On a conjecture about (k,t)-choosability

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Abstract

A (k,t)-list assignment L of a graph G is a list of k colors available at each vertex v in G such that $|\bigcup_{v\in V(G)} L(v)| = t$. A proper coloring c such that $c(v) \in L(v)$ for each $v \in V(G)$ is said to be an L-coloring. We say that a graph G is L-colorable if G has an L-coloring. A graph G is (k,t)-choosable if G is L-colorable for every (k,t)-list assignment L.

Let G be a graph with n vertices and G does not contain $C_5 \vee K_{k-2}$ and K_{k+1}. We prove that G is $(k, kn - k^2 - 2k)$-choosable for $k \geq 3$ and G is not $(k, kn - k^2 - 2k)$-choosable for $k = 2$, which solves a conjecture posed by Chareonpanitseri, Punnim, and Uiyyasathian [W. Chareonpanitseri, N. Punnim, C. Uiyyasathian, On (k,t)-choosability of Graphs: Ars Combinatoria., 99, (2011) 321-333].

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1 Introduction

A *graph* is an order pair $G = (V(G), E(G))$, where $V(G)$ is a finite set of vertices and $E(G)$ is a set of unordered pairs of distinct vertices. A graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For $X \subseteq V(G)$, a graph $G - X$ is obtained by deleting all vertices of X from G. For $S \subseteq V(G)$, a subgraph of G induced by S, denoted by $G[S]$, is the graph obtained by deleting all vertices of $V(G) - S$ from G. In this paper, we denote a complete graph of order k, an independent set of size k' and a cycle with k vertices by K_k, S_k and C_k, respectively. A graph G is called K_k-free if G does not contain K_k as a subgraph.

For each vertex v in a graph G, let $L(v)$ denote a list of colors available at v. A *k-list assignment* L of a graph G is a list assignment L such that $|L(v)| = k$ for each $v \in V(G)$. A *(k,t)-list assignment* of a graph G is a k-list assignment L such that $|\bigcup_{v \in V(G)} L(v)| = t$. A proper coloring c such that $c(v) \in L(v)$ for each $v \in V(G)$ is said to be a *list coloring* or an *L-coloring.* If a graph G has an L-coloring, then we say that G is *L-colorable.* A graph G is *k-choosable* if every k-list assignment of G gives a list coloring. The *list chromatic number*, denoted by $\chi_l(G)$, is the minimum k such that G is k-choosable. If a graph G is L-colorable for every (k,t)-list assignment L, then G is *(k,t)-choosable.* Let $S \subseteq V(G)$. For a list assignment L of G, we denote the restriction of L to S by $L|_S$ and we denote $\bigcup_{v \in S} L(v)$ by $L(S)$.

The concept of list coloring was introduced by Vizing [11] and by Erdős, Rubin, and Taylor [4]. In 1979, Erdős et al. [4] established a characterization of 2-choosable graphs, especially, a characterization of bipartite graphs which are 2-choosable. In addition, several researchers studied and gave some properties of list coloring on specific classes of graphs. For example, Borowiecki et al. [1] studied the list coloring of cartesian products of graphs, in particular, they provided the bound on list chromatic number of cartesian products of graphs. In [7], [10], [9], [13], [14], [15], they studied and gave the concept of choosability on plane graphs and planar graphs. In 2011, Charoenpanitseri et al. [2] established the concept of (k,t)-choosability of graphs, they proved that an n-vertex graph G is (k,t)-choosable if $t \geq kn - k^2 + 1$. They also provided the bound on t to the K_{k+1}-free graph to be (k,t)-choosable in the following theorem.

Theorem 1.1. *Let $k \geq 3$. A K_{k+1}-free graph with n vertices is (k,t)-choosable for $t \geq kn - k^2 - 2k + 1$.*

Moreover, they found that an n-vertex graph containing $C_5 \vee K_{k-2}$ is not (k,t)-choosable for $k \geq 2$ and $k \leq t \leq kn - k^2 - 2k$. This implies that for $k \leq t \leq kn - k^2 - 2k$, an n-vertex graph containing $C_5 \vee K_{k-2}$ or K_{k+1} is not (k,t)-choosable. After that, they gave the following conjecture.

