On $(64, 28, 12)$ Difference Sets

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Abstract.
There are 267 nonisomorphic groups of order 64. It was known that 259 of these groups admit $(64, 28, 12)$ difference sets and the other eight groups do not admit $(64, 28, 12)$ difference sets. Despite of this result, no research investigates the problem of finding all $(64, 28, 12)$ difference sets in a certain group of order 64. In this paper, we find all $(64, 28, 12)$ difference sets in 111 groups of order 64. 106 of these groups are nonabelian. The other five are $\mathbb{Z}_{16} \times \mathbb{Z}_4$, $\mathbb{Z}_{16} \times \mathbb{Z}_2^2$, $\mathbb{Z}_8 \times \mathbb{Z}_8$, $\mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2$, and $\mathbb{Z}_8 \times \mathbb{Z}_2^3$. In these 111 groups we get 74922 nonequivalent $(64, 28, 12)$ difference sets. These difference sets provide at least 105 nonisomorphic symmetric $(64, 28, 12)$ designs. Most of our work have been done by programs using the software GAP.

AMS Subject Classification: 05B10.

Key words. Groups of order 64, difference set, symmetric design, GAP.

1 Introduction

A (v, k, λ) difference set is a subset D of size k in a group G of order v with the property that for every nonidentity g in G, there are exactly λ ordered pairs $(x, y) \in D \times D$ such that

$$xy^{-1} = g.$$

One may identify the set D with an element $\hat{D}$ in the group ring $\mathbb{Z}(G)$. In this case write

$$\hat{D} = \sum_{g \in D} g$$

*This research is supported by the Deanship of Academic Research at the University of Jordan

and

$$\hat{D}^{(-1)} = \sum_{g \in D} g^{-1}.$$

We also write $\hat{G}$ for $\sum_{g \in G} g$. D is a difference set if the group ring element $\hat{D}$ satisfies the equation

$$\hat{D}\hat{D}^{(-1)} = (k - \lambda)1_G + \lambda\hat{G}. \tag{1}$$

If a group G has a difference set D then $\{gD : g \in G\}$ is the set of blocks of a symmetric (v, k, λ) design with point set G. On this design G acts by left multiplication as a sharply transitive automorphism group. Conversely, any symmetric design with a sharply transitive automorphism group acting on points may be constructed as the set of left translates of a difference set. A difference set is called cyclic (abelian, nonabelian) if the group G is cyclic (abelian, nonabelian). Difference sets were first introduced in cyclic groups in the study of projective plans, see [6] and [12]. Most of the progress in the study of difference sets has occurred in abelian groups; indeed the term "difference" comes from the abelian (additive) version of the formula in the definition.

For a basic introduction on difference sets and more details, the reader may consult [5], [14], [17].

Difference sets with parameters $(q^{d+1}(\frac{q^{d+1}-1}{q-1} + 1), q^d \frac{q^{d+1}-1}{q-1}, q^d \frac{q^d-1}{q-1})$, where $q = p^f$ is a prime power, are known as McFarland difference sets. For further discussion on McFarland difference sets, see [8] and [18].

Difference sets with parameters $(4N^2, 2N^2 - N, N^2 - N)$ are known as Menon-Hadamard difference sets. More details on these difference sets can be found in [7] and [13].

The intersection between McFarland and Menon-Hadamard difference sets happens when $q = 2$. In that case we get the parameters $(2^{2d+2}, 2^{2d+1} - 2^d, 2^{2d} - 2^d)$. When $d = 1$ we have the $(16, 6, 2)$ difference sets. There are 14 groups of order 16. Among them there are 12 groups that admit $(16, 6, 2)$ difference sets. Kibler found *all* the $(16, 6, 2)$ difference sets in these groups and there are three nonisomorphic symmetric $(16, 6, 2)$ designs that rise from these difference sets, see [16]. When $d = 2$ we have the $(64, 28, 12)$ difference sets. In this paper we will construct *all* difference sets in 111 groups of order 64. Similar studies have been done for the $(96, 20, 4)$ difference sets. In a series of papers the existence of $(96, 20, 4)$ difference sets have been resolved in most groups, see [1, 3, 9, 10].

A homomorphism f from G onto G' induces, by linearity, a homomorphism from $\mathbb{Z}[G]$ onto $\mathbb{Z}[G']$. If the kernel of f is the subgroup U, let T be a complete set of distinct representatives of cosets of U and, for $g \in T$, set

$t_g := |gU \cap D|$. The multiset $\{t_g : g \in T\}$ is the collection of "intersection numbers" of D with respect to U. The image of $\hat{D}$ under the function f is $f(\hat{D}) = \sum_{g \in T} t_g f(g)$.

This group ring element satisfies the equation

$$f(\hat{D})f(\hat{D})^{(-1)} = (k - \lambda)1_{G'} + \lambda|U|\hat{G'} \tag{2}$$

in the group ring $\mathbb{Z}[G']$.

Because the size of D equals k we have the equation

$$\sum_{g \in T} t_g = k. \tag{3}$$

Because the coefficient of the identity element is the same on both sides of the equation 2 we have the equation

$$\sum_{g' \in T} {t_g}^2 = (k - \lambda) + \lambda|U|. \tag{4}$$

The contraction of D to a smaller homomorphic image often provides useful conditions on the existence of a difference set in the original group. Usually, we use the software GAP (Groups, Algorithms and Programming), see [11], to construct the images of difference sets (find the intersection numbers) recursively. In the $(64, 28, 12)$ case, we use the intersection numbers of the group of order two to find them for groups of order four then we use intersection numbers of groups of order four to find them for groups of order eight. We proceed up till we find all the intersection numbers in groups of order 32.

We say $D_1 \in \mathbb{Z}(G)$, $D_2 \in \mathbb{Z}(G)$ are equivalent if there is an element $g \in G$ and an automorphism φ of G such that $D_1 = g\varphi(D_2)$. We say that two difference sets are inequivalent if either they are subsets of nonisomorphic groups or if they are subsets in a common group G but are not equivalent in G (as defined above.) Inequivalent difference sets *may* give rise to isomorphic designs.

