Endomorphism monoids of generalized split graphs *

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Abstract: We call a graph G a *generalized split graph* if there exists a core K of G such that $V(G) \setminus V(K)$ is an independent set of G. Let G be a generalized split graph with a partition $V(G) = K\dot{\cup}S$, where K is a core of G and S is an independent set. We prove that G is end-regular if and only if for any $a, b \in S$, $\phi \in \mathrm{Aut}(K)$, the inclusion $\phi(N(a)) \subsetneqq N(b)$ doesn't hold, and that G is end-orthodox if and only if G is end-regular and for any $a, b \in S$, $N(a) \neq N(b)$.

Key Words: Core; Generalized split graph; End-regular; End-orthodox

0 Introduction

Endomorphism monoids of graphs has been investigated for quite some time. The research in this field aims at revealing the relationship between semigroup theory and graph theory and at advancing application of one to the other. Refs. [1, 4, 13, 14] may serve as a survey. A semigroup is called *regular* if every element has a pseudo-inverse. According to [10], the most coherent part of semigroup theory at the present time is the part concerned with the structure of regular semigroups of various kinds. In [18], the following question was posed: Which graphs have a regular endomorphism monoid? To

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answer this question, we say a graph to be *end-regular* if it has a regular endomorphism monoid. As is pointed out in [6] that end-regular graphs do have some kind of graphical symmetry although they are defined in the semigroup sense. This viewpoint can also be seen by the characterization of end-regular split graphs that a split graph G is end-regular if and only if there is a maximal complete subgraph K of G such that $V(G) \setminus V(K)$ is independent and all the vertices in $V(G) \setminus V(K)$ have the same degrees ([16, Corollary 2.14 and Theorem 3.3] or Corollary 2.9). However it seems difficult to obtain a characterization of all end-regular graphs. In [15], a regular endomorphism of a graph was characterized by means of idempotents. In [3] and [19], Fan and Wilkeit characterized independently end-regular bipartite graphs. In [11], the authors gave all non-local rings whose zero-divisor graphs are end-regular.

We call a graph G a *generalized split graph* if there exists a core K of G such that $V(G) \setminus V(K)$ is an independent set. The class of generalized split graphs is much larger than the class of split graphs. In this paper, through a path completely distinct from the one in [16], we obtain an explicit characterization of generalized split graphs which are end-regular and thus provide a large class of end-regular graphs. Finally, generalized split graphs which are end-orthodox are determined.

1 Basic notions

Our graphs are finite undirected graphs without loops and multiple edges. Let G be a graph, we denote by $V(G)$ (or just G) and $E(G)$ its vertex set and edge set, respectively. The *neighborhood* of a vertex u, denoted by $N(u)$, is the set of vertices adjacent to u. A graph H is called a *subgraph* of G if $V(H) \subseteqq V(G)$ and $E(H) \subseteqq E(G)$. Moreover, if for any $a, b \in V(H)$, $\{a, b\} \in E(G)$ implies $\{a, b\} \in E(H)$, then H is called an *induced subgraph* of G. Let $S \subseteqq V(G)$. The induced subgraph H with $V(H) = S$ is denoted by $[S]$. A graph G is *complete* if any two of its vertices are adjacent. A subset $S \subseteq V(G)$ is said to be *independent* if $\{a, b\} \notin E(G)$ for any $a, b \in S$.

Let G and H be graphs. A mapping $f : V(G) \to V(H)$ is called a *homomorphism* from G to H if for any $a, b \in V(G)$, $\{a, b\} \in V(G)$ implies $\{f(a), f(b)\} \in E(H)$. Moreover, if f is bijective and its inverse mapping is also a homomorphism, then f is called an *isomorphism* from G to H. A homomorphism (resp. an isomorphism) from G to itself is called an *endomorphism* (resp. *automorphism*) of G. An endomorphism f is said to be *half-strong* if $f(a)$ is adjacent to $f(b)$ implies that there exist $c \in f^{-1}(f(a))$ and $d \in f^{-1}(f(b))$ such that c is adjacent to d. By End(G) and Aut(G), we denote the set of all the endomorphisms and automorphisms of G respectively. It is well-known that End(G) is a monoid (i.e., a semigroup with an identity element) and Aut(G) is a group with respect to the composition of mappings. If K is a subgraph of G, then a homomorphism $f : G \to K$ such that $f(k) = k$ for all $k \in V(K)$ is called a *retraction* of G onto K and K a *retract* of G. A subgraph K of G is called a *core* of G if there is a homomorphism $G \to K$ but no homomorphism $G \to H$ for any proper subgraph H of K. A graph which is its own core will be called simply *a core*. Cores have been called *minimal graphs*, *unretractive graphs* in the literature. We give the following natural definition.