Conjecture An n-vertex graph G is $(k, kn - k^2 - 2k)$-choosable if G does not contain $C_5 \vee K_{k-2}$ and K_{k+1}.

In this paper, we will show that the conjecture is not true for $k = 2$ but it holds for $k \geq 3$.

2 (k,t)-choosability for $k = 2$ or 3.

In this section, we focus on (k,t)-choosability for $k = 2$ or 3. We will give an example to show that the conjecture does not hold for $k = 2$. After that, we will establish some important results for proving the conjecture. Now, we begin with Example 1 which shows that the conjecture is not true for $k = 2$.

Example 1. Let $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3\}$ be partite sets of complete bipartite graph $K_{3,3}$. We show that $K_{3,3}$, is not $(2,4)$-choosable. Let L be a $(2,4)$-list assignment shown in Figure 1. Assume that x_1 is colored by 1. Then vertices y_1 and y_2 must be colored by 2 and 3, respectively. Thus there is no an available color for the vertex x_2. The case x_1 is colored by 2 is similar. Hence, $K_{3,3}$ is not $(2,4)$-choosable.

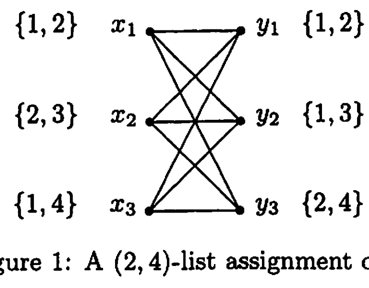

Figure 1: A $(2,4)$-list assignment of $K_{3,3}$.

Next, we consider the graphs with 7 vertices and having no $C_5 \vee K_1$ and K_4. Showing that the 7-vertex graphs having no $C_5 \vee K_1$ and K_4 are $(3,6)$-choosable, is an important part for proving the conjecture.

The following theorems and lemmas are essential tools to prove that the 7-vertex graphs having no $C_5 \vee K_1$ and K_4 are $(3,6)$-choosable.

Corollary 2.1. *[2] Let L be a list assignment of a graph G. If $|L(S)| \geq |S|$ for all $S \subseteq V(G)$, then G is L-colorable. Moreover, there exists an L-coloring such that each vertex of G assigned by distinct colors.*

Theorem 2.2. *[6] Let L be a list assignment of a graph G and let $S \subseteq V(G)$ be a maximal non-empty subset such that $|L(S)| < |S|$. If $G[S]$ is $L|_S$-colorable, then G is L-colorable.*

Lemma 2.3. *[2] Let G be an n-vertex graph. If $k \geq n-2$ and G is K_{k+1}-free, then G is (k,t)-choosable for any positive integer t.*

Lemma 2.4. *[2] Let G be a K_{k+1}-free graph with $k+3$ vertices. G is either $K_{k-1} \vee S_4$ or $C_5 \vee K_{k-2}$ if and only if $G - \{u,v\}$ contains K_k for every pair of nonadjacent vertices u, v.*

Theorem 2.5. *[4] The complete bipartite graph $K_{k,s}$ is k-choosable if and only if $s < k^k$.*

Corollary 2.6. *The complete bipartite graph $K_{2,n}$ is 2-choosable if and only if $n \leq 3$.*

Lemma 2.7. *Let G be a graph of order n where $n \geq 6$. If G does not contain K_{n-2} and $C_5 \vee K_{n-5}$, then G is $(n-3)$-colorable.*

Proof. Let $k = n-3$. Assume that G does not contain K_{k+1} and $C_5 \vee K_{k-2}$. We show that G is k-colorable.

Case 1 : G is $K_{k-1} \vee S_4$.

Then we use $k-1$ colors to color K_{k-1} and we assign a color different from colors in K_{k-1} to vertices in S_4. Thus G is k-colorable.

Case 2 : G is not $K_{k-1} \vee S_4$.

Since G is $C_5 \vee K_{k-2}$-free, Lemma 2.4 implies that there are two non-adjacent vertices u, v such that $G - \{u,v\}$ does not contain K_k. Since $k-1 = (k+1)-2$ and $G[V(G) - \{u,v\}]$ has no K_k, Lemma 2.3 implies that $G - \{u,v\}$ is $(k-1)$-colorable. Therefore G is k-colorable. $\square$

By Lemma 2.7, it follows that the chromatic number of 7-vertex graphs having no $C_5 \vee K_1$ and K_4 is at most 4. Thus we can divide the graphs which we are interested into two cases that is, the graphs with chromatic number at most 3 and the graphs with chromatic number 4.

