BILATERAL AND BILINEAR GENERATING FUNCTIONS FOR THE GENERALIZED ZERNIKE OR DISC POLYNOMIALS

RABİA AKTAŞ*, FATMA TAŞDELEN AND NURAY YAVUZ

ABSTRACT. In this paper, we obtain some generating functions for the generalized Zernike or disk polynomials $P_{m,n}^{\alpha}(z,z^*)$ which are investigated by Wünsche [13]. We derive various families of bilinear and bilateral generating functions. Furthermore, some special cases of the results presented in this study are indicated. Also, it is possible to obtain multilinear and multilateral generating functions for the polynomials $P_{m,n}^{\alpha}(z,z^*)$.

1. INTRODUCTION

Zernike polynomials were introduced by Zernike in [14] when discussing his phase-contrast method in application to circular concave mirrors. They are taken into account in a few monographs on optics and on geometrical optical imaging. The Zernike polynomials are considered only in a very few number of mathematical monographs and representations of orthogonal polynomials and special functions. The most natural generalization of the Zernike polynomials is to 2D polynomials which are orthogonal in the unit disc $zz^* < 1$ with weight function $(1-zz^*)^{\alpha}$, where $\alpha > -1$ and the special case $\alpha = 0$ is equivalent to the usual Zernike polynomials. However, different notations and variables are favorable. A comprehensive representation of 2D polynomials with taking into the account the disc polynomials which is the first class of the considered seven classes of 2D polynomials and some their properties were given by Koornwinder [3]. The monograph of Dunkl and Xu [2] takes shortly into account the disc polynomials (see also [11]).

In a recent paper [13], Wünsche has introduced generalized Zernike or disc polynomials $P_{m,n}^{\alpha}(z,z^*)$, generalizing to pairs of complex conjugate

Key words and phrases. Jacobi polynomials; generalized Zernike polynomials; disc polynomials; generating function; bilinear generating function; bilateral generating function.

2000 Math. Subject Classification. 33C45.

* Corresponding author.

variables (z, z^*), the real domain relation of the Zernike polynomials to Jacobi polynomials and then, Wünsche has given some their properties, such as lowering and raising operators and differential equations for the disc polynomials and disc functions, and generating functions. In [13], it was hoped that the generalized Zernike polynomials could find new applications in quantum optics.

The generalized Zernike or disc polynomials $P^{\alpha}_{m,n}(z, z^*)$ have been introduced in [13] by the following definition

$$P^{\alpha}_{m,n}(z, z^*) = \begin{cases} \frac{n!\alpha!}{(n+\alpha)!} z^{m-n} P_n^{(\alpha,m-n)}(2zz^*-1) & ; \quad m \geq n \\ \frac{m!\alpha!}{(m+\alpha)!} z^{*n-m} P_m^{(\alpha,n-m)}(2zz^*-1) & ; \quad n > m \end{cases} \tag{1.1}$$

$$(\, m, n \in \mathbb{N}_0 =: \{0\} \cup \mathbb{N} = \{1, 2, ...\})$$

or equivalently,

$$P^{\alpha}_{m,n}(z, z^*) = \begin{cases} \frac{n!}{(\alpha+1)_n} z^{m-n} P_n^{(\alpha,m-n)}(2zz^*-1) & ; \quad m \geq n \\ \frac{m!}{(\alpha+1)_m} z^{*n-m} P_m^{(\alpha,n-m)}(2zz^*-1) & ; \quad n > m \end{cases}$$

where $(\lambda)_k := \lambda(\lambda+1)\ldots(\lambda+k-1) = \dfrac{\Gamma(\lambda+k)}{\Gamma(\lambda)}$ and $(\lambda)_0 := 1$ denotes the Pochhammer symbol and $P_n^{(\alpha,\beta)}(u)$ is Jacobi polynomials of degree n. Here, z and z^* are complex conjugate variables: $z = x + iy$ and $z^* = x - iy$.

Many relations for disc polynomials or generalized Zernike polynomials can be obtained from corresponding relations for Jacobi polynomials. In order to obtain generating functions for the polynomials $P^{\alpha}_{m,n}(z, z^*)$, we recall some properties satisfied by Jacobi polynomials. Jacobi polynomials have the following finite series forms [7]:

$$P_n^{(\alpha,\beta)}(u) = \sum_{k=0}^{n} \frac{(\alpha+1)_n (\alpha+\beta+1)_{n+k}}{k!\,(n-k)!\,(\alpha+1)_k (\alpha+\beta+1)_n} \left(\frac{u-1}{2}\right)^k \tag{1.2}$$

or equivalently,

$$P_n^{(\alpha,\beta)}(u) = \sum_{k=0}^{n} \frac{(\alpha+1)_n (\beta+1)_n}{k!\,(n-k)!\,(\alpha+1)_k (\beta+1)_{n-k}} \left(\frac{u-1}{2}\right)^k \left(\frac{u+1}{2}\right)^{n-k}. \tag{1.3}$$