Definition 1.1. A graph G is called a *generalized split graph* if there is a core K of G such that $V(G) \setminus V(K)$ is an independent set of G.

We have known following graphs are cores.

(1) The χ-critical graphs such as complete graphs, cycles of odd length, etc. ([8] or [7, Chapter 6.2]) (2) The join of two cores ([12, Theorem 2.2]). (3) The lexicographic products $C_{2n+1}[X]$ and $K_n[Y]$, where Y is a core ([12, Theorem 3.11]). (4) The complements of cycles of odd length ([17, Propositin 2.2]).

A generalized split graph is a split graph if and only if its core is a complete graph. Hence the generalized split graph is a large class of graphs containing split graphs properly.

Recall from [10] that an element a in a monoid S is *regular* if there exists b in S such that $aba = a$, a monoid S is *regular* if each element in S is regular. An element e in S is an *idempotent* if $e = e^2$. A regular monoid S is *orthodox* if the product of any two idempotents is an idempotent. A graph G is said to be *end-regular* (resp. *end-orthodox*) if its endomorphism monoid $\text{End}(G)$ is regular (resp. orthodox). By definition, an end-regular graph is end-orthodox if and only if for any retractions f and g of G, the composition fg is also a retraction of G.

For undefined concepts in graph theory and semigroup theory, one may refer to [7] and [10] respectively.

The following observations about a core are useful.

Lemma 1.2. (1) *If K is a core of G, then K is a core.*
(2) *Every graph G has a core of G, which is a retract, so an induced subgraph of G.*
(3) *The core of a graph is unique, up to isomorphism.*
(4) *A subgraph H of G is a core of G if and only if it is a core and there is a homomorphism from G to H.*
(5) *Let f be an endomorphism of G. Then every core of $[f(V(G))]$ is a core of G.*

Proof. (1)-(4) See [9, Observations] or [7, Chapter 6.2].

(5) Let K be a core of $[f(V(G))]$. Then there is a homomorphism $[f(V(G))] \to K$, and so a homomorphism: $G \to K$. Now, the result follows from (4). □

2 End-regular graphs

The main result of this section is given in Theorem 2.7. We begin with a characterization of generalized split graph, which provides a way to construct such graphs from a core.

Proposition 2.1. *A graph G is a generalized split graph if and only if its vertex set $V(G)$ can be partitioned into disjoint sets S and K, i.e., $V(G) = K\dot{\cup}S$, such that S is an independent set of G, K is a core and $W(a)$ is not empty for all $a \in S$, where $W(a) = \{x \in K | N(a) \subseteqq N(x)\}$.*

Proof. Assume that G is a generalized split graph. Then there exists a core of G, say K, such that $V(G) \setminus V(K)$ is an independent set of G. Set $S = V(G) \setminus V(K)$. By Lemma 1.2, there is a retraction $e : G \to K$. Given any $a \in S$ and $x \in N(a)$, we obtain $\{x, e(a)\} = \{e(x), e(a)\} \in E(G)$ and so $x \in N(e(a))$, which implies $e(a) \in W(a)$ and thus $W(a)$ is not empty. Conversely, fix $w(a) \in W(a)$ for any $a \in S$ and define $f : V(G) \to V(K)$ by $f(x) = w(x)$ if $x \in S$ and $f(x) = x$ otherwise. Then f is a homomorphism from G onto K and so K is a core of G, as desired. $\square$

In what follows, we will call a partition $V(G) = K\dot{\cup}S$ to be a *split partition* of G if S is an independent set and K is a core of G. The proof of the following proposition is similar to that of [16, Proposition 2.2].

Proposition 2.2. *Let G be a generalized split graph with a split partition $V(G) = K\dot{\cup}S$. If there exist $u, v \in S$ and $\phi \in Aut(G)$ such that $\phi(N(u)) \subsetneqq N(v)$, then G is not end-regular.*

Proof. Let $\{W(a) : a \in S\}$ be as in Proposition 2.1 and fix $w(a) \in W(a)$ for any $a \in S$. Define $f : V(G) \to V(G)$ as follows:

$$f(x)=\begin{cases}\phi(x), & \text{if } x\in K\\ v, & \text{if } x=u\\ \phi(w(x)), & \text{if } x\in S\setminus\{u\}.\end{cases} \tag{1}$$

It is routine to check that $f \in \mathrm{End}(G)$. Suppose that f is half strong. Fix $y \in N(v)\setminus \phi(N(u))$. Since $\{y,v\}\in E(G)$ and $f^{-1}(v)=\{u\}$, there exists $x\in f^{-1}(y)\cap N(u)$, which implies $y=f(x)\in f(N(u))=\phi(N(u))$, a contradiction. Hence f is not half strong and it follows that G is not end-regular by [16, Lemma 2.1]. □

For convenience, we often use the notion $x\sim y$ to denote $\{x,y\}\in E(G)$. The cardinality of a set S is denoted by $|S|$.