Next, we show that 7-vertex graphs with chromatic number at most 3 and having no K_4 and $C_5 \vee K_1$ are $(3,6)$-choosable. Since every graph with these properties is a subgraph of complete 3-partite graph $K_{1,1,5}$ or $K_{1,2,4}$

or $K_{1,3,3}$ or $K_{2,2,3}$, it suffices to show that $K_{1,1,5}$, $K_{1,2,4}$, $K_{1,3,3}$, and $K_{2,2,3}$ are $(3,6)$-choosable. Before showing this, the three following theorems are needed.

Theorem 2.8. *[3] If $m \leq 2s+1$, then $\chi_l(K_{m,2*(k-s-1),1*s}) = k$.*

Theorem 2.9. *[3] If k is odd, then $\chi_l(K_{4,2*(k-1)}) = k$. Otherwise $\chi_l(K_{4,2*(k-1)}) = k+1$.*

Theorem 2.10. *[5] If $k \geq 3$, then $\chi_l(K_{3*2,2*(k-2)}) = k$.*

Lemma 2.11. *$K_{1,1,5}$, $K_{1,2,4}$, $K_{1,3,3}$, and $K_{2,2,3}$ are $(3,6)$-choosable.*

Proof. Let $m = 5$, $k = 3$, and $s = 2$ in Theorem 2.8. Then we have $\chi_l(K_{1,1,5}) = 3$. It follows that $K_{1,1,5}$ is 3-choosable. By Theorem 2.9, we have $\chi_l(K_{2,2,4}) = 3$. Since $K_{1,2,4}$ is a subgraph of $K_{2,2,4}$, we have $K_{1,2,4}$ is 3-choosable. Consider complete 3-partite graphs $K_{1,3,3}$ and $K_{2,2,3}$. By Theorem 2.10, we have $\chi_l(K_{2,3,3}) = 3$. Since $K_{1,3,3}$ and $K_{2,2,3}$ are subgraphs of $K_{2,3,3}$, we have $K_{1,3,3}$ and $K_{2,2,3}$ are 3-choosable. □

We now consider the 7-vertex graphs having no $C_5 \vee K_1$ and K_4 with chromatic number 4. We will prove that those graphs are $(3,6)$-choosable. The next lemma determines the minimum degree and maximum degree of the above graphs.

Lemma 2.12. *Let G be a graph with 7 vertices and $\chi(G) = 4$. If G does not contain K_4 and $C_5 \vee K_1$, then $\delta(G) \geq 3$ and $\Delta(G) = 4$.*

Proof. We first show that $\delta(G) \geq 3$. Suppose that $\delta(G) \leq 2$. Let v be a vertex in G such that $d(v) = \delta(G)$. Consider $G - v$. Then $|V(G-v)| = 6$ and so $G - v$ is 3-colorable by Lemma 2.7. Since $d(v) \leq 2$, then $\chi(G) = 3$, a contradiction. So $\delta(G) \geq 3$.

We next show that $\Delta(G) = 4$. If $\Delta(G) = 3$, then G is a 3-regular graph because $\delta(G) \geq 3$. But then the number of odd vertices is odd since G has 7 vertices which is a contradiction. Thus $\Delta(G) \geq 4$. Suppose that $\Delta(G) \geq 5$.

Case 1 : $\Delta(G) = 6$.

Let $v \in V(G)$ be such that $d(v) = 6$. If $G-v$ is bipartite, then $\chi(G) \leq 3$. If $G - v$ is not bipartite, then $G - v$ contains C_3 or C_5 which contradicts to G does not contain K_4 and $C_5 \vee K_1$. Thus Case 1 cannot occur.

Case 2 : $\Delta(G) = 5$.

Let $v \in V(G)$ be such that $d(v) = 5$. Then $|N(v)| = 5$. Since $|G| = 7$, there exists $u \in V(G)$ such that $uv \notin E(G)$. Thus u and v can be colored by the same color. If $G[N(v)]$ is bipartite, then $\chi(G) \leq 3$. If $G[N(v)]$ is not bipartite, then $G[N(v)]$ contains C_3 or C_5 which contradicts to G does not contain K_4 and $C_5 \vee K_1$. So Case 2 cannot occur either.