The adjacency matrix of a symmetric design has a set of invariant factors (or "elementary divisors", associated with the Smith Normal Form of the matrix.) If A is the adjacency matrix of the design, there are invertible matrices P and Q such that PAQ is a diagonal matrix. P and Q may be chosen so that the (i, i) entry of PAQ divides the $(i+1, i+1)$ entry of PAQ (for all i, $1 \leq i \leq v - 1$). These numbers in the main diagonal of PAQ are the invariant factors of A. For all the $(64, 28, 12)$ difference sets that we find in this paper, we build the incidence matrices of the symmetric $(64, 28, 12)$ designs that rise from these difference sets. Say one of these incidence matrices is A, then the command ElementaryDivisorsMat(A) (in

GAP) provides the invariant factors of the incidence matrix A. For all the incidence matrices of the symmetric $(64, 28, 12)$ designs that we find in this paper, the invariant factors consist of 1s, 2s, 4s, 8s, 16s and 112s. Hence, according to what we find for the parameters $(64, 28, 12)$, we may assume that PAQ is the direct sum of six scalar matrices:

$$PAQ = I_{r_1} \oplus 2I_{r_2} \oplus 4I_{r_3} \oplus 8I_{r_4} \oplus 16I_{r_5} \oplus 112I_{r_6}.$$

The sum of the matrix sizes, $r_1 + r_2 + r_3 + r_4 + r_5 + r_6$, is 64. The rank of A over $GF(2)$ is r_1, the size of the identity matrix in this direct sum. We will abbreviate the list of invariant factors with the 6-tuple $(r_1, r_2, r_3, r_4, r_5, r_6)$. For example, there are at least two distinct difference sets in the abelian group GAP[64, 2] $\cong \mathbb{Z}_8 \times \mathbb{Z}_8$; one has invariant factors abbreviated by $(16, 8, 16, 8, 15, 1)$ while the other has invariant factors abbreviated by $(15, 14, 6, 14, 14, 1)$. The 2-rank for the first difference set is 16 and for the second one is 15. Note that the $(16, 8, 16, 8, 15, 1)$ invariant factors mean that the 1 occurs sixteen times in the main diagonal of PAQ, the 2 occurs eight times in the main diagonal of PAQ, the 4 occurs sixteen times in the main diagonal of PAQ, the 8 occurs eight times in the main diagonal of PAQ, the 16 occurs fifteen times in the main diagonal of PAQ, and the 112 occurs once in the main diagonal of PAQ. Although nonisomorphic designs may have the same invariant factors, if the designs have different invariant factors, they must be nonisomorphic. So the two designs that have the distinct invariant factors $(16, 8, 16, 8, 15, 1)$ and $(15, 14, 6, 14, 14, 1)$ are nonisomorphic. We found 105 different patterns of invariant factors and so have at least 105 distinct, nonisomorphic symmetric $(64, 28, 12)$ designs. (See [17], Appendix C, for a discussion of invariant factors of incidence matrices and the underlying design.)

Among the 105 different invariant factor patterns, there are nineteen distinct 2-ranks. These nineteen different 2-ranks are the consecutive integers from 8 to 26.

There are 267 nonisomorphic groups of order 64. These groups are listed in GAP SmallGroups library as $[64, 1], \ldots, [64, 267]$. In this paper we will denote these groups by GAP[64,i], $1 \leq i \leq 267$. So when we indicate GAP[64,10] we mean group number 10 of order 64 in the GAP library.

In the next section, we present previous results on $(64, 20, 4)$ difference sets.

2 Summary of previous results on $(64, 28, 12)$ difference sets.

Turyn showed that an abelian group of order 2^{2d+2} and exponent greater than 2^{d+2} does not admit a difference set, see [20]. Turyn's exponent bound rules out the existence of $(64, 28, 12)$ difference sets in GAP[64,1] $\cong \mathbb{Z}_{64}$ and GAP[64,50] $\cong \mathbb{Z}_{32} \times \mathbb{Z}_2$.

Dillon showed that the existence of difference sets in dihedral groups gives difference sets in cyclic groups, see [8]. This is sometimes called "Dillon's dihedral trick". Dillon's dihedral trick rules out the existence of $(64, 28, 12)$ difference sets in every group of order 64 that has $\mathbb{D}_{32}$ as a factor group, where $\mathbb{D}_{32}$ is the dihedral group of order 32. Groups that have $\mathbb{D}_{32}$ as a factor group are GAP[64,i], where $i \in \{38, 47, 52, 53, 54, 186\}$. Indeed groups that ruled out using Turyn's exponent bound and Dillon's dihedral trick are the only groups that do not admit $(64, 28, 12)$ difference sets. Next we will present the existence results of $(64, 28, 12)$ difference sets in all other groups.

McFarland constructed $(q^{d+1}(\frac{q^{d+1}-1}{q-1}+1), q^d\frac{q^{d+1}-1}{q-1}, q^d\frac{q^d-1}{q-1})$ difference sets in abelian groups which have an elementary abelian normal subgroup of order q^{d+1}, where $q = p^f$, p is a prime and d is a positive integer, see [18]. McFarland's construction gives $(64, 28, 12)$ difference sets in GAP[64,55] $\cong \mathbb{Z}_4^3$, GAP[64,83]$\cong \mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2$, and GAP[64,183]$\cong \mathbb{Z}_{16} \times \mathbb{Z}_2^2$, GAP[64,192]$\cong \mathbb{Z}_4^2 \times \mathbb{Z}_2^2$, GAP[64,246]$\cong \mathbb{Z}_8 \times \mathbb{Z}_2^3$, GAP[64,260]$\cong \mathbb{Z}_4 \times \mathbb{Z}_2^4$, and GAP[64,267]$\cong \mathbb{Z}_2^6$.

Dillon generalized McFarland's construction to work for a larger set of groups. He constructed McFarland difference sets in groups that have an elementary abelian normal subgroup of order q^{d+1} in its center, see [8]. Dillon's construction gives $(64, 28, 12)$ difference sets in GAP[64,i], where $i \in \{$ 17, 21, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 84, 87, 95, 96, 103, 106, 107, 193, 194, 195, 196, 197, 202, 203, 204, 205, 207, 208, 209, 211, 212, 247, 250, 251, 252, 261, 262, 263 $\}$. Note that all of these groups are nonabelian and have $\mathbb{Z}_2^3$ in their center.

Arasu constructed $(64, 28, 12)$ difference sets in the last two abelian groups GAP[64,26] $\cong \mathbb{Z}_{16} \times \mathbb{Z}_4$ and GAP[64,2] $\cong \mathbb{Z}_8 \times \mathbb{Z}_8$, see [4].

In an unpublished work by Dillon, he was able to construct $(64, 28, 12)$ difference sets in 258 groups. This left the existence of $(64, 28, 12)$ difference sets undecided in the single group GAP[64,51] $\cong \langle x, y : x^{32} = y^2 = 1,\ yxy = x^{17}\rangle$. This group is called the Modular group. The exponent of the Modular group is 32.

Liebler and Smith constructed $(64, 28, 12)$ difference set in the Modular group, see [15]. This was the first example which demonstrated that Turyn's exponent bound for abelian groups can be violated in the nonabelian case.

All of this work can be summarize as follows. Among the 267 groups of order 64, there are 259 groups that admit $(64, 28, 12)$ difference sets and there are 8 groups that do not admit $(64, 28, 12)$ difference sets. As we mention before the groups that do not admit $(64, 28, 12)$ difference sets are the ones that have $\mathbb{Z}_{32}$ or $\mathbb{D}_{32}$ as a factor group.

Despite all of this work, no research have been done to find all $(64, 28, 12)$

difference sets in a certain group of order 64. In this paper, we use the software GAP to construct all (64, 28, 12) difference sets in 111 groups.