Proposition 2.3. *Let G be a graph. Then for any $f\in End(G)$ and any core K of G, $[f(K)]$ is also a core of G, and the restriction map $f_K : K\to [f(K)]$ is an isomorphism.*

Proof. Let $f\in \mathrm{End}(G)$ and K be a core of G. By Lemma 1.2(2), there exists a retraction e from G to K. Consider $e: G\to K$ as an endomorphism of G, we obtain $fe\in\mathrm{End}(G)$, and by Lemma 1.2(5), $[fe(G)]$ contains a core of G, say H. As $f(K)=fe(G)$, $f(K)$ contains $V(H)$, which implies $f(K)=V(H)$ by the fact $|K|=|H|$. Hence $H=[f(K)]$ by Lemma 1.2(2) and so $[f(K)]$ is also a core of G. Finally f_K is an isomorphism since $K\cong [f(K)]$ and $f_K: K\to [f(K)]$ is a surjective homomorphism. □

Remark 2.4. Let u,v be distinct vertices of a graph G. If $N(u)\subseteq N(v)$, then G is certainly not a core since $f: G\to G$ defined by $f(u)=v$ and $f(x)=x$ otherwise, is a retraction from G to $G\setminus\{u\}$. In other words, if u,v are vertices of a core K with $N(u)\subseteq N(v)$, then $u=v$.

Lemma 2.5. *Let G be a generalized split graph with a split partition $V(G)=K\dot{\cup}S$. Fix $w(a)\in W(a)$ for any $a\in S$. Assume H is a core of G and denote $V(G)\setminus V(H)$ by T. Let e be a retraction from G to*

H. Then the following statements hold.

(1) *For any $a \in S$, at most one of vertices a and $w(a)$ lies in H.*

(2) *For any $a \in S$, if $w(a) \in T$ and $a \in H$, then $e(w(a)) = a$ and $N(w(a)) \cap H = N(a) \cap H$.*

(3) *For any $a_1 \neq a_2 \in S$, if $w(a_1) = w(a_2) \in T$, then at most one of vertices a_1 and a_2 lies in H.*

(4) *For any $a \in S$, set $S(a) = \{b \in S | w(b) = w(a)\}$ and $T(a) = S(a) \cup \{w(a)\}$. Then $|T(a) \cap T| = |S(a)|$. Moreover, $T \subseteqq S \cup \{w(a) | a \in S\}$.*

(5) *For any $a \in S$, if $w(a) \in T$, then there exists $b \in S \cap H$ such that $w(b) = w(a)$.*

(6) *After indexing properly, we can write $S = \{a_1, a_2, \cdots, a_n\}$ and $T = \{b_1, b_2, \cdots, b_n\}$ such that $N(a_i) \subseteqq N(b_i)$ for $1 \leq i \leq n$ and if $b_i \neq a_i$, then $a_i \in H$ and $b_i \notin S$.*

Let S, T be as in (6) and let $i, j \in \{1, 2, \cdots, n\}$.

(7) *If $b_i \neq a_i$, then $a_i = e(b_i)$ and $N(b_i) \cap H = N(a_i) \cap H$.*

(8) *If $b_i \sim b_j$, then either $b_i = a_i$ or $b_j = a_j$. If assume further that $b_i = a_i$, then $b_j \neq a_j$ and $a_j \sim e(b_i)$.*

(9) *If $b_i = a_i$, then $N(a_i) \subseteqq N(e(a_i))$.*

Proof. (1) Assume that $\{a, w(a)\} \subseteqq H$. By $N(a) \cap H \subseteqq N(w(a)) \cap H$, we obtain $a = w(a)$ by Remark 2.4, a contradiction.

(2) Given any $x \in N(w(a)) \cap H$, we obtain that $x = e(x) \sim e(w(a))$, which implies $N(w(a)) \cap H \subseteqq N(e(w(a))) \cap H$. It follows that $N(a) \cap H \subseteqq N(e(w(a))) \cap H$ and so $a = e(w(a))$ by Remark 2.4 and now the equation $N(w(a)) \cap H = N(a) \cap H$ is clear.