The classical Jacobi polynomials $P_n^{(\alpha,\beta)}(u)$ are generated by (see [9], [12]):

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(u)\, t^n = 2^{\alpha+\beta} R^{-1} (1-t+R)^{-\alpha} (1+t+R)^{-\beta} \tag{1.4}$$

where $R = \left(1 - 2ut + t^2\right)^{1/2}$.

In this paper, we obtain some generating functions for the polynomials $P_{m,n}^{\alpha}(z,z^*)$ which are investigated by Wünsche. Furthermore, we derive various families of bilinear and bilateral generating functions for the polynomials $P_{m,n}^{\alpha}(z,z^*)$. Some applications of the results obtained in section 3 are presented.

2. Generating Functions for the Generalized Zernike Polynomials

In this section, we give some generating functions for $P_{m,n}^{\alpha}(z,z^*)$.

Theorem 2.1. *For the polynomials $P_{m,n}^{\alpha}(z,z^*)$, we have the following generating functions*

$$\sum_{n,m=0}^{\infty}(\alpha+1)_n\ P_{n+m,n}^{\alpha}(z,z^*)\frac{t^{m+n}}{m!n!} = R^{-1}\left(\frac{2}{1-t+R}\right)^{\alpha}\exp\left(\frac{2tz}{1+t+R}\right) \tag{2.1}$$

and

$$\sum_{n,m=0}^{\infty}(\alpha+1)_m\ P_{m,n+m}^{\alpha}(z,z^*)\frac{t^{m+n}}{m!n!} = R^{-1}\left(\frac{2}{1-t+R}\right)^{\alpha}\exp\left(\frac{2tz^*}{1+t+R}\right)$$

where $R=\left(1-2\left(2zz^*-1\right)t+t^2\right)^{1/2}$.

Proof. For $m\geq n$, by (1.1) and (1.4), we can write

$$\begin{aligned}
&\sum_{n,m=0}^{\infty}(\alpha+1)_n\ P_{n+m,n}^{\alpha}(z,z^*)\frac{t^{m+n}}{m!n!}\\
&= \sum_{m=0}^{\infty}\left\{\sum_{n=0}^{\infty}P_n^{(\alpha,m)}(2zz^*-1)\,t^n\right\}\frac{(tz)^m}{m!}\\
&= R^{-1}\left(\frac{2}{1-t+R}\right)^{\alpha}\sum_{m=0}^{\infty}\frac{1}{m!}\left(\frac{2tz}{1+t+R}\right)^m\\
&= R^{-1}\left(\frac{2}{1-t+R}\right)^{\alpha}\exp\left(\frac{2tz}{1+t+R}\right)
\end{aligned}$$

which completes the proof. For $n>m$, we obtain the second generating function. □

Theorem 2.2. *The polynomials $P^{\alpha}_{m,n}(z,z^*)$ are generated by*

$$\sum_{m,n=0}^{\infty} (\alpha+m+1)_n \ (\beta)_m \ P^{\alpha}_{n+m,n}(z,z^*) \frac{t^m}{m!}\frac{s^n}{n!}$$
$$= (1-s)^{-\alpha-1} H_4\left(\alpha+1,\beta;\alpha+1,\alpha+1;\frac{s(zz^*-1)}{(1-s)^2},\frac{tz}{1-s}\right) \quad (2.2)$$

and

$$\sum_{m,n=0}^{\infty} (\alpha+n+1)_m \ (\beta)_n \ P^{\alpha}_{m,n+m}(z,z^*) \frac{t^n}{n!}\frac{s^m}{m!}$$
$$= (1-s)^{-\alpha-1} H_4\left(\alpha+1,\beta;\alpha+1,\alpha+1;\frac{s(zz^*-1)}{(1-s)^2},\frac{tz^*}{1-s}\right)$$

for

$$\left(|s|<1\ ,\ \left|\frac{s}{(1-s)^2}\right|<r_1\ ,\ \left|\frac{t}{1-s}\right|<r_2\ ,\ 4\,r_1=(r_2-1)^2\right)$$

where Horn's H_4 function is defined by (see [9]*)*

$$H_4(\alpha,\beta;\gamma,\delta;x,y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{2m+n}(\beta)_n}{(\gamma)_m(\delta)_n}\frac{x^m}{m!}\frac{y^n}{n!}$$
$$\left(|x|<r_1\ ,\ \ |y|<r_2\ ,\ 4\,r_1=(r_2-1)^2\right).$$