3 All (64, 28, 12) Difference sets in 111 groups

In our search for difference sets, we find the intersection numbers (images) of putative (64, 28, 12) difference sets in various factor groups. We use GAP to find the 2, 4, 8, 16, and 32-images of putative (64, 28, 12) difference sets in all groups of order 2, 4, 8, 16, and 32. Our search is recursive. We use the 2-images to find the 4-images then the 4-images to find the 8-images up till we find the 32-images. These images tell us how the elements of putative (64, 28, 12) difference sets distribute in the cosets of these factor groups. We present the following example to explain that. The example shows how we utilize the 4-images to build the 8-images.

Example 1
Suppose G is a group of order 64 that has normal subgroups U and K of order 16 and 8 respectively with $G/U \cong \mathbb{Z}_4 \cong \langle x : x^4 = 1\rangle$ and $G/K \cong \mathbb{Z}_4 \times \mathbb{Z}_2 \cong \langle x, y : x^4 = y^2 = 1, xy = yx\rangle$. If δ is a 4-image of a putative (64, 28, 12) difference set in G/U then δ is equivalent to $8+8x+8x^2+4x^3$ or $10 + 6x + 6x^2 + 6x^3$. The numbers $[8, 8, 8, 4]$ and $[10, 6, 6, 6]$ are the intersection numbers. For instance, the $[8, 8, 8, 4]$ means that a putative (64, 28, 12) difference set intersects the cosets U, xU, x^2U, x^3U in 8, 8, 8, 4 elements respectively. We want to explain how our programs in GAP utilize these 4-images to find the 8-images in G/K. Suppose that $\delta_1 = a_0+a_1x+a_2x^2+a_3x^3+a_4y+a_5xy+a_6x^2y+a_7x^3y$ is an 8-image of a putative (64, 28, 12) difference set in G/K. The natural homomorphism (say φ) from G/K into G/U fixes x and maps y to the identity. The image of δ_1 under φ will be $\varphi(\delta_1) = (a_0 + a_4) + (a_1 + a_5)x + (a_2 + a_6)x^2 + (a_3 + a_7)x^3$. Observe that $\varphi(\delta_1)$ is a 4-image in G/U and hence $\varphi(\delta_1)$ is equal to $8+8x+8x^2+4x^3$ or $10 + 6x + 6x^2 + 6x^3$. Suppose that $\varphi(\delta_1)$ is equal to $8 + 8x + 8x^2 + 4x^3$ then we get the four equations $a_0 + a_4 = 8$, $a_1 + a_5 = 8$, $a_2 + a_6 = 8$, $a_3 + a_7 = 4$. The a_i's are intersection numbers and hence the a_i's are positive integers. Thus the possible values for $[a_0, a_4]$, $[a_1, a_5]$, and $[a_2, a_6]$ are $[8, 0]$, $[7, 1]$, $[6, 2]$, $[5, 3]$, $[4, 4]$, $[3, 5]$, $[2, 6]$, $[1, 7]$, and $[0, 8]$. The possible values for $[a_3, a_7]$ are $[4, 0]$, $[3, 1]$, $[2, 2]$, $[1, 3]$, or $[0, 4]$. Indeed, one can translate by y, if necessary, to reduce the possible values for $[a_3, a_7]$ to $[4, 0]$, $[3, 1]$, $[2, 2]$. Our programs in GAP will run over all of these possibilities and check which one of them give an 8-image. For instance, the previous possibilities give $4 + 4x + 4x^2 + 4x^3 + 4y + 4xy + 4x^2y + 0x^3y$ and $8 + 8x + 8x^2 + 4x^3 + 0y + 0xy + 0x^2y + 0x^3y$ as candidates for an 8-image in G/K. The first one is an 8-image whereas the second one is not an 8-image. Our programs will find all candidates then will test which one of them is an

actual 8-image. The programs will spit the nonequivalent 8-images. Similar work will be done if $\varphi(\delta_1)$ is equal to $10+6x+6x^2+6x^3$.

When we run our programs to find 2, 4, 8, 16, and 32-images of putative $(64, 28, 12)$ difference sets, as we explained in Example 1, we run into a problem. In the following example, we explain this problem and explain how we resolve it.

Example 2
Consider the group $H = \langle x, z, w, t : x^4 = y^2 = z^2 = t^2 = [t, w] = [t, z] = 1, zxz = x^{-1}, wxw = x^{-1}, txt = x^{-1}, wzw = zx^2\rangle$. The order of H is 32 and $\langle x^2\rangle$ is the unique normal subgroup of order 2 in H. The quotient group $H/\langle x^2\rangle$ is isomorphic to $\mathbb{Z}_2^4$. The order of the full automorphism group of $\mathbb{Z}_2^4$ (denote the full automorphism group of $\mathbb{Z}_2^4$ by $\mathrm{Aut}(\mathbb{Z}_2^4)$) is 20160. Whereas the order of the full automorphism group of H (denote the full automorphism group of H by $\mathrm{Aut}(H)$) is 1152.
The group $\mathrm{Aut}(H)$ induces an automorphism group on $\mathbb{Z}_2^4$ (denote it by $Aut_{\mathbb{Z}_2^4}(H)$) of order less than or equal 1152. In finding the nonequivalent 16-images in $\mathbb{Z}_2^4$, we use the full automorphism group of $\mathbb{Z}_2^4$ which is $\mathrm{Aut}(\mathbb{Z}_2^4)$. Then in finding the 32-images in H, we use the 16-images in $\mathbb{Z}_2^4$ that are nonequivalent under the full automorphism group $\mathrm{Aut}(\mathbb{Z}_2^4)$. The problem is that some of the nonequivalent 16-images under the automorphism group $Aut_{\mathbb{Z}_2^4}(H)$ might be equivalent under the full automorphism group $\mathrm{Aut}(\mathbb{Z}_2^4)$. This result in losing some of the 32-images in H. To overcome this problem, we have to use the automorphism group $Aut_{\mathbb{Z}_2^4}(H)$ in finding the nonequivalent 16-images in $\mathbb{Z}_2^4$. This way we guarantee that we are not going to loss any of the 32-images in H.
Hence in finding 2, 4, 8, 16, 32, and 64-images (which is the difference set), we have to use the right automorphism group to get a complete list of the nonequivalent images.

As we explain in Example 1 and Example 2, we run our programs to build all the nonequivalent 2, 4, 8, 16 and 32-images of putative $(64, 28, 12)$ difference sets in all groups of order 2, 4, 8, 16 and 32. More details on these programs can be found in [2] and [19]. The last step is to go from 32-images to 64-images which is the $(64, 28, 12)$ difference set. Using the 32-images in only one factor group to build all the $(64, 28, 12)$ difference sets is computationally impossible in a reasonable amount of time. The following example demonstrates that.