(3) If both a_1 and a_2 lie in H, then $a_1 = e(w(a_1)) = e(w(a_2)) = a_2$, a contradiction by (2).

(4) If $w(a) \notin T$, then $S(a) \subseteqq T$ by (1), and so $|T(a) \cap T| = |S(a)|$. If $w(a) \in T$, then at most one of $S(a)$ lies in H by (3) and so $|T(a) \cap T| \geq |S(a)|$. Hence $|T(a) \cap T| \geq |S(a)|$ for any $a \in S$.

Let S_1 be a subset of S such that $S = \cup_{a \in S_1} S(a)$ and $S(a) \cap S(b) = \emptyset$ for any distinct $a, b \in S_1$. Then $|S| = \Sigma_{a \in S_1} |S(a)|$ and

$|T| \geq \Sigma_{a \in S_1} |T(a) \cap T| \geq \Sigma_{a \in S_1} |S(a)|$. As $|S| = |T|$, we have that $\Sigma_{a \in S_1} |T(a) \cap T| = \Sigma_{a \in S_1} |S(a)|$, which implies $|S(a)| = |T(a) \cap T|$ for any $a \in S$. Moreover, $T = \cup_{a \in S_1} (T(a) \cap T)$ and so $T \subseteqq \cup_{a \in S_1} T(a) = S \cup \{w(a) | a \in S\}$.

(5) Since $|T(a) \cap T| = |S(a)|$, there exists $b \in S(a) \setminus T$, as desired.

(6) It follows from (4).

(7) Note that in this case, $b_i \in W(a_i)$. Now, the result follows from (2).

(8) If $b_i \neq a_i$ and $b_j \neq a_j$, then $a_i = e(b_i)$ and $a_j = e(b_j)$ by (7) and then $a_i \sim a_j$, a contradiction. Now, assume that $b_i = a_i$. Then $b_j \neq a_j$ and so $a_j = e(b_j) \sim e(b_i)$.

(9) Let $x \in N(a_i)$. If $x \in N(a_i) \cap H$, then $x = e(x) \sim e(a_i)$, implying $x \in N(e(a_i))$. Suppose $x \in N(a_i) \cap T$. Then $x = b_j$ for some j, and so $a_j \sim e(b_i) = e(a_i)$ by (8). As $N(a_j) \subseteqq N(b_j)$, we obtain that $e(a_i) \in N(b_j)$, which implies $x = b_j \in N(e(a_i))$, completing the proof. $\square$

Lemma 2.6. *Let G, K, H, S, T be as in Lemma 2.5. Then the following statements hold for any $i \in \{1, 2, \cdots, n\}$.*
(1) If $b_i \neq a_i$, then $N(b_i) \cap H = N(a_i) \cap H = N(a_i)$.
(2) There exists an isomorphism φ from H to K such that $\varphi(x) = x$ if $x \in V(H) \cap V(K)$.

Proof. Without loss of generality, we can assume

$$T = \{a_1, a_2, \cdots, a_k, b_{k+1}, b_{k+2}, \cdots, b_n\}$$

such that $b_i \neq a_i$ for $i = k+1, k+2, \cdots, n$ by Lemma 2.5(6).

(1) By Lemma 2.5(8), $[T]$ is a bipartite graph with parts $\{a_1, a_2, \cdots, a$ and $\{b_{k+1}, b_{k+2}, \cdots, b_n\}$ and so $|E([T])| = |N(b_{k+1}) \cap T| + |N(b_{k+2}) \cap T| + \cdots + |N(b_n) \cap T|$. Counting the number of $E(G)$, we obtain that $|E(G)| = |E(K)| + |N(a_1)| + |N(a_2)| + \cdots + |N(a_n)|$ and $|E(G)| = |E(H)| + |N(a_1)| + \cdots + |N(a_k)| + |N(b_{k+1})| + \cdots + |N(b_n)| - |E([T])|$. Observing that $|N(b_i)| = |N(b_i) \cap H| + |N(b_i) \cap T|$ for $i = k+1, \cdots, n$, we obtain that $|N(a_{k+1})| + |N(a_{k+2})| + \cdots + |N(a_n)| = |N(b_{k+1}) \cap H| +$

$|N(b_{k+2}) \cap H| + \cdots + |N(b_n) \cap H|$, and it follows that $N(b_i) \cap H = N(a_i)$ from the fact $N(b_i) \cap H \subseteqq N(a_i)$ for $i \geq k+1$ by Lemma 2.5(7).