Proof. For $m \geq n$, using (1.1) and (1.2), we have

$$\sum_{m,n=0}^{\infty} (\alpha+m+1)_n \ (\beta)_m \ P_{n+m,n}^{\alpha}(z,z^*) \frac{t^m}{m!}\frac{s^n}{n!}$$

$$= \sum_{m,n=0}^{\infty} \frac{(\alpha+m+1)_n \ (\beta)_m}{(\alpha+1)_n} P_n^{(\alpha,m)}(2zz^*-1)\frac{(tz)^m s^n}{m!}$$

$$= \sum_{m,n=0}^{\infty}\sum_{k=0}^{n} \frac{(\beta)_m (\alpha+m+1)_{n+k}}{(\alpha+1)_k} \frac{(zz^*-1)^k}{k!\,(n-k)!} \frac{(tz)^m s^n}{m!}$$

$$= \sum_{m=0}^{\infty}\sum_{n,k=0}^{\infty} \frac{(\beta)_m (\alpha+m+1)_{n+2k}}{(\alpha+1)_k} \frac{(zz^*-1)^k}{k!n!} \frac{(tz)^m s^{n+k}}{m!}$$

$$= \sum_{m,k=0}^{\infty} \left\{ \sum_{n=0}^{\infty} \frac{(\alpha+m+2k+1)_n}{n!} s^n \right\} \frac{(\beta)_m (\alpha+m+1)_{2k}}{(\alpha+1)_k \ k!m!}$$

$$\times (zz^*-1)^k (tz)^m s^k$$

$$= (1-s)^{-\alpha-1} \sum_{m,k=0}^{\infty} \frac{(\alpha+1)_{m+2k} \ (\beta)_m}{(\alpha+1)_k \ (\alpha+1)_m \ m!k!} \left(\frac{tz}{1-s}\right)^m \left(\frac{s(zz^*-1)}{(1-s)^2}\right)^k$$

$$= (1-s)^{-\alpha-1} H_4\left(\alpha+1,\beta;\alpha+1,\alpha+1;\frac{s(zz^*-1)}{(1-s)^2},\frac{tz}{1-s}\right).$$

For $n > m$, the second relation can be easily shown. □

Other generating functions for the polynomials $P_{m,n}^{\alpha}(z,z^*)$ can be obtained as follows:

Theorem 2.3. *For the polynomials $P_{m,n}^{\alpha}(z,z^*)$, we have*

$$\sum_{m,n=0}^{\infty} (\alpha+1)_n \ (\beta)_m \ P_{n+m,n}^{\alpha}(z,z^*) \frac{t^m}{m!}\frac{s^n}{n!}$$

$$= F_{14} : F_F(1,1,1,\alpha+1,\beta,\alpha+1;\alpha+1,1,1;s(zz^*-1),tz,szz^*)$$

and

$$\sum_{m,n=0}^{\infty} (\alpha+1)_m \ (\beta)_n \ P_{m,n+m}^{\alpha}(z,z^*) \frac{t^n}{n!}\frac{s^m}{m!}$$

$$= F_{14} : F_F(1,1,1,\alpha+1,\beta,\alpha+1;\alpha+1,1,1;s(zz^*-1),tz^*,szz^*)$$

$$(|s| < r_1 \ , \ |t| < r_2 \ , \ r_1 = r_2(1-r_2)) \tag{2.3}$$

where F_{14} is Lauricella hypergeometric function of three variables (and also this function can be given by Saran's notation F_F [8]) which is defined *by* [4] *(see also* [9]*)*

$$F_{14} \quad : \quad F_F\left(\alpha_1,\alpha_1,\alpha_1,\beta_1,\beta_2,\beta_1;\gamma_1,\gamma_2,\gamma_2;x,y,z\right) = \sum_{m,n,p=0}^{\infty}\frac{(\alpha_1)_{m+n+p}(\beta_1)_{m+p}(\beta_2)_n}{(\gamma_1)_m(\gamma_2)_{n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}$$

$$\left(\ |x|<r_1\ ,\quad |y|<r_2\ ,\ |z|<r_3\ ,\ (1-\ r_2)(r_2-r_3)=r_1r_2\ \right) \tag{2.4}$$

Proof. It is enough to use (1.1) and (1.3). Also, the condition (2.3) is obtained from (2.4). □

3. Bilinear and Bilateral Generating Functions

In recent years by making use of the familiar group-theoretic (Lie algebraic) method a certain mixed trilateral finite-series relationships have been proved for orthogonal polynomials (see, for instance, [9]). This section presents several families of bilinear and bilateral generating functions for the generalized Zernike or disc polynomials $P^{\alpha}_{m,n}(z,z^*)$ given by (1.1) without using Lie algebraic techniques but, with the help of the similar method as considered in [6],[10].