Example 3
Take G to be the nonabelian group GAP[64,184] $\cong \langle x, y, z : x^{16} = y^2 =$

$z^2 = 1,\ yxy = x^9, [x,z] = 1, [y,z] = 1\rangle$. Observe that the subgroup $H = \langle z : z^2 = 1\rangle$ is a normal subgroup of G and $G/H \cong \langle x, y : x^{16} = y^2,\ yxy = x^9\rangle$. The group G/H is isomorphic to GAP[32,17]. As we explain in Example 1, we find all the 32-images of a putative $(64, 28, 12)$ difference set in G/H. We get 71 nonequivalent 32-images in G/H. Let us look at one of these images. We have $\delta = 2+x+2x^2+x^3+0x^4+x^5+0x^6+x^7+0x^8+x^9+2x^{10}+x^{11}+2x^{12}+x^{13}+x^{13}+0x^{14}+x^{15}+2y+xy+0x^2y+x^3y+2x^4y+x^5y+0x^6y+x^7y+0x^8y+x^9y+0x^{10}y+x^{11}y+0x^{12}y+x^{13}y+0x^{14}y+x^{15}y$ is one of the 32-images in G/H. The cosets of G/H are of size two. Hence the twos in δ mean that putative difference sets contain the coset. For instance, $2x^2$ in δ means that putative difference sets contain the coset $x^2H = \{x^2, x^2z\}$. The zeros in δ mean that putative difference sets contain no elements from the coset. For instance, $0x^4$ means that putative difference sets contain no elements from the coset $x^4H = \{x^4, x^4z\}$. The ones in δ mean that putative difference sets contain one element from the coset. For instance, x in δ means that putative difference sets contain exactly one element from the coset $xH = \{x, xz\}$. Indeed, in all the 32-images in all groups of order 32 we get 6 twos, 16 ones, and 10 zeros. Hence putative difference sets contain 12 elements (six cosets) that correspond to the 6 twos and it will contain no elements from the cosets that correspond to the zeros. For the 16 ones, every one will give two possibilities. Hence the 71 32-images give $71 \cdot 2^{16} = 4653056$ candidates for $(64, 28, 12)$ difference sets and the programs have to check every candidates for a difference set. Hence the search space is huge and it take long time to check which one of these candidates is a $(64, 28, 12)$ difference set. Because of that we said using 32-images in only one factor group to build all the $(64, 28, 12)$ difference sets is computationally impossible in a reasonable amount of time. We note that the number of the nonequivalent 32-images in GAP[32,17] (which is 71) is relatively small. In some of the other groups, the number of the nonequivalent 32-images is more than 500.

In some cases, if the group G has two factor groups of order 32 then one can reduce the search space and construct all $(64, 28, 12)$ difference sets in G. The following example explain that.

Example 4

As in Example 3, take G to be the nonabelian group GAP[64,184] $\cong \langle x, y, z : x^{16} = y^2 = z^2 = 1,\ yxy = x^9, [x,z] = 1, [y,z] = 1\rangle$. Observe that the subgroups $H = \langle z : z^2 = 1\rangle$ and $K = \langle x^8z : (x^8z)^2 = 1\rangle$ are two normal subgroups of G with GAP[32,17] $\cong G/H \cong \langle x, y : x^{16} = y^2,\ yxy = x^9\rangle$ and GAP[32,17] $\cong G/K \cong \langle xz, y : (xz)^{16} = y^2,\ y(xz)y = (xz)^9 = x^9z\rangle$. We have 71 nonequivalent 32-images in G/H and 71 nonequivalent 32-images in G/K. Our programs will take each one of the 32-images in G/H and

look for the 32-images in G/K that are compatible with this image. To do that, our programs have to find all the equivalent and nonequivalent 32-images in G/K. This require translating the 71 32-images by the elements of G/K and applying the automorphisms of G/K on these images. This will blow the list of the 71 nonequivalent 32-images in G/K into a list of 25784 equivalent and nonequivalent 32-images in G/K. Indeed, this step take long time if the number of the 32-images is large or the size of the automorphism group is big. Our programs blow the list of the 32-images in a reasonable amount of time for the groups GAP[32,i] where $i \in T = \{3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 14, 15, 16, 17, 19, 20, 27, 36, 39, 40, 42, 43\}$. If H is a group of order 64 that has two factor groups of order 32 where one of these factor groups is isomorphic to GAP[32,i] where $i \in T$ then we find all the $(64, 28, 12)$ difference sets in H. We have 111 groups that have this property. We find all $(64, 28, 12)$ difference sets in these groups. Next, we explain how we use the two factor groups G/H and G/K to find all the $(64, 28, 12)$ difference sets in G.
Take $\delta = 2+x+2x^2+x^3+0x^4+x^5+0x^6+x^7+0x^8+x^9+2x^{10}+x^{11}+2x^{12}+x^{13}+x^{13}+0x^{14}+x^{15}+2y+xy+0x^2y+x^3y+2x^4y+x^5y+0x^6y+x^7y+0x^8y+x^9y+0x^{10}y+x^{11}y+0x^{12}y+x^{13}y+0x^{14}y+x^{15}y$ to be one of the 71 nonequivalent 32-images in G/H and take $\delta_1 = \sum_{i=0}^{15} a_i(xz)^i + \sum_{i=0}^{15} b_i(xz)^i y$ to be any 32-image in G/K. In δ the first 2 means that a hypothetical difference set should contain the coset $\{1, z\}$ and $0x^8$ means that a hypothetical difference set should contain no elements from the coset $\{x^8, x^8z\}$. In δ_1, a_0 means that a hypothetical difference set will intersect the coset $\{1, x^8z\}$ in a_0 elements and a_8x^8 means that a hypothetical difference set will intersect the coset $\{x^8, z\}$ in a_8 elements. But the hypothetical difference set contain $\{1, z\}$ and contain no elements from $\{x^8, x^8z\}$ and hence $a_0 = a_8 = 1$. Hence any 32-image in G/K will have $a_0 = a_8 = 1$. Hence our programs will throw away all the 32-images in G/K that have $a_0 \neq 1$ or $a_8 \neq 1$. Our programs continue in this manner and throw away all the 32-images in G/K that are not compatible with δ. We mention before in this example that we have 25784 32-images in G/K. If we throw away all the 32-images in G/K that are not compatible with δ then this will leave only 48 32-images in G/K. These 32-images in G/K will impose restrictions on the 32-images in G/H. This will reduce the search space and it will make it possible in a reasonable amount of time.

As we explain in the last four examples, our programs find all $(64, 28, 12)$ difference sets in 111 groups of order 64. These 111 groups are the ones that have two factor groups of order 32 and one of these factor groups is isomorphic to GAP[32, i] where $i \in \{3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 14,$

15, 16, 17, 19, 20, 27, 36, 39, 40, 42, 43}. This gives 74922 nonequivalent difference sets in these groups. We obtain the incidence matrices of the symmetric $(64, 28, 12)$ designs that rise from these difference sets. By computing the invariant factors of these matrices we get 105 distinct sets of invariant factors. Thus we get at least 105 nonisomorphic symmetric $(64, 28, 12)$ designs. In the other 148 groups either we have a unique group of order 32 or the number of 32-images is large. As we explain in Example 3 and Example 4 that makes the computations not feasible at that time.