(2) Note that $V(H) = (V(H) \cap V(K)) \cup \{a_{k+1}, a_{k+2}, \cdots, a_n\}$ and $V(K) = (V(H) \cap V(K)) \cup \{b_{k+1}, b_{k+2}, \cdots, b_n\}$, we can define a bijective map $\varphi : V(H) \to V(K)$ by $\varphi(a_i) = b_i$ for $i = k+1, \cdots, n$ and $\varphi(x) = x$ otherwise. Let $x \in V(H) \cap V(K)$ and $i \in \{k+1, \cdots, n\}$. If $\{a_i, x\} \in E(H)$, then $x \in N(a_i) = N(b_i) \cap H$ and so $\{\varphi(x), \varphi(a_i)\} = \{x, b_i\} \in E(K)$. Since $\{a_{k+1}, a_{k+2}, \cdots, a_n\}$ is an independent set of K, we obtain that φ is an isomorphism as required. □

Now, we are ready to prove our main result of this section.

Theorem 2.7. *Let G be a generalized split graph with a split partition $V(G) = K\dot{\cup}S$. Then G is end-regular if and only if for any $a, b \in S$, any $\phi \in Aut(K)$, the inclusion $\phi(N(a)) \subsetneqq N(b)$ doesn't hold, i.e., either $\phi(N(a)) \setminus N(b) \neq \emptyset$ or $\phi(N(a)) = N(b)$.*

Proof. The "only if" part follows from Proposition 2.2. For the "if" part, we need to show that End(G) is a regular monoid. Let $f \in$ End(G). Then $H = [f(K)]$ is a core again by Proposition 2.3. Let e be a retraction from G to H and set $T = V(G) \setminus V(H)$. Write S and T as in Lemma 2.5(6). Since $f_1 : K \to H$ defined by $f_1(x) = f(x)$ for any $x \in K$ is an isomorphism, there exists a homomorphism $g_1 : H \to K$ such that $f_1 g_1 = 1_H, g_1 f_1 = 1_K$. For any $a \in f(V(G)) \cap T$, fix a vertex $t(a) \in S$ such that $f(t(a)) = a$. Now, we define a map $g : V(G) \to V(G)$ as follows:

$$g(x) = \begin{cases} g_1(x), & \text{if } x \in H \\ t(x), & \text{if } x \in f(V(G)) \cap T \\ g_1(e(x)), & \text{if } x \in T \setminus f(V(G)). \end{cases} \quad (2)$$

We first show that $g \in$ End(G). Let $x, y \in V(G)$ with $x \sim y$. We need to check that $g(x) \sim g(y)$ in each case. Let φ be as in Lemma 2.6(2).

Case 1: $x, y \in V(H)$. It is clear that $g(x) \sim g(y)$ in this case.

Case 2: $x \in H$ and $y \in f(V(G)) \cap T$. Then $y = b_i$ for some i and $g(y) = a_k \in S$ for some k. As $f(a_k) = b_i$, $f_1(N(a_k)) = f(N(a_k)) \subseteq N(b_i) \cap H = N(a_i) \cap H$ by Lemma 2.5(7). Assume on the contrary that $\{g(x), g(y)\} \notin E(G)$. Then $g(x) \notin N(a_k)$ and so $x = f_1 g(x) \notin f_1(N(a_k))$ since f_1 is one-to-one. Since $x \in N(y) \cap H = N(b_i) \cap H = N(a_i) \cap H$, we obtain $f_1(N(a_k)) \subsetneqq N(a_i) \cap H \subseteq V(K) \cap V(H)$. It follows that $\varphi f_1(N(a_k)) \subsetneqq N(a_i) \cap H \subseteq N(a_i)$ by Lemma 2.6(2), which is impossible since $\varphi f_1 \in \mathrm{Aut}(K)$. Hence $\{g(x), g(y)\} \in E(G)$.

Case 3: $x \in H, y \in T \setminus f(V(G))$. Then $y = b_j$ for some j and $x \in N(b_j) \cap H \subseteq N(e(b_j)) \cap H$ by Lemma 2.5(9) and (7), which implies $x \sim e(b_j)$. Since $g(x) = g_1(x)$ and $g(b_j) = g_1(e(b_j))$, we obtain $g(x) \sim g(y)$.