We begin by stating the following theorem.

Theorem 3.1. *Corresponding to an identically nonvanishing function $\Omega_\mu(y_1,\ldots,y_s)$ of s complex variables $y_1,\ldots,y_s$ $(s\in\mathbb{N})$ and of complex order μ, let*

$$\Lambda_{\mu,\nu}(y_1,\ldots,y_s;\xi):=\sum_{k=0}^{\infty}a_k\Omega_{\mu+\nu k}(y_1,\ldots,y_s)\xi^k \tag{3.1}$$

$$(a_k\neq 0,\ \ \mu,\nu\in\mathbb{C}).$$

Then, we have

$$\sum_{n_1,n_2=0}^{\infty}\sum_{k=0}^{[n_1/p]}a_k(\alpha+n_1-pk+1)_{n_2}\ P^{\alpha}_{n_1+n_2-pk,n_2}(z,z^*) \tag{3.2}$$

$$\times(\beta)_{n_1-pk}\,\Omega_{\mu+\nu k}(y_1,\ldots,y_s)\eta^k\frac{t^{n_1-pk}}{(n_1-pk)!}\frac{w^{n_2}}{n_2!}$$

$$=\ (1-w)^{-\alpha-1}H_4\left(\alpha+1,\beta;\alpha+1,\alpha+1;\frac{w\,(zz^*-1)}{(1-w)^2},\frac{tz}{1-w}\right)$$

$$\times\Lambda_{\mu,\nu}(y_1,\ldots,y_s;\eta)$$

provided that each member of (3.2) exists.

Proof. For convenience, let T denote the first member of the assertion (3.2) of Theorem 3.1. Straightforward calculations give

$$\begin{aligned} T &= \sum_{n_1,n_2=0}^{\infty} \sum_{k=0}^{\infty} a_k\,(\alpha+n_1+1)_{n_2}\,(\beta)_{n_1}\,P^{\alpha}_{n_1+n_2,n_2}(z,z^*) \\ &\quad \times\ \Omega_{\mu+\nu k}(y_1,\ldots,y_s)\eta^k\,\frac{t^{n_1}}{n_1!}\frac{w^{n_2}}{n_2!} \\ &= \sum_{k=0}^{\infty} a_k \Omega_{\mu+\nu k}(y_1,\ldots,y_s)\eta^k \\ &\quad \times \sum_{n_1,n_2=0}^{\infty} (\alpha+n_1+1)_{n_2}\,(\beta)_{n_1}\,P^{\alpha}_{n_1+n_2,n_2}(z,z^*)\,\frac{t^{n_1}}{n_1!}\frac{w^{n_2}}{n_2!}. \end{aligned}$$

If we use Theorem 2.2, then the proof of Theorem 3.1 is completed. □

In a similar manner, by appealing to the formula (2.1), we are led fairly easily to the following theorem.

Theorem 3.2. *Corresponding to an identically nonvanishing function $\Phi_\mu(u,v)$ of complex variables u,v and of complex order μ, let*

$$\begin{aligned} &\Psi_{\mu,\nu_1,\nu_2}(u,v;\tau) \\ &:= \sum_{k_1,k_2=0}^{\infty} b_{k_1,k_2}\ \Phi_{k_1\nu_1+k_2\nu_2+\mu\ ,k_2}(u,v)\tau^{k_1+k_2} \qquad (3.3) \\ &\qquad (b_{k_1,k_2} \neq 0\ ,\ \mu,\nu_1,\nu_2\in\mathbb{C}) \end{aligned}$$

and

$$\begin{aligned} &\Theta^{\mu,\nu_1,\nu_2}_{n_1,n_2,p}(z,z^*;u,v;\zeta) \\ &:= \sum_{k_1=0}^{[n_1/p]}\sum_{k_2=0}^{[n_2/p]} \frac{b_{k_1,k_2}\,(\alpha+1)_{n_2-pk_2}}{(n_1-pk_1)!\,(n_2-pk_2)!}\,P^{\alpha}_{n_1+n_2-(pk_1+pk_2),n_2-pk_2}(z,z^*) \\ &\times \Phi_{k_1\nu_1+k_2\nu_2+\mu,k_2}(u,v)\ \zeta^{k_1+k_2} \qquad (3.4) \end{aligned}$$