Table 1 lists some of these nonequivalent $(64, 28, 12)$ difference sets. In the first column of Table 1, we list the group number according to GAP SmallGroups library then we list the number of nonequivalent $(64, 28, 12)$ difference sets in this group, we denote this number by n_1. Then we list the number of distinct invariant factors of the symmetric $(64, 28, 12)$ designs that rise from these nonequivalent difference sets. We denote this number by n_2. Hence n_2 represents the least number of the nonisomorphic symmetric $(64, 28, 12)$ designs that rise from these difference sets.

For instance, In the first column of Table 1, we have the numbers 3, 43, then 2 and this means that we are talking about group GAP[64, 3] and there are 43 nonequivalent $(64, 28, 12)$ difference sets in this group and the number of distinct invariant factors of the symmetric $(64, 28, 12)$ designs that rise from these nonequivalent difference sets is 2. The number 2 also tells us that there are at least 2 nonisomorphic symmetric $(64, 28, 12)$ designs rise from these difference sets.

If G is one of these 111 groups, then the command *Elements*(G) in GAP gives a list of the elements of G. In the second column of Table 1, numbers were written to represent the elements. For instance, if, in the group GAP[64,i], the number j occurs, then this number j represents the j-th element in the list of the elements of G as they appear in GAP using the command *Elements*(*SmallGroup*$(64, i)$). In the last column, we list the invariant factors of the incidence matrix of the symmetric $(64, 28, 12)$ design that rise from the difference set that listed in the second column.

In Table 1, for each one of the 111 groups we list at least one of the difference sets. For some groups, we list more than one difference set because we want to cover all the 105 distinct invariant factors.

Table 1. List of some (64, 28, 12) Difference Sets

G, n_1, n_2	Elements of the Difference Set	Invariant Factors
2, 21, 2	1, 2, 3, 4, 5, 9, 10, 12, 14, 15, 17, 22, 23, 29, 31, 33, 35, 36, 41, 42, 44, 45, 49, 53, 55, 57, 60, 62	(16, 8, 16, 8, 15, 1)
3, 43, 2	1, 2, 3, 4, 5, 6, 8, 9, 10, 13, 16, 18, 20, 21, 25, 27, 32, 33, 35, 36, 42, 45, 51, 52, 54, 56, 60, 62	(15, 14, 6, 14, 14, 1)
4, 442, 19	1, 2, 3, 4, 5, 7, 8, 11, 19, 21, 24, 25, 26, 29, 34, 37, 38, 39, 43, 46, 48, 51, 53, 54, 57, 58	(12, 10, 20, 10, 11, 1)
5, 512, 21	1, 2, 3, 4, 5, 7, 9, 11, 19, 21, 23, 24, 28, 34, 35, 36, 37, 38, 39, 44, 45, 46, 53, 54, 57, 59, 60, 64	(10, 12, 20, 12, 9, 1)
6, 288, 15	1, 2, 3, 4, 5, 7, 8, 11, 12, 13, 16, 17, 21, 24, 25, 32, 36, 41, 43, 45, 46, 53, 54, 55, 56, 57, 58, 60	(13, 12, 14, 12, 12, 1)
7, 760, 17	1, 2, 3, 4, 5, 7, 8, 11, 12, 13, 14, 16, 19, 25, 32, 35, 39, 40, 42, 45, 47, 53, 54, 56, 59, 60, 61, 64	(14, 8, 20, 8, 13, 1)
8, 552, 18	1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 14, 22, 24, 26, 28, 30, 38, 39, 40, 41, 43, 44, 47, 50, 56, 59, 60, 62	(12, 8, 24, 8, 11, 1)
9, 1736, 41	1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 14, 22, 24, 26, 28, 30, 38, 39, 40, 41, 43, 44, 47, 50, 56, 59, 60, 62	(9, 14, 18, 14, 8, 1)
9, 1736, 41	1, 2, 3, 4, 5, 6, 7, 8, 9, 13, 16, 17, 24, 27, 29, 34, 36, 38, 39, 40, 44, 49, 51, 52, 54, 55, 58, 62	(11, 17, 8, 17, 10, 1)
9, 1736, 41	1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 13, 14, 16, 22, 25, 27, 29, 30, 33, 41, 42, 49, 52, 54, 55, 56, 59, 64	(17, 6, 18, 6, 16, 1)
9, 1736, 41	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 20, 25, 26, 27, 29, 31, 34, 37, 39, 42, 44, 45, 46, 51, 52, 59, 60, 63	(13, 13, 12, 13, 12, 1)
10, 300, 11	1, 2, 3, 4, 5, 6, 8, 10, 12, 13, 14, 17, 18, 25, 26, 27, 28, 31, 33, 36, 38, 42, 48, 50, 51, 53, 55, 57	(13, 10, 18, 10, 12, 1)
11, 522, 12	1, 2, 3, 4, 5, 6, 9, 10, 13, 16, 19, 21, 23, 27, 30, 32, 33, 36, 40, 44, 45, 49, 51, 55, 56, 57, 60, 63	(24, 4, 8, 4, 23, 1)
12, 57, 2	1, 2, 3, 4, 5, 6, 8, 10, 13, 15, 16, 17, 18, 20, 23, 26, 32, 38, 39, 43, 45, 46, 52, 53, 54, 55, 59, 61	(15, 14, 6, 14, 14, 1)
13, 480, 19	1, 2, 3, 4, 5, 7, 8, 9, 11, 13, 16, 17, 21, 25, 27, 28, 29, 36, 41, 49, 51, 53, 54, 55, 56, 57, 59, 64	(16, 6, 20, 6, 15, 1)
14, 309, 16	1, 2, 3, 4, 5, 6, 8, 9, 11, 12, 23, 24, 26, 28, 31, 32, 38, 40, 41, 42, 45, 47, 52, 53, 54, 57, 58, 64	(18, 8, 12, 8, 17, 1)
15, 64, 2	1, 2, 3, 4, 5, 7, 8, 9, 13, 15, 19, 21, 25, 30, 31, 32, 34, 39, 43, 45, 49, 50, 51, 57, 58, 59, 60, 64	(15, 14, 6, 14, 14, 1)
16, 56, 2	1, 2, 3, 4, 5, 7, 8, 9, 11, 14, 15, 19, 21, 23, 28, 35, 36, 37, 39, 46, 47, 48, 55, 56, 57, 58, 59, 60	(15, 14, 6, 14, 14, 1)