Hence, $\Delta(G) = 4$. This completes the proof. □

By [8], there are 22 graphs with degree sequence as in Lemma 2.12. Among them there are 7 graphs which have no K_4 and $C_5 \vee K_1$. So we consider only these 7 following graphs as shown in Figure 2.

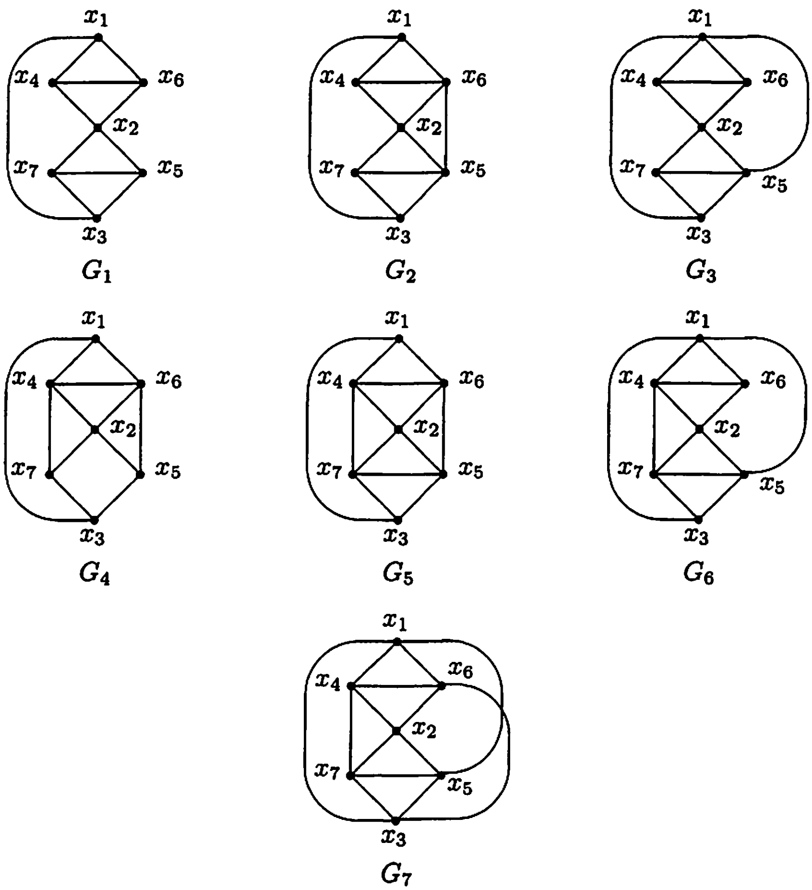

Figure 2: 7-vertex graphs with chromatic number 4 and having no K_4 and $C_5 \vee K_1$.

We want to show that these 7 graphs are (3, 6)-choosable. Since G_1, G_2, and G_4 are subgraphs of G_5 while G_3 and G_6 are subgraphs of G_7, it suffices to show that G_5 and G_7 are (3, 6)-choosable. Before proving that, we establish two following lemmas.

Lemma 2.13. *$K_{2,2,2}$ is L-colorable if $|L(V(K_{2,2,2}))| = 5$ and each list has size 3 except one list of size 2.*

Proof. Let $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$ and $Z = \{z_1, z_2\}$ be partite sets of $K_{2,2,2}$. Let L be a list assignment of $K_{2,2,2}$ such that $|L(V(K_{2,2,2}))| = 5$ and each list has size 3 except one list of size 2. Without loss of generality, we may assume that $|L(x_1)| = 2$. Then $|L(u)| = 3$ for each $u \in V(K_{2,2,2}) - \{x_1\}$.

Case 1 : $L(x_1) \cap L(x_2) \neq \emptyset$.

Let $a \in L(x_1) \cap L(x_2)$. Consider $G' = K_{2,2,2} - \{x_1, x_2\}$. Put $L'(u) = L(u) - \{a\}$ for each $u \in V(G')$. Then $|L'(u)| \geq 2$ for each $u \in V(G')$. Since G' is $K_{2,2}$, G' is L'-colorable by Lemma 2.6. It follows that $K_{2,2,2}$ is L-colorable.