Case 4: $x, y \in T$. Then $x = b_i$ and $y = b_j$ for some i, j. By Lemma 2.5(8), we can assume $b_i = a_i$ and $b_j \neq a_j$ without loss of generality. As $N(a_i) \cap H \subseteq N(a_i) \setminus \{b_j\}$, we claim that $a_i \notin f(V(G))$. For otherwise, $f(a_k) = a_i$ for some k and $f_1(N(a_k)) \subseteq N(a_i) \cap H \subseteq N(a_i) \setminus \{b_j\}$, which implies $\varphi f_1(N(a_k)) \subsetneqq N(a_i)$, a contradiction.

If $b_j \notin f(V(G))$, then $g(b_j) = g_1(e(b_j)) \sim g_1(e(a_i)) = g(a_i)$. That is $g(x) \sim g(y)$.

Suppose that $b_j \in f(V(G))$. Then $t(b_j) \in S$ and $t(b_j) = a_l$ for some l. Since $f(a_l) = b_j$, we obtain that $f_1(N(a_l)) \subseteq N(b_j) \cap H = N(a_j) \subseteq V(H) \cap V(K)$ by Lemma 2.6(1). In the same reason as above, it is impossible that $f_1(N(a_l)) \subsetneqq N(a_j)$ and so $f_1(N(a_l)) = N(a_j)$. It follows that $N(a_l) = g_1(N(a_j))$. On the other hand, as $a_i \sim b_j$, we obtain $e(a_i) \sim e(b_j) = a_j$, i.e., $e(a_i) \in N(a_j)$. Hence $g_1(e(a_i)) \in g_1(N(a_j)) = N(a_l)$. Note that $g(y) = g(b_j) = a_l$ and $g(x) = g(a_i) = g_1(e(a_i))$, we conclude that $g(x) \sim g(y)$.

Thus we prove that $g(x) \sim g(y)$ for any $x, y \in G$ with $x \sim y$, and so $g \in \mathrm{End}(G)$.

Finally, we check that $fgf(x) = f(x)$ for any $x \in G$. If $f(x) \in H$, then $fgf(x) = f_1 g_1 f(x) = 1_K(f(x)) = f(x)$. If $f(x) \in T$, then

$fgf(x) = f(t(f(x)) = (ft)(f(x)) = f(x)$. Hence f is regular and G is end-regular. $\square$

Let x be a vertex of G. The number of vertices adjacent to x is called the *degree* of x and denoted by $d(x)$. We say that a generalized split graph with $V(G) = K\dot{\cup}S$ satisfies Condition (α) if for any $a, b \in S$, $\phi \in \text{Aut}(K)$, the inclusion $\phi(N(a)) \subsetneqq N(b)$ doesn't hold, and satisfies Condition (β) if for any $a, b \in S$, $d(a) = d(b)$. Clearly, Condition (β) always implies Condition (α) and the converse implication is true if K is a complete graph. However, Condition (α) doesn't imply (β) in general as shown by the following example.

Example 2.8. Let $K = C_5 + K_3$, where C_5 is the cycle of length 5. Write $V(C_5) = \{1, 2, 3, 4, 5\}$ and $V(K_3) = \{6, 7, 8\}$. By [12, Theorem 2.2], K is a core. Construct a graph G such that $V(G) = V(K) \cup \{a, b\}$ and $E(G) = E(K) \cup \{\{a, i\} | i = 1, 2, 3, 4\} \cup \{\{b, j\} | i = 6, 7, 8\}$. It is not hard to see that G is a generalized split graph with $V(G) = V(K) \cup \{a, b\}$, which satisfies ($\alpha$), but not ($\beta$).

Since Condition (β) can be checked more easily than Condition (α), the following corollary may be useful.

Corollary 2.9. *Let G be a generalized split graph with a split partition $V(G) = K\dot{\cup}S$. If $d(a) = d(b)$ for all $a, b \in S$, then G is end-regular.*

A graph G is called a *spider graph* if its vertex-set can be partitioned into disjoint sets K and S, satisfying the following three conditions: (1) K is a complete set and S is an independent set; (2) $|S| = |K|$; (3) there exists a bijective mapping $\phi : S \to K$ such that either: (i) $N(s) = \{\phi(s)\}$ for $s \in S$ or (ii) $N(s) = K \setminus \{\phi(s)\}$ for $s \in S$ (cf. [2]). We call G *a generalized spider graph* if the condition that K is a complete set is replaced by the condition that $[K]$ is a core of G in the definition above.

Corollary 2.10. *Any generalized spider graph is end-regular.*

Proof. This follows from Corollary 2.9 immediately.