G, n_1, n_2	Elements of the Difference Set	Invariant Factors
17, 952, 19	1, 2, 3, 4, 5, 8, 10, 11, 12, 13, 16, 17, 20, 22, 24, 25, 35, 39, 41, 45, 47, 49, 50, 52, 58, 60, 64	(11, 10, 22, 10, 10, 1)
20, 1665, 37	1, 2, 3, 4, 5, 8, 10, 11, 12, 13, 16, 18, 20, 22, 25, 31, 32, 35, 39, 40, 43, 44, 45, 46, 47, 49, 58, 62	(16, 12, 8, 12, 15, 1)
20, 1665, 37	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 18, 19, 26, 30, 32, 37, 40, 43, 48, 49, 52, 55, 58, 59, 60, 61, 63	(16, 4, 24, 4, 15, 1)
20, 1665, 37	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 15, 20, 24, 29, 32, 35, 37, 38, 42, 50, 51, 53, 54, 57, 58, 60, 63	(16, 2, 28, 2, 15, 1)
21, 836, 13	1, 2, 3, 4, 5, 6, 8, 13, 16, 19, 22, 23, 24, 31, 36, 38, 41, 43, 44, 45, 49, 51, 53, 54, 57, 58, 59, 63	(12, 12, 16, 12, 11, 1)
22, 540, 20	1, 2, 3, 4, 5, 6, 9, 10, 11, 14, 15, 16, 18, 24, 27, 31, 37, 38, 39, 40, 42, 46, 50, 51, 53, 58, 59, 63	(20, 8, 8, 8, 19, 1)
23, 785, 32	1, 2, 3, 4, 5, 6, 8, 11, 12, 18, 20, 23, 26, 32, 34, 35, 36, 37, 38, 40, 43, 45, 46, 53, 54, 57, 59, 61	(8, 14, 20, 14, 7, 1)
23, 785, 32	1, 2, 3, 4, 5, 6, 7, 8, 9, 13, 16, 17, 24, 25, 26, 34, 36, 38, 39, 40, 46, 47, 52, 54, 55, 60, 62, 64	(18, 2, 24, 2, 17, 1)
23, 785, 32	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 21, 23, 24, 25, 29, 30, 37, 40, 42, 44, 45, 48, 51, 52, 53, 55, 58, 62	(10, 15, 14, 15, 9, 1)
23, 785, 32	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 14, 15, 21, 28, 29, 32, 33, 38, 40, 42, 47, 49, 54, 55, 58, 60, 63	(11, 11, 20, 11, 10, 1)
24, 988, 27	1, 2, 3, 4, 5, 6, 9, 12, 13, 14, 16, 17, 23, 27, 30, 34, 35, 37, 39, 41, 42, 44, 51, 52, 55, 59, 62, 64	(20, 6, 12, 6, 19, 1)
26, 11, 2	1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 17, 21, 26, 32, 35, 37, 41, 42, 44, 46, 50, 54, 55, 57, 61, 63, 64	(18, 10, 8, 10, 17, 1)
27, 24, 3	1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 14, 19, 20, 23, 25, 26, 31, 38, 39, 48, 49, 50, 51, 56, 57, 58, 59, 60	(17, 10, 10, 10, 16, 1)
29, 226, 12	1, 2, 3, 4, 5, 8, 11, 12, 13, 15, 17, 21, 22, 24, 25, 26, 27, 30, 34, 40, 41, 47, 50, 51, 58, 60, 61, 62	(14, 12, 12, 12, 13, 1)
29, 226, 12	1, 2, 3, 4, 5, 6, 8, 10, 13, 15, 20, 21, 22, 25, 27, 28, 29, 32, 33, 38, 39, 42, 45, 47, 54, 57, 59, 60	(15, 8, 18, 8, 14, 1)
39, 440, 16	1, 2, 3, 4, 5, 7, 8, 9, 11, 14, 15, 17, 18, 21, 29, 31, 36, 39, 43, 46, 47, 48, 49, 52, 58, 59, 62, 64	(14, 11, 14, 11, 13, 1)
39, 440, 16	1, 2, 3, 4, 5, 6, 8, 10, 12, 13, 14, 15, 17, 23, 26, 31, 34, 41, 42, 44, 45, 46, 54, 56, 57, 60, 61, 63	(18, 9, 10, 9, 17, 1)
39, 440, 16	1, 2, 3, 4, 5, 6, 8, 9, 11, 14, 20, 23, 24, 26, 29, 31, 38, 39, 40, 41, 44, 45, 47, 49, 52, 56, 60, 62	(22, 2, 16, 2, 21, 1)
44, 30, 4	1, 2, 3, 4, 5, 6, 8, 10, 11, 13, 14, 17, 18, 21, 25, 29, 31, 36, 37, 38, 40, 45, 49, 51, 55, 56, 60, 62	(17, 10, 10, 10, 16, 1)
48, 56, 2	1, 2, 3, 4, 5, 6, 8, 9, 11, 12, 14, 16, 20, 23, 26, 28, 32, 37, 39, 40, 41, 43, 46, 50, 60, 61, 62, 64	(22, 4, 12, 4, 21, 1)

G, n_1, n_2	Elements of the Difference Set	Invariant Factors
60, 375, 15	1, 2, 3, 4, 5, 6, 8, 9, 13, 15, 19, 20, 25, 26, 27, 29, 30, 31, 32, 33, 37, 39, 48, 50, 55, 56, 57, 63	(14, 6, 24, 6, 13, 1)
66, 720, 25	1, 2, 3, 4, 5, 8, 9, 11, 14, 15, 17, 19, 22, 28, 29, 31, 36, 40, 42, 43, 44, 46, 47, 48, 51, 52, 59, 60	(8, 15, 18, 15, 7, 1)
67, 954, 28	1, 2, 3, 4, 5, 7, 8, 9, 13, 14, 18, 22, 24, 25, 29, 30, 31, 32, 34, 35, 37, 41, 46, 49, 54, 56, 57, 62	(12, 14, 12, 14, 11, 1)
73, 746, 13	1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 19, 21, 22, 24, 28, 32, 36, 40, 42, 43, 44, 45, 53, 57, 63, 64	(20, 4, 16, 4, 19, 1)
74, 1247, 29	1, 2, 3, 4, 5, 9, 10, 12, 13, 14, 15, 18, 19, 22, 23, 28, 29, 31, 34, 35, 36, 42, 44, 45, 57, 62, 63, 64	(12, 9, 22, 9, 11, 1)
75, 1618, 22	1, 2, 3, 4, 5, 6, 8, 9, 11, 12, 17, 18, 21, 22, 27, 32, 35, 36, 37, 38, 40, 42, 43, 44, 54, 58, 60, 61	(18, 6, 16, 6, 17, 1)
76, 1203, 29	1, 2, 3, 4, 5, 6, 7, 9, 11, 13, 15, 17, 19, 20, 22, 26, 44, 45, 48, 49, 51, 53, 55, 57, 58, 62, 63, 64	(11, 15, 12, 15, 10, 1)
83, 452, 25	1, 2, 3, 4, 5, 6, 8, 9, 10, 14, 18, 21, 25, 30, 31, 32, 33, 34, 35, 37, 38, 39, 45, 51, 52, 54, 57, 60	(13, 14, 10, 14, 12, 1)
84, 565, 20	1, 2, 3, 4, 5, 6, 8, 9, 10, 14, 16, 18, 25, 26, 30, 32, 33, 34, 37, 39, 40, 42, 48, 50, 51, 54, 55, 62	(12, 16, 8, 16, 11, 1)
84, 565, 20	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 15, 16, 18, 19, 21, 25, 33, 37, 42, 48, 49, 50, 55, 57, 59, 60, 61, 62	(25, 4, 6, 4, 24, 1)
87, 806, 33	1, 2, 3, 4, 5, 6, 8, 9, 13, 14, 19, 20, 24, 25, 26, 31, 34, 35, 38, 39, 47, 50, 55, 56, 57, 58, 60, 61	(14, 10, 16, 10, 13, 1)
90, 859, 35	1, 2, 3, 4, 5, 6, 8, 10, 15, 16, 17, 18, 23, 25, 26, 29, 33, 36, 38, 42, 45, 50, 54, 56, 57, 58, 60, 61	(11, 14, 14, 14, 10, 1)
92, 412, 31	1, 2, 3, 4, 5, 6, 8, 10, 11, 13, 16, 17, 18, 26, 29, 35, 42, 43, 44, 49, 50, 51, 52, 57, 58, 59, 60, 64	(10, 13, 18, 13, 9, 1)
93, 1358, 36	1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 18, 20, 25, 26, 29, 31, 35, 37, 40, 48, 49, 54, 55, 56, 57, 62, 63	(10, 14, 16, 14, 9, 1)
93, 1358, 36	1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 13, 16, 17, 18, 21, 24, 32, 35, 36, 38, 39, 42, 43, 44, 45, 49, 62, 64	(21, 8, 6, 8, 20, 1)
95, 414, 15	1, 2, 3, 4, 5, 6, 8, 11, 13, 14, 16, 18, 20, 21, 26, 29, 35, 42, 44, 46, 47, 51, 53, 54, 56, 58, 59, 60	(9, 15, 16, 15, 8, 1)
96, 1565, 33	1, 2, 3, 4, 5, 6, 8, 11, 13, 14, 18, 21, 24, 29, 36, 37, 38, 39, 43, 44, 45, 47, 48, 51, 55, 56, 57, 59	(10, 16, 12, 16, 9, 1)
97, 1170, 41	1, 2, 3, 4, 5, 6, 9, 10, 13, 16, 17, 19, 20, 28, 29, 32, 33, 34, 36, 40, 43, 44, 45, 51, 52, 53, 56, 64	(12, 13, 14, 13, 11, 1)
98, 1138, 38	1, 2, 3, 4, 5, 7, 8, 10, 11, 13, 14, 18, 19, 21, 23, 26, 30, 38, 41, 44, 46, 53, 54, 55, 56, 59, 61, 64	(18, 4, 20, 4, 17, 1)