Case 2 : $L(x_1) \cap L(x_2) = \emptyset$.

Without loss of generality, assume that $|L(x_1)| = 2$ and $|L(x_2)| = 3$. Since there are 5 colors and $|L(y_1)| = |L(y_2)| = 3$, it follows that $L(y_1) \cap L(y_2) \neq \emptyset$. Let $a \in L(y_1) \cap L(y_2)$. Consider $G' = K_{2,2,2} - \{y_1, y_2\}$. Put $L'(u) = L(u) - \{a\}$ for each $u \in V(G')$. Since $L(x_1) \cap L(x_2) = \emptyset$, either $|L'(x_1)| = 2 = |L'(x_2)|$ or $|L'(x_1)| = 1$ and $|L'(x_2)| = 3$. In either case, we can color each vertex in G' by increasing order of the size of lists. So $K_{2,2,2}$ is L-colorable. □

Lemma 2.14. *Let G be a graph in Figure 3. If each list has size 2 except exactly two lists of size 3 and $|L(x_4)| = 2$ or $|L(x_5)| = 2$, then G is L-colorable.*

Proof. Let L be a list assignment of G such that each list has size 2 except exactly two lists of size 3 and $|L(x_4)| = 2$ or $|L(x_5)| = 2$.

Case 1 : $|L(x_1)| = 3$.

Then $|L(x_i)| \geq 2$ for $2 \leq i \leq 5$. Since $G[\{x_2, x_3, x_4, x_5\}]$ is $K_{2,2}$, it is 2-choosable by Lemma 2.6. After we color $G[\{x_2, x_3, x_4, x_5\}]$, we can color x_1 because $|L(x_1)| = 3$ and $d(x_1) = 2$.

Case 2 : $|L(x_1)| = 2$.

If $|L(x_4)| = 3$ or $|L(x_5)| = 3$, then it is easy to verify that G is L-colorable.

Suppose $|L(x_4)| = 2$ and $|L(x_5)| = 2$, then $|L(x_2)| = |L(x_3)| = 3$ by the hypothesis of Lemma 2.14. If $L(x_1) \cap L(x_4) \neq \emptyset$, then we can choose a color $a \in L(x_1) \cap L(x_4)$ to color both vertices x_1 and x_4, and we can color the vertices x_5, x_3 and x_2 orderly. Thus G is L-colorable. If $L(x_1) \cap L(x_4) = \emptyset$, then we can choose a color $a \in (L(x_1) \cup L(x_4)) - L(x_2)$, say $a \in L(x_1)$. Assign a to vertices x_1, and color the vertices x_3, x_5, x_4 and x_2 orderly. Thus G is L-colorable. □

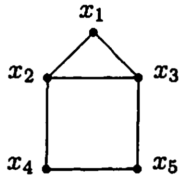

Figure 3: A graph G in Lemma 2.14.

Lemma 2.15. *G_5 is $(3,6)$-choosable.*

Proof. Let L be a $(3,6)$-list assignment of G_5.

Case 1 : $L(x_4) \cap L(x_5) \neq \emptyset$ or $L(x_6) \cap L(x_7) \neq \emptyset$.

By the symmetry of G_5, we may assume that $L(x_4) \cap L(x_5) \neq \emptyset$ and $1 \in L(x_4) \cap L(x_5)$. Consider $G' = G_5 - \{x_4, x_5\}$. Put $L^{'}(u) = L(u) - \{1\}$ for each $u \in V(G')$. Since G' is C_5, a graph G' is $L^{'}$-colorable if there are vertices u and v such that $L^{'}(u) \neq L^{'}(v)$ or $|L^{'}(u)| = 3$ which implies that G_5 is L-colorable. Now, we suppose that $L^{'}(u) = L^{'}(v)$ and $|L^{'}(u)| = 2$ for $u, v \in V(G')$. Without loss of generality, assume that $L^{'}(u) = \{2,3\}$ for each $u \in V(G')$. Thus $L(u) = \{1,2,3\}$ for each $u \in V(G')$. Since G' is C_5, G' is L-colorable. Since there are 6 colors and $1 \in L(x_4) \cap L(x_5)$, it follows that there are $a \in L(x_4) - \{1,2,3\}$ and $b \in L(x_5) - \{1,2,3\}$. Thus we can assign colors a to the vertex x_4 and b to the vertex x_5. It follows that G_5 is L-colorable. This completes the proof of Case 1.