Note that a tree of diameter 2 or 3 is either a star graph or a double star graph, we immediately obtain part of the main result of [19] (cf. [19, Theorem 3.4.]).

Corollary 2.11. *Any tree of diameter 2 or 3 is end-regular.*

It is clear for a split graph G with $V(G) = K\dot{\cup}S$, K is a core of G if and only if K is a maximal complete subgraph of G, so we immediately obtain the following corollary, which is actually the combination of the main results of [16] i.e., [16, Corollary 2.14 and Theorem 3.3].

Corollary 2.12. *Let G be a split graph with $V(G) = K\dot{\cup}S$ and $|K| = n$, where S is an independent set. Suppose that K is a maximal complete subgraph of G. Then G is end-regular if and only if there exists $r \in \{0, 1, 2, \cdots, n-1\}$ such that $d(x) = r$ for any $x \in S$.*

3 End-orthodox graphs

In this section, we will determine which generalized split graphs are end-orthodox. We begin with a general result about end-orthodox graphs.

Lemma 3.1. *If a graph G contains pairwise distinct vertices a, b, c such that $N(a) \subsetneqq N(b) \subsetneqq N(c)$, then G is not end-orthodox.*

Proof. Define f to be the retraction that maps a to b and fixes all other vertices, and define g to be the retraction that maps b to c and fixes the other vertices. Now $fg(a) = b$ and $fg(b) = c$, so fg is not a retraction, proving the lemma. □

Corollary 3.2. *Let G be a generalized split graph with a split partition $V(G) = K\dot{\cup}S$. If there exist distinct $a, b \in S$ such that $N(a) \subsetneqq N(b)$, then G is not end-orthodox.*

Proof. By Proposition 2.1, there is $c \in V(K)$ such that $N(b) \subsetneqq N(c)$. Now, the result follows from Lemma 3.1. □

Lemma 3.3. *Assume that G is a generalized split graph with a split partition $V(G) = K\dot{\cup}S$ such that $N(a) \setminus N(b) \neq \emptyset$ for any $a \neq b \in S$. Let H be any core of G and denote $V(G) \setminus V(H)$ by T. Set $W(a) = \{x \in K | N(a) \subsetneqq N(x)\}$ for any $a \in S$. If there is $\hat{a} \in S$ such that $|W(\hat{a})| \geq 2$, then $\hat{a} \in T$ and $W(\hat{a}) \subsetneqq H$.*

Proof. Let e be a retraction from G to H. Assume that $\hat{a} \in H$. Then $W(\hat{a}) \subsetneqq T$ by Lemma 2.5(1). Let u_1, u_2 be distinct vertices in $W(\hat{a})$. We obtain $u_1 = b_i$ and $u_2 = b_j$ for some $i \neq j$ by Lemma 2.5(6) and it follows that $\hat{a} = e(b_i) = e(b_j) = a_i = a_j$ by Lemma 2.5(2) and (7), a contradiction. Hence $\hat{a} \in T$.

Now, assume that $W(\hat{a}) \nsubseteq H$. Take $u \in W(\hat{a}) \setminus H$. Then there is $b \in S \cap H$ such that $u \in W(b)$ by Lemma 2.5(5). Note that $N(u) \cap H = N(b)$ by Lemma 2.6(1) and that $N(u) \setminus S \subsetneqq N(u) \cap H$ by Lemma 2.5(8), we obtain $N(\hat{a}) = N(\hat{a}) \setminus S \subsetneqq N(u) \setminus S \subsetneqq N(u) \cap H = N(b)$, a contradiction again, as desired. □

The following lemma gives a description of a retraction on a generalized split graph.

Lemma 3.4. *Let G be a generalized split graph with a split partition $V(G) = K\dot{\cup}S$ such that $N(a) \setminus N(b) \neq \emptyset$ for any $a \neq b \in S$. Set $W(a) = \{x \in K | N(a) \subsetneqq N(x)\}$ and fix $w(a) \in W(a)$ for any $a \in S$. Set $S' = \{w(a) | a \in S\}$. Let f be a retraction of G. Then the following statements hold.*

(1) *The retraction f fixes the vertices of $V(G) \setminus (S \cup S')$.*

(2) *For any $a \in S$, if $|W(a)| \geq 2$, then f fixes the vertices of $W(a)$; if $|W(a)| = 1$, then $f(w(a)) = w(a)$ or $f(w(a)) = a$. Moreover, if $f(w(a)) = a$, then $f(a) = a$.*