G, n_1, n_2	Elements of the Difference Set	Invariant Factors
99, 396, 31	1, 2, 3, 4, 5, 6, 8, 9, 12, 13, 14, 18, 21, 23, 27, 29, 31, 36, 37, 38, 39, 48, 49, 52, 55, 56, 57, 60	(15, 10, 14, 10, 14, 1)
101, 1763, 42	1, 2, 3, 4, 5, 6, 8, 11, 12, 14, 15, 16, 17, 18, 22, 24, 27, 30, 34, 37, 41, 45, 50, 51, 56, 59, 62, 63	(16, 10, 12, 10, 15, 1)
103, 590, 24	1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 14, 15, 17, 18, 21, 22, 30, 31, 38, 41, 43, 45, 47, 50, 53, 60, 63, 64	(17, 8, 14, 8, 16, 1)
106, 441, 19	1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 17, 23, 25, 29, 36, 39, 40, 46, 47, 48, 49, 51, 52, 54, 55, 59, 61	(20, 8, 8, 8, 19, 1)
107, 256, 17	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 16, 21, 22, 23, 28, 33, 41, 43, 45, 47, 48, 49, 52, 58, 61, 62, 63	(19, 8, 10, 8, 18, 1)
108, 594, 29	1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 13, 16, 17, 19, 25, 28, 30, 31, 34, 42, 43, 44, 46, 47, 52, 57, 60, 64	(14, 16, 4, 16, 13, 1)
109, 1078, 40	1, 2, 3, 4, 5, 6, 7, 8, 9, 12, 21, 23, 26, 27, 32, 36, 42, 45, 47, 48, 49, 50, 52, 55, 57, 60, 63, 64	(21, 6, 10, 6, 20, 1)
110, 568, 22	1, 2, 3, 4, 5, 6, 8, 9, 11, 12, 14, 15, 16, 18, 21, 28, 29, 38, 39, 43, 45, 47, 52, 53, 54, 56, 57, 62	(17, 12, 6, 12, 16, 1)
112, 818, 27	1, 2, 3, 4, 5, 6, 7, 8, 11, 14, 15, 18, 23, 26, 27, 29, 30, 31, 33, 37, 38, 39, 41, 45, 51, 55, 56, 62	(18, 12, 4, 12, 17, 1)
115, 1090, 42	1, 2, 3, 4, 5, 6, 7, 9, 12, 15, 17, 18, 20, 23, 27, 34, 35, 42, 43, 44, 46, 48, 49, 50, 54, 55, 57, 59	(22, 6, 8, 6, 21, 1)
115, 1090, 42	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 20, 23, 26, 28, 30, 31, 34, 36, 42, 43, 44, 45, 47, 48, 49, 51, 54, 57	(15, 11, 12, 11, 14, 1)
118, 268, 30	1, 2, 3, 4, 5, 6, 8, 9, 10, 13, 16, 18, 21, 24, 30, 31, 32, 35, 39, 44, 45, 46, 48, 51, 52, 57, 59, 64	(15, 12, 10, 12, 14, 1)
119, 820, 47	1, 2, 3, 4, 5, 6, 8, 11, 13, 14, 16, 18, 21, 24, 29, 35, 39, 44, 45, 46, 47, 48, 51, 53, 54, 56, 57, 61	(19, 4, 18, 4, 18, 1)
120, 2119, 46	1, 2, 3, 4, 5, 6, 8, 9, 12, 14, 15, 17, 19, 25, 26, 30, 40, 42, 43, 44, 45, 48, 50, 51, 52, 56, 58, 62	(14, 13, 10, 13, 13, 1)
120, 2119, 46	1, 2, 3, 4, 5, 6, 8, 9, 10, 13, 14, 16, 17, 19, 20, 26, 27, 29, 32, 36, 37, 45, 49, 52, 55, 56, 57, 60	(23, 6, 6, 6, 22, 1)
121, 1704, 47	1, 2, 3, 4, 5, 6, 8, 9, 10, 14, 16, 17, 18, 21, 30, 31, 32, 34, 36, 40, 44, 45, 48, 51, 52, 54, 55, 63	(13, 11, 16, 11, 12, 1)
123, 306, 25	1, 2, 3, 4, 5, 6, 8, 9, 10, 14, 16, 17, 18, 21, 25, 26, 32, 34, 36, 40, 48, 49, 50, 51, 52, 54, 55, 63	(14, 9, 18, 9, 13, 1)
126, 548, 18	1, 2, 3, 4, 5, 6, 8, 11, 12, 13, 14, 18, 19, 20, 22, 35, 42, 46, 48, 49, 52, 53, 55, 57, 60, 64	(11, 12, 18, 12, 10, 1)
128, 198, 19	1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 14, 15, 17, 18, 26, 29, 30, 31, 35, 42, 44, 52, 53, 54, 55, 56, 57, 60	(10, 14, 16, 14, 9, 1)