Case 2 : $L(x_4) \cap L(x_5) = \emptyset$ and $L(x_6) \cap L(x_7) = \emptyset$.

Without loss of generality, we may assume that $L(x_4) = \{1,2,3\}$ and $L(x_5) = \{4,5,6\}$.

Subcase 2.1 : $L(x_3) \cap L(x_4) \neq \emptyset$.

Without loss of generality, we may assume that $1 \in L(x_3) \cap L(x_4)$. Then color vertices x_3 and x_4 by 1. Consider $G' = G_5 - \{x_3, x_4\}$. Put $L^{'}(u) = L(u) - \{1\}$ for each $u \in V(G')$.

Subcase 2.1.1 : $1 \in L(x_6)$.
Then $1 \notin L(x_7)$ because $L(x_6) \cap L(x_7) = \emptyset$. Thus $|L'(x_7)| = 3$. We now define a list coloring c as follows. We assign any color from $L'(x_6)$ to the vertex x_6 and then assign any colors from $L'(x_1) - \{c(x_6)\}$ and $L'(x_2) - \{c(x_6)\}$ to vertices x_1 and x_2, respectively. Next, we assign a color from $L'(x_5) - \{c(x_2), c(x_6)\}$ to the vertex x_5. For the vertex x_7, it obtains a color from $L'(x_7) - \{c(x_2), c(x_5)\}$. So G' is L'-colorable. It follows that G_5 is L-colorable.
Subcase 2.1.2 : $1 \notin L(x_6)$.
Then $|L'(x_6)| = 3$. Since $L(x_6) \cap L(x_7) = \emptyset$ and there are 6 colors, it follows that $1 \in L(x_7)$ and so $|L'(x_7)| = 2$. We now define a list coloring c as follows. We first assign any color from $L'(x_7)$ to the vertex x_7 and then assign any color from $L'(x_2) - \{c(x_7)\}$ to the vertex x_2. Next, the vertex x_5 obtains a color from $L'(x_5) - \{c(x_2), c(x_7)\}$. For the vertex x_6, it can be colored by a color from $L'(x_6) - \{c(x_2), c(x_5)\}$. Thus the vertex x_1 obtains a color from $L'(x_1) - \{c(x_6)\}$. So G' is L'-colorable. It follows that G_5 is L-colorable. This proves Subcase 2.1.2 and completes the proof of Subcase 2.1.

The case $L(x_3) \cap L(x_6) \neq \emptyset$ or $L(x_1) \cap L(x_5) \neq \emptyset$ or $L(x_1) \cap L(x_7) \neq \emptyset$ is similar to Subcase 2.1. Next, we may assume that $L(x_3) \cap L(x_6) = \emptyset$ and $L(x_1) \cap L(x_5) = \emptyset$ and $L(x_1) \cap L(x_7) = \emptyset$.
Subcase 2.2 : $L(x_3) \cap L(x_4) = \emptyset$.
Since $L(x_4) = \{1, 2, 3\}$, $L(x_3) = \{4, 5, 6\}$. So $L(x_6) = \{1, 2, 3\}$ because $L(x_3) \cap L(x_6) = \emptyset$. Since $L(x_6) \cap L(x_7) = \emptyset$, $L(x_7) = \{4, 5, 6\}$. It follows that $L(x_1) = \{1, 2, 3\}$ because $L(x_1) \cap L(x_7) = \emptyset$. Since $L(x_1) \cap L(x_5) = \emptyset$, $L(x_5) = \{4, 5, 6\}$. We now assign any color from $L(x_2)$ to the vertex x_2. Thus we can color the remaining vertices of G_5. It follows that G_5 is L-colorable. This proves Subcase 2.2 and completes the proof of our lemma. □