where $n_1,n_2,p\in\mathbb{N}$. Then, we have

$$\begin{aligned} &\sum_{n_1,n_2=0}^{\infty} \Theta^{\mu,\nu_1,\nu_2}_{n_1,n_2,p}\left(z,z^*;u,v;\frac{\eta}{t^p}\right)t^{n_1+n_2} \\ &= \Psi_{\mu,\nu_1,\nu_2}(u,v;\eta)\ R^{-1}\left(\frac{2}{1-t+R}\right)^{\alpha}\exp\left(\frac{2tz}{1+t+R}\right) \qquad (3.5) \end{aligned}$$

where

$$R=\left(1-2\,(2zz^*-1)\,t+t^2\right)^{1/2}$$

provided that each member of (3.5) exists.

Proof. For convenience, let T denote the first member of the assertion (3.5). Then, upon substituting for the polynomials

$$\Theta_{n_1,n_2,p}^{\mu,\nu_1,\nu_2}\left(z,z^*;u,v;\frac{\eta}{t^p}\right)$$

from the definition (3.4) into the left-hand side of (3.5), we obtain

$$\begin{aligned} T &= \sum_{n_1,n_2=0}^{\infty}\sum_{k_1=0}^{[n_1/p]}\sum_{k_2=0}^{[n_2/p]} \frac{b_{k_1,k_2}\,(\alpha+1)_{n_2-pk_2}}{(n_1-pk_1)!\,(n_2-pk_2)!} \\ &\quad \times P^{\alpha}_{n_1+n_2-(pk_1+pk_2),n_2-pk_2}(z,z^*)\ \Phi_{k_1\nu_1+k_2\nu_2+\mu,k_2}(u,v) \\ &\quad \times \eta^{k_1+k_2}\,t^{n_1+n_2-pk_1-pk_2}. \end{aligned} \tag{3.6}$$

Upon inverting the order of summation in (3.6), if we replace n_1 by n_1+pk_1 and n_2 by n_2+pk_2 and then we use Theorem 2.1, we may write

$$\begin{aligned} T &= \sum_{n_1,n_2=0}^{\infty}\sum_{k_1,k_2=0}^{\infty} \frac{b_{k_1,k_2}\,(\alpha+1)_{n_2}}{n_1!n_2!} P^{\alpha}_{n_1+n_2,n_2}(z,z^*) \\ &\quad \times \Phi_{k_1\nu_1+k_2\nu_2+\mu,k_2}(u,v)\eta^{k_1+k_2}t^{n_1+n_2} \\ &= \sum_{n_1,n_2=0}^{\infty}(\alpha+1)_{n_2}\,P^{\alpha}_{n_1+n_2,n_2}(z,z^*)\,\frac{t^{n_1+n_2}}{n_1!n_2!} \\ &\quad \times \sum_{k_1,k_2=0}^{\infty} b_{k_1,k_2}\ \Phi_{k_1\nu_1+k_2\nu_2+\mu,k_2}(u,v)\eta^{k_1+k_2} \\ &= \Psi_{\mu,\nu_1,\nu_2}(u,v;\eta)\ R^{-1}\left(\frac{2}{1-t+R}\right)^{\alpha}\exp\left(\frac{2tz}{1+t+R}\right) \end{aligned}$$

which completes the proof. □

4. Further Consequences

We can give many applications of our theorems obtained in the previous section with help of appropriate choices of the multivariable functions

$$\Omega_{\mu+\nu k}(y_1,\ldots,y_s)\ (k\in\mathbb{N}_0\,,\ s\in\mathbb{N})$$

in terms of simpler function of one and more variables. For example, if we set

$$s=r\quad\text{and}\quad \Omega_{\mu+\nu k}(y_1,\ldots,y_r)=h_{\mu+\nu k}^{(\gamma_1,\ldots,\gamma_r)}(y_1,\ldots,y_r)$$

in Theorem 3.1, where the multivariable Lagrange-Hermite polynomials (see [1])

$$h_n^{(\alpha_1,\ldots,\alpha_r)}(x_1,\ldots,x_r)$$

are generated by

$$\prod_{j=1}^{r}\left\{\left(1-x_j t^j\right)^{-\alpha_j}\right\}=\sum_{n=0}^{\infty} h_n^{(\alpha_1,\ldots,\alpha_r)}\left(x_1,\ldots,x_r\right)t^n \tag{4.1}$$

where $|t|<\min\left\{|x_1|^{-1},|x_2|^{-1/2},\ldots,|x_r|^{-1/r}\right\}$. Then, we obtain the following result which provides a class of bilateral generating functions for the multivariable Lagrange-Hermite polynomials and the polynomials $P_{m,n}^{\alpha}(z,z^*)$ defined by (1.1).

