Construction

Let $n = 4k + 2$ for some integer k. For $i = 0, 1$, let $V_i = \{0_i, 1_i, \ldots, (2k)_i\}$ be a copy of the the ring $\mathbb{Z}_{2k+1}$. Define Γ_n to be the 3-hypergraph with vertex set $V = V_0 \cup V_1$ and edge set $E = E_1 \cup E_2 \cup E_3$, where

$$\begin{aligned} E_1 &= V_0^{(3)}, \\ E_2 &= \{\{a_0, b_0, c_1\} \,:\, a, b \in \mathbb{Z}_{2k+1},\, a \neq b,\, c = \frac{a+b}{2}\}, \\ E_3 &= \{\{a_0, b_1, c_1\} \,:\, a, b, c \in \mathbb{Z}_{2k+1},\, a \neq \frac{b+c}{2}\}. \end{aligned}$$

Note that 2 is invertible in $\mathbb{Z}_{2k+1}$, hence dividing by 2 in the definitions of E_2 and E_3 is well defined.

First we show that Γ_n is 2-subset-regular, i.e. we show that each pair of vertices is contained in exactly $(n-2)/2 = 2k$ edges.

There are four types of pairs of vertices to consider:

(a) A pair a_0, b_0, where $a, b \in \mathbb{Z}_{2k+1}$, $a \neq b$. This pair is contained in $2k-1$ edges of E_1 and in a unique edge in E_2. As it is contained in none of the edges of E_3, the pair is in total of $2k$ edges.

(b) A pair a_1, b_1, where $a, b \in \mathbb{Z}_{2k+1}$, $a \neq b$. This pair appears only in $(2k+1) - 1 = 2k$ edges of E_3.

(c) A pair a_0, a_1, where $a \in \mathbb{Z}_{2k+1}$. This pair also appears only in the edges of E_3. In fact, it appears precisely in the $2k$ edges of the form $\{a_0, a_1, b_1\}$, $b \in \mathbb{Z}_{2k+1} \setminus \{a\}$.

(d) A pair a_0, c_1, where $a, c \in \mathbb{Z}_{2k+1}$, $a \neq c$. This pair is contained in the edge $\{a_0, c_1, (2c-a)_0\}$ of E_2, and in the $2k-1$ edges of the form $\{a_0, c_1, b_1\}$ where $b \in \mathbb{Z}_{2k+1} \setminus \{2a-c\}$, of E_3. Hence this pair is in exactly $2k$ edges of Γ_n.

This proves that Γ_n is a 2-subset-regular hypergraph. To prove that it is self-complementary, note that the mapping $\phi : V \to V$ defined by $\phi(a_i) = a_{i+1}$, for $i = 0, 1$, with addition in the subscript being modulo 2, is an antimorphism of Γ_n. □

We remark that Γ_n is not a vertex-transitive hypergraph if $n > 6$ (check the complete sub-hypergraphs of order $n/2$). However, in Γ_6 every pair of vertices appears in exactly two edges. Hence Γ_6 can be considered as a triangular embedding of a complete graph K_6 into a surface (see [2] for a detailed account on graph embeddings). In fact, Γ_6 represents a regular triangulation of the projective plane by K_6. As a consequence, Γ_6 is vertex-transitive.

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