Lemma 2.16. *G_7 is $(3, 6)$-choosable.*

Proof. Note that x_i is not adjacent to x_j if and only if $i - j \equiv \pm 1 (\text{mod} 7)$. Let L be a $(3, 6)$-list assignment of G_7.
Case 1 : There exists a unique $i \in \{1, 2, ..., 7\}$ and a color a such that $a \in L(x_i)$.
Without loss of generality, we may assume that $a \in L(x_1)$ and then we assign a color a to x_1. Thus $L(x_6) \cap L(x_7) \neq \emptyset$. Without loss of generality, let $2 \in L(x_6) \cap L(x_7)$ Consider $G' = G_7 - \{x_1, x_6, x_7\}$. Put $L'(u) =$

$L(u)-\{2\}$ for each $u \in V(G')$. Then $|L'(u)| \geq 2$ for each $u \in V(G')$. Since G' is P_4, G' is L'-colorable. Hence, G_7 is L-colorable. This proves Case 1.

Case 2 : There exists a color a such that a appears exactly in 2 lists.

Recall that x_i is not adjacent to x_j if and only if $i-j \equiv \pm 1 (\text{mod} 7)$. We only need consider three subcases as follows.

Subcase 2.1 : $a \in L(x_1) \cap L(x_2)$.

Consider $G' = G_7 - \{x_1, x_2\}$. Put $L'(u) = L(u) - \{a\}$ for each $u \in V(G')$. Then $|L'(u)| = 3$ for each $u \in V(G')$. By Lemma 2.14, G' is L'-colorable. So G_7 is L-colorable.

Subcase 2.2 : $a \in L(x_1) \cap L(x_3)$.

Consider $G' = G_7 - \{x_1\}$. Put $L'(u) = L(u) - \{a\}$ for each $u \in V(G')$. Then $|L'(x_3)| = 2$ and $|L'(u)| = 3$ for each $u \in V(G') - \{x_3\}$. Since G' is a subgraph of $K_{2,2,2}$, G' is L'-colorable by Lemma 2.13. So G_7 is L-colorable.

Subcase 2.3 : $a \in L(x_1) \cap L(x_4)$.

Consider $G' = G_7 - \{x_1\}$. Put $L'(u) = L(u) - \{a\}$ for each $u \in V(G')$. Then $|L'(x_4)| = 2$ and $|L'(u)| = 3$ for each $u \in V(G') - \{x_4\}$. Since G' is a subgraph of $K_{2,2,2}$, G' is L'-colorable by Lemma 2.13. So G_7 is L-colorable. This completes the proof of Case 2.

Case 3 : There exists a color a such that a appears exactly in 4, 5, or 6 lists.

Recall that x_i is not adjacent to x_j if and only if $i-j \equiv \pm 1 (\text{mod} 7)$. We only need consider three subcases as follows.

If a appears in exactly 4 lists, we may assume that $a \in L(x_1) \cap L(x_2) \cap L(x_3) \cap L(x_4)$ or $a \in L(x_1) \cap L(x_2) \cap L(x_3) \cap L(x_5)$ or $a \in L(x_1) \cap L(x_2) \cap L(x_4) \cap L(x_5)$ or $a \in L(x_1) \cap L(x_2) \cap L(x_4) \cap L(x_6)$.

If a appears exactly in 5 lists, we may assume that $a \in L(x_1) \cap L(x_2) \cap L(x_3) \cap L(x_4) \cap L(x_5)$ or $a \in L(x_1) \cap L(x_2) \cap L(x_3) \cap L(x_4) \cap L(x_6)$ or $a \in L(x_1) \cap L(x_2) \cap L(x_3) \cap L(x_5) \cap L(x_6)$.

If a appears exactly in 6 lists, we may assume that $a \in L(x_1) \cap L(x_2) \cap L(x_3) \cap L(x_4) \cap L(x_6) \cap L(x_7)$.

Then we color vertices x_1 and x_2 by a. Consider $G' = G_7 - \{x_1, x_2\}$. Put $L'(u) = L(u) - \{a\}$ for each $u \in V(G')$. Then $|L'(u)| \geq 2$ for each $u \in V(G')$. By Lemma 2.14, G' is L'-colorable. Hence, G_7 is L-colorable. This proves Case 3.

Case 4 : Each color appears exactly in 3 lists or 7 lists.