(3) *For any $a \in S$, either $f(a) = a$ or $f(a) \in W(a)$. Moreover, if $f(a) \in W(a)$, then f fixes the vertices of $W(a)$.*

Proof. Assume that $H = [f(V(K))]$ and set $T = V(G) \setminus V(H)$. Then H is a core of G by Proposition 2.3. Without loss of generality, we can assume by Lemma 2.5(4) that

$$S = \{a_1, a_2, \cdots, a_k, a_{k+1}, a_{k+2}, \cdots, a_n\}$$
$$T = \{a_1, a_2, \cdots, a_k, w(a_{k+1}), w(a_{k+2}), \cdots, w(a_n)\}$$

(1) By Lemma 2.5(4), we obtain if $x \notin S \cup S'$ then $x \in V(K) \cap V(H)$, and the result follows immediately.

(2) If $|W(a)| \geq 2$, then $W(a) \subseteqq H \subseteqq f(V(G))$ by Lemma 3.3 and so f fixes on vertices of $W(a)$. Suppose that $|W(a)| = 1$. If $f(w(a)) \neq w(a)$, then $w(a) \notin H$ and so $w(a) = w(a_i)$ for some $i \geq k+1$. Note that $V(K) = (V(K) \cap V(H)) \cup \{w(a_{k+1}), w(a_{k+2}), \cdots, w(a_n)\}$ and $V(H) = (V(K) \cap V(H)) \cup \{a_{k+1}, a_{k+2}, \cdots, a_n\}$. Hence $f(w(a)) = a_j$ for some $j \geq k+1$. Note that f fixes the vertices of $V(K) \cap V(H)$ and that $N(a) = N(w(a)) \cap V(H)$ by Lemma 2.6(1), we obtain $N(a) = N(a) \cap V(K) = N(w(a)) \cap V(K) \cap V(H) = N(a_j) \cap V(K) \cap V(H) = N(a_j) \cap V(H)$, and so $N(a_j) \subseteqq N(a)$, which implies $a = a_j$, i.e., $f(w(a)) = a$. At last, if $f(w(a)) = a$, then $f(a) = f(f(a)) = f(w(a)) = a$.

(3) Let $a \in S$ such that $f(a) \neq a$. We claim that $N(a) \subseteqq N(f(a))$. Let $x \in N(a)$. If $x \in V(H) \cap V(K)$, then $x = e(x) \in N(e(a))$. If $x \notin V(H) \cap V(K)$, then $x = w(a_j)$ for some $j \geq k+1$ (as $V(K) = (V(K) \cap V(H)) \cup \{w(a_{k+1}), w(a_{k+2}), \cdots, w(a_n)\}$) and so $a_j = f(w(a_j)) \in N(f(a))$ by (2), which implies $f(a) \in N(a_j) \subseteqq N(w(a_j)) = N(x)$. Hence $x \in N(f(a))$ and thus $N(a) \subseteqq N(f(a))$, proving the claim. By the assumption that $N(a) \setminus N(b) \neq \emptyset$ for any $a \neq b$, we have that $f(a) \in S$ and $f(a) \in W(a)$.

Now, suppose $f(a) \in W(a)$. If $|W(a)| \geq 2$, then f fixes the vertices of $W(a)$ by (2). If $|W(a)| = 1$, then $W(a) = \{w(a)\}$ and $f(a) = w(a)$, which implies $f(w(a)) = w(a)$. □

Theorem 3.5. *Let G be a generalized split graph with a split partition $V(G) = K \dot\cup S$. Then G is end-orthodox if and only if for any* *distinct* $a, b \in S$ *and any* $\phi \in Aut(K)$, $N(a) \neq N(b)$ *and the inclusion* $\phi(N(a)) \subsetneqq N(b)$ *doesn't hold.*

Proof. By Theorem 2.7 and Corollary 3.2, the "only if" part is obvious. Conversely, by Theorem 2.7 and the definition of end-orthodox graphs, we only need to prove that the composition of any two retractions of G is a retraction. This can be checked routinely by virtue of Lemma 3.4 and the result follows. □

The following corollaries follow immediately from Theorem 3.5.

Corollary 3.6. *Any generalized spider graph is end-orthodox.*

Corollary 3.7. *Let* G *be a tree of diameter 2 or 3. Then* G *is a end-orthodox if and only if* G *is a path.*

We see that [5, Proposition 3.2.] follows immediately from Corollary 3.7, which said if $K_{1,n}$ is a star, $n \geq 1$ then End($K_{1,n}$) is orthodox if and only if $n \leq 2$.

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