Corollary 4.1. *If* $\Lambda_{\mu,\nu}(y_1,\ldots,y_r;\xi):=\sum_{k=0}^{\infty}a_k h_{\mu+\nu k}^{(\gamma_1,\ldots,\gamma_r)}(y_1,\ldots,y_r)\xi^k$ *where* $a_k\neq 0$, $\nu,\mu\in\mathbb{C}$; *then we have*

$$\begin{aligned}
&\sum_{n_1,n_2=0}^{\infty}\sum_{k=0}^{[n_1/p]} a_k\left(\alpha+n_1-pk+1\right)_{n_2} P_{n_1+n_2-pk,n_2}^{\alpha}\left(z,z^*\right) \\
&\times(\beta)_{n_1-pk}\, h_{\mu+\nu k}^{(\gamma_1,\ldots,\gamma_r)}(y_1,\ldots,y_r)\eta^k\frac{t^{n_1-pk}}{(n_1-pk)!}\frac{w^{n_2}}{n_2!} \\
=\;&(1-w)^{-\alpha-1}H_4\left(\alpha+1,\beta;\alpha+1,\alpha+1;\frac{w\left(zz^*-1\right)}{(1-w)^2},\frac{tz}{1-w}\right) \\
&\times\Lambda_{\mu,\nu}(y_1,\ldots,y_r;\eta)
\end{aligned}\tag{4.2}$$

provided that each member of (4.2) exists.

Remark 4.1. *Using the generating relation (4.1) for the multivariable Lagrange-Hermite polynomials and taking* $a_k=1$, $\mu=0$ *and* $\nu=1$, *we have*

$$\begin{aligned}
&\sum_{n_1,n_2=0}^{\infty}\sum_{k=0}^{[n_1/p]}\left(\alpha+n_1-pk+1\right)_{n_2}(\beta)_{n_1-pk}\,P_{n_1+n_2-pk,n_2}^{\alpha}\left(z,z^*\right) \\
&\times h_k^{(\gamma_1,\ldots,\gamma_r)}(y_1,\ldots,y_r)\eta^k\frac{t^{n_1-pk}}{(n_1-pk)!}\frac{w^{n_2}}{n_2!} \\
=\;&(1-w)^{-\alpha-1}H_4\left(\alpha+1,\beta;\alpha+1,\alpha+1;\frac{w\left(zz^*-1\right)}{(1-w)^2},\frac{tz}{1-w}\right) \\
&\times\prod_{j=1}^{r}\left\{\left(1-y_j\eta^j\right)^{-\gamma_j}\right\}
\end{aligned}$$

where

$$\left(|\eta|<\min\left\{|y_1|^{-1},|y_2|^{-1/2},\ldots,|y_r|^{-1/r}\right\}\right).$$

Choosing $\Phi_{k_1\nu_1+k_2\nu_2+\mu\ ,k_2}(u,v) = P^{\gamma}_{k_1\nu_1+k_2\nu_2+\mu\ ,k_2}(u,u^*)$, $(\mu,\nu_1,\nu_2 \in \mathbb{C})$, where u and u^* are complex conjugate variables, in Theorem 3.2, we obtain the following class of bilinear generating functions for the polynomials $P^{\alpha}_{m,n}(z,z^*)$.

Corollary 4.2. *If*

$$\Psi_{\mu,\nu_1,\nu_2}(u,u^*;\tau) := \sum_{k_1,k_2=0}^{\infty} b_{k_1,k_2}\, P^{\gamma}_{k_1\nu_1+k_2\nu_2+\mu\ ,k_2}(u,u^*)\,\tau^{k_1+k_2}$$

$$(b_{k_1,k_2} \neq 0\ ,\ \mu,\nu_1,\nu_2 \in \mathbb{C})$$

and

$$\Theta^{\mu,\nu_1,\nu_2}_{n_1,n_2,p}(z,z^*;u,u^*;\zeta) := \sum_{k_1=0}^{[n_1/p]}\sum_{k_2=0}^{[n_2/p]} \frac{b_{k_1,k_2}\,(\alpha+1)_{n_2-pk_2}}{(n_1-pk_1)!\,(n_2-pk_2)!}\, P^{\alpha}_{n_1+n_2-(pk_1+pk_2),n_2-pk_2}(z,z^*) \times P^{\gamma}_{k_1\nu_1+k_2\nu_2+\mu\ ,k_2}(u,u^*)\,\zeta^{k_1+k_2}$$