Let x and y be the number of colors that appear in exactly 3 lists and 7 lists, respectively. Then $x + y = 6$ and $3x + 7y = 21$. Observe that this system of equations has no integer solution. So this case cannot occur.

Hence, G_7 is $(3,6)$-choosable, as required. ☐

Corollary 2.17. *If G is a 7-vertex graph having no $C_5 \vee K_1$ and K_4, then G is $(3,6)$-choosable.*

Proof. This corollary follows from Lemmas 2.7, 2.11, 2.15, and 2.16. ☐

3 Proof of the conjecture.

We prove the conjecture in this section. Two following lemmas are needed.

Lemma 3.1. *[2] Let $A_1, A_2, ..., A_n$ be k-sets and $J \subseteq \{1, 2, ..., n\}$. If $|\bigcup_{i=1}^{n} A_i| \geq p$, then $|\bigcup_{i \in J} A_i| \geq p - (n - |J|)k$.*

Lemma 3.2. *[2] If a $(k+3)$-vertex graph is K_{k+1}-free, then it is (k,t)-choosable for $t \geq k+1$.*

Theorem 3.3. *Let G be a graph with n vertices and $k \geq 3$. If G does not contain K_{k+1} and $C_5 \vee K_{k-2}$, then G is (k,t)-choosable for $t = kn - k^2 - 2k$.*

Proof. Suppose that G does not contain K_{k+1} and $C_5 \vee K_{k-2}$. Let $S \subseteq V(G)$ be such that $|L(S)| < |S|$. We prove that $G[S]$ is $L|_S$-colorable in order to utilize Theorem 2.2. By Lemma 3.1, $|L(S)| \geq t - (n - |S|)k = kn - k^2 - 2k - nk + |S|k = |S|k - k^2 - 2k$. Thus $|S| > |L(S)| \geq |S|k - k^2 - 2k$. It follows that $|S| < k + 3 + \frac{3}{k-1}$. So $|S| \leq k+4$ for $k = 3$ and $|S| \leq k+3$ for $k \geq 4$. Note that $k \leq |L(S)| < |S| \leq k+4$. If $|S| \leq k+2$, then $G[S]$ is $L|_S$-colorable by Lemma 2.3. Now we consider $|S| = k+3$ or $k+4$.
Case 1 : $|S| = k+3$.

Notice that $k \leq |L(S)| < |S| = k+3$. If $|L(S)| \geq k+1$, then $G[S]$ is $L|_S$-colorable by Lemma 3.2. We now suppose that $|L(S)| = k$. Since $G[S]$ does not contain K_{k+1} and $C_5 \vee K_{k-2}$, we have $G[S]$ is k-colorable by Lemma 2.7. This implies that $G[S]$ is $L|_S$-colorable.
Case 2 : $|S| = k+4$.

Then $k = 3$ which implies that $|S| = 7$. Since $|S| > |L(S)| \geq |S|k - k^2 - 2k = 7(3) - 3^2 - 2(3) = 6$, it follows that $|L(S)| = 6$. Since $k = 3$ and G does not contain $C_5 \vee K_{k-2}$ and K_{k+1}, it follows that G has no $C_5 \vee K_1$ and K_4. So $G[S]$ does not contain $C_5 \vee K_1$ and K_4. By Corollary 2.17, $G[S]$ is $(3,6)$-choosable. This implies that $G[S]$ is $L|_S$-colorable.

Hence, G is L-colorable by Theorem 2.2. ☐

4 Open problem.

Chareonpanitseri et al. [2] studied the concept of (k,t)-choosability. They provided the characterization of graphs to be $(k, kn-k^2-2k+1)$-choosable. Moreover, they gave a conjecture on n-vertex graphs to be $(k, kn-k^2-2k)$-choosable. In this paper, we showed that $K_{3,3}$ is not $(2,4)$-choosable and we proved that an n-vertex graph having no $C_5 \vee K_1$ and K_4 is $(k, kn-k^2-2k)$-choosable for $k \geq 3$. That is, the conjecture does not hold for $k=2$ and it is true for $k \geq 3$. However, there is no result on $(k, kn-k^2-2k-1)$-choosability of n-vertex graphs up to date. This leads to the following open problem.

Open Problem What is a characterization of graph that is $(k, kn-k^2-2k-1)$-choosable?

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