where $n_1,n_2,p \in \mathbb{N}$. Then, we have

$$\sum_{n_1,n_2=0}^{\infty} \Theta^{\mu,\nu_1,\nu_2}_{n_1,n_2,p}\left(z,z^*;u,u^*;\frac{\eta}{t^p}\right) t^{n_1+n_2} = \Psi_{\mu,\nu_1,\nu_2}(u,u^*;\eta)\, R^{-1}\left(\frac{2}{1-t+R}\right)^{\alpha} \exp\left(\frac{2tz}{1+t+R}\right)$$

where

$$R = \left(1-2\,(2zz^*-1)\,t+t^2\right)^{1/2}.$$

Remark 4.2. *Using Theorem 2.1 and taking $b_{k_1,k_2} = \frac{(\gamma+1)_{k_2}}{k_1!k_2!}$, $\mu = 0$ and $\nu_1 = \nu_2 = 1$, we have*

$$\sum_{n_1,n_2=0}^{\infty}\sum_{k_1=0}^{[n_1/p]}\sum_{k_2=0}^{[n_2/p]} \frac{(\gamma+1)_{k_2}\,(\alpha+1)_{n_2-pk_2}}{k_1!k_2!\,(n_1-pk_1)!\,(n_2-pk_2)!} \times P^{\alpha}_{n_1+n_2-(pk_1+pk_2),n_2-pk_2}(z,z^*)\,P^{\gamma}_{k_1+k_2\ ,k_2}(u,u^*) \times \eta^{k_1+k_2}\,t^{n_1+n_2-pk_1-pk_2}$$
$$= (RS)^{-1}\left(\frac{2}{1-t+R}\right)^{\alpha}\left(\frac{2}{1-\eta+S}\right)^{\gamma}\exp\left(\frac{2tz}{1+t+R}+\frac{2\eta u}{1+\eta+S}\right)$$

where

$$R = \left(1-2\,(2zz^*-1)\,t+t^2\right)^{1/2}$$
$$S = \left(1-2\,(2uu^*-1)\,\eta+\eta^2\right)^{1/2}.$$

If we set $\Phi_{k_1\nu_1+k_2\nu_2+\mu\ ,k_2}(u,v) = {}_2P^{(\delta-k_2)}_{k_1\nu_1+k_2\nu_2+\mu\ ,k_2}(u,v)$, $(\mu,\nu_1,\nu_2 \in \mathbb{C})$, in Theorem 3.2, where two-variable analogues of Jacobi polynomials [3]

$$ {}_2P^{\delta}_{m,n}(u,v) = P^{(\delta+n+\frac{1}{2},\delta+n+\frac{1}{2})}_{m-n}(u)\left(1-u^2\right)^{n/2} P^{(\delta,\delta)}_{n}\left(\tfrac{v}{\sqrt{1-u^2}}\right),\quad m \geq n \geq 0 $$

are generated by [5]

$$ \begin{aligned} &\sum_{m,n=0}^{\infty} \frac{(\lambda)_n}{(\sigma)_n}\ {}_2P^{\delta-n}_{m+n,n}(u,v)\,t^{m+n} \\ = \ & \frac{2^{2\delta+1}}{\rho}\left((1+\rho)^2-t^2\right)^{-\delta-\frac{1}{2}} \\ & \times F_1\left[\lambda,-\delta,-\delta;\sigma;-\tfrac{tv}{2}-\tfrac{t\sqrt{1-u^2}}{2},-\tfrac{tv}{2}+\tfrac{t\sqrt{1-u^2}}{2}\right] \\ & \left(\rho=\left\{1-2ut+t^2\right\}^{1/2}\right), \end{aligned} $$

then we obtain the following result which provides a class of bilateral generating functions for the polynomials ${}_2P^{\delta}_{m,n}(u,v)$ and the generalized Zernike polynomials.

Corollary 4.3. *If*

$$ \begin{aligned} &\Psi_{\mu,\nu_1,\nu_2}(u,v;\tau) \\ :&= \sum_{k_1,k_2=0}^{\infty} b_{k_1,k_2}\ {}_2P^{(\delta-k_2)}_{k_1\nu_1+k_2\nu_2+\mu\ ,k_2}(u,v)\tau^{k_1+k_2} \\ (b_{k_1,k_2} &\neq 0\ ,\ \mu,\nu_1,\nu_2 \in \mathbb{C}) \end{aligned} $$

and

$$ \begin{aligned} &\Theta^{\mu,\nu_1,\nu_2}_{n_1,n_2,p}(z,z^*;u,v;\zeta) \\ := &\sum_{k_1=0}^{[n_1/p]}\sum_{k_2=0}^{[n_2/p]} \frac{b_{k_1,k_2}\,(\alpha+1)_{n_2-pk_2}}{(n_1-pk_1)!\,(n_2-pk_2)!} P^{\alpha}_{n_1+n_2-(pk_1+pk_2),n_2-pk_2}(z,z^*) \\ &\times\ {}_2P^{(\delta-k_2)}_{k_1\nu_1+k_2\nu_2+\mu\ ,k_2}(u,v)\ \zeta^{k_1+k_2} \end{aligned} $$

where $n_1,n_2,p \in \mathbb{N}$. *Then, we have*

$$ \begin{aligned} &\sum_{n_1,n_2=0}^{\infty} \Theta^{\mu,\nu_1,\nu_2}_{n_1,n_2,p}\left(z,z^*;u,v;\frac{\eta}{t^p}\right) t^{n_1+n_2} \\ = \ & \Psi_{\mu,\nu_1,\nu_2}(u,v;\eta)\ R^{-1}\left(\frac{2}{1-t+R}\right)^{\alpha}\exp\left(\frac{2tz}{1+t+R}\right) \end{aligned} $$

where

$$ R = \left(1-2\left(2zz^*-1\right)t+t^2\right)^{1/2}. $$

Furthermore, for every suitable choice of the coefficients a_k and b_{k_1,k_2} $(k, k_1, k_2 \in \mathbb{N}_0)$, if the multivariable functions $\Omega_{\mu+\nu k}(y_1, ..., y_s)$ $(s \in \mathbb{N})$ and $\Phi_{k_1\nu_1+k_2\nu_2+\mu\ ,k_2}(u, v)$ are expressed as an appropriate product of several simpler functions, the assertions of Theorems 3.1 and 3.2 can be applied in order to derive various families of multilinear and multilateral generating functions for the polynomials $P^{\alpha}_{m,n}(z, z^*)$.

REFERENCES

[1] A. Altın and E. Erkuş, On a multivariable extension of the Lagrange-Hermite polynomials, *Integral Transforms Spec. Funct.*, 17 (2006), 239-244.

[2] C.F. Dunkl and Y. Xu, Orthogonal polynomials of several variables, Cambridge univ. press, New York, 2001.

[3] T.H. Koornwinder, Two variable analogues of the classical orthogonal polynomials, *Theory and application of special functions*, Acad. Press. Inc., New York, 1975.

[4] G. Lauricella, Sulle funzioni ipergeometriche a piú variabili, *Rend. Circ. Mat. Palermo*, 7 (1893), 111-158.

[5] P.B. Malave and B.R. Bhonsle, Some generating functions of two variable analogue of jacobi polynomials of class II, *Ganita,* 31 (1980), no. 1-2, 29-37.

[6] E. Özergin, M.A. Özarslan and H. M. Srivastava, Some families of generating functions for a class of bivariate polynomials, *Math. Comput. Modelling,* 50 (2009), no. 7-8, 1113 1120.

[7] E. D. Rainville, Special Functions, The Macmillan Company, New York, 1960.

[8] S. Saran, Hypergeometric functions of three variables, *Ganita*, 5 (1954), 77-91.

[9] H. M. Srivastava and H. L. Manocha, A Treatise on Generating Functions, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, 1984.

[10] H. M. Srivastava, F. Taşdelen and B. Şekeroğlu, Some families of generating functions for the q-Konhauser polynomials, *Taiwanese J. Math.*, 12 (2008), no. 3, 841 850.

[11] P. K. Suetin, Orthogonal polynomials in two variables, Gordon and Breach Science Publishers, Moscow, 1988.

[12] G. Szegö, Orthogonal polynomials, fourth ed., Amer. Math. Soc. Colloq. Publ.,vol. 23, 1975.

[13] A. Wünsche, Generalized Zernike or disc polynomials, *J. Comput. Appl. Math.*, 174 (2005), 135-163.

[14] F. Zernike, Beugungstheorie des schneidenverfahrens und seiner verbesserten form, der phasenkontrastmethode, *Physica*, 1 (1934), 689-704.

Rabia Aktaş (Corresponding Author), Fatma Taşdelen and Nuray Yavuz
Ankara University,
Faculty of Science,
Department of Mathematics,
Tandoğan TR-06100, Ankara,
Turkey.
E-Mail Address(R. Aktaş): raktas@science.ankara.edu.tr
E-Mail Address(F. Taşdelen): tasdelen@science.ankara.edu.tr
E-Mail Address(N. Yavuz): asis_nuray@hotmail.com