G, n_1, n_2	Elements of the Difference Set	Invariant Factors
129, 914, 33	1, 2, 3, 4, 5, 6, 8, 9, 12, 13, 14, 15, 17, 25, 26, 29, 33, 36, 39, 41, 42, 43, 46, 48, 55, 59, 61, 63	(16, 9, 14, 9, 15, 1)
130, 698, 27	1, 2, 3, 4, 5, 6, 7, 8, 9, 13, 16, 18, 24, 25, 30, 33, 37, 38, 41, 45, 46, 47, 48, 54, 57, 58, 62, 63	(19, 6, 14, 6, 18, 1)
131, 442, 33	1, 2, 3, 4, 5, 6, 8, 9, 10, 16, 17, 18, 21, 24, 30, 31, 32, 34, 40, 44, 45, 46, 48, 51, 52, 55, 59, 64	(11, 16, 10, 16, 10, 1)
132, 857, 53	1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 13, 15, 17, 19, 25, 28, 29, 35, 36, 37, 40, 42, 46, 47, 51, 52, 53, 63	(13, 15, 8, 15, 12, 1)
132, 857, 53	1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 14, 15, 17, 22, 26, 28, 39, 40, 42, 46, 51, 55, 56, 58, 60, 61, 62, 63	(23, 8, 2, 8, 22, 1)
132, 857, 53	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 16, 17, 20, 22, 23, 30, 32, 34, 36, 39, 40, 43, 44, 45, 48, 50, 53, 56	(24, 8, 0, 8, 23, 1)
132, 857, 53	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 15, 17, 18, 22, 24, 31, 33, 37, 38, 41, 42, 45, 50, 58, 60, 63, 64	(21, 10, 2, 10, 20, 1)
132, 857, 53	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 14, 18, 20, 21, 26, 29, 32, 35, 39, 40, 46, 48, 53, 54, 55, 58, 64	(20, 12, 0, 12, 19, 1)
133, 1196, 43	1, 2, 3, 4, 5, 6, 8, 10, 11, 13, 15, 16, 19, 23, 24, 26, 33, 38, 39, 42, 43, 47, 52, 53, 55, 57, 58, 60	(12, 17, 6, 17, 11, 1)
140, 136, 22	1, 2, 3, 4, 5, 6, 8, 9, 11, 12, 14, 17, 20, 22, 26, 28, 29, 35, 42, 43, 44, 46, 48, 52, 54, 59, 62, 64	(10, 17, 10, 17, 9, 1)
141, 460, 22	1, 2, 3, 4, 5, 6, 8, 9, 11, 12, 14, 16, 17, 18, 20, 25, 26, 28, 29, 37, 39, 43, 44, 52, 59, 61, 62, 64	(12, 16, 8, 16, 11, 1)
142, 682, 36	1, 2, 3, 4, 5, 6, 8, 9, 11, 14, 15, 17, 18, 19, 21, 25, 29, 37, 40, 43, 44, 51, 53, 55, 57, 58, 59, 62	(24, 2, 12, 2, 23, 1)
144, 904, 34	1, 2, 3, 4, 5, 6, 8, 9, 11, 12, 14, 17, 18, 26, 29, 30, 32, 34, 35, 36, 38, 42, 43, 52, 54, 57, 59, 63	(11, 13, 16, 13, 10, 1)
145, 2168, 42	1, 2, 3, 4, 5, 6, 8, 9, 12, 13, 15, 17, 20, 23, 25, 26, 37, 43, 44, 46, 48, 49, 50, 54, 56, 60, 61, 62	(22, 8, 4, 8, 21, 1)
146, 622, 34	1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 14, 17, 20, 21, 26, 29, 31, 32, 45, 48, 49, 51, 55, 58, 59, 61, 63, 64	(16, 9, 14, 9, 15, 1)
147, 184, 24	1, 2, 3, 4, 5, 6, 8, 9, 10, 13, 16, 17, 18, 21, 29, 30, 31, 32, 33, 37, 44, 48, 52, 53, 55, 56, 57, 60	(17, 9, 12, 9, 16, 1)
148, 1778, 47	1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 21, 22, 24, 34, 38, 40, 42, 50, 51, 52, 54, 55, 59, 60, 62, 63	(26, 2, 8, 2, 25, 1)
149, 1368, 39	1, 2, 3, 4, 5, 6, 7, 8, 9, 12, 21, 26, 27, 30, 32, 35, 37, 42, 44, 45, 48, 50, 53, 57, 60, 61, 62, 64	(21, 4, 14, 4, 20, 1)
149, 1368, 39	1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 21, 26, 29, 30, 35, 37, 42, 43, 46, 48, 50, 51, 53, 57, 58, 60, 64	(19, 5, 16, 5, 18, 1)
150, 288, 21	1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 14, 16, 21, 23, 29, 31, 32, 34, 38, 40, 42, 46, 49, 51, 53, 54, 58, 63	(21, 6, 10, 6, 20, 1)

G, n_1, n_2	Elements of the Difference Set	Invariant Factors
155, 401, 32	1, 2, 3, 4, 5, 6, 8, 9, 11, 12, 13, 14, 15, 22, 30, 32, 35, 40, 43, 44, 47, 50, 51, 52, 56, 58, 59, 62	(14, 15, 6, 15, 13, 1)
156, 867, 40	1, 2, 3, 4, 5, 6, 8, 9, 11, 13, 14, 19, 28, 29, 30, 31, 32, 35, 36, 38, 42, 43, 47, 49, 52, 59, 60, 64	(14, 14, 8, 14, 13, 1)
157, 564, 32	1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 16, 18, 21, 22, 25, 26, 27, 33, 40, 52, 55, 57, 60, 61, 63, 64	(24, 6, 4, 6, 23, 1)
157, 564, 32	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 15, 18, 20, 25, 28, 30, 34, 39, 41, 42, 46, 48, 54, 55, 59, 60, 63	(16, 11, 10, 11, 15, 1)
158, 1476, 43	1, 2, 3, 4, 5, 6, 7, 8, 9, 12, 13, 18, 19, 20, 24, 27, 30, 32, 33, 36, 40, 42, 44, 46, 54, 61, 63, 64	(24, 0, 16, 0, 23, 1)
159, 1200, 43	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 16, 19, 21, 22, 24, 27, 45, 46, 48, 49, 51, 52, 53, 58, 59, 60, 63	(11, 14, 14, 14, 10, 1)
160, 2664, 42	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 13, 16, 18, 21, 22, 25, 26, 30, 33, 34, 39, 40, 44, 46, 50, 52, 63, 64	(13, 16, 6, 16, 12, 1)
161, 192, 26	1, 2, 3, 4, 5, 6, 8, 9, 11, 13, 14, 19, 22, 23, 26, 28, 30, 31, 32, 36, 38, 45, 49, 50, 52, 54, 60, 64	(15, 13, 8, 13, 14, 1)
162, 434, 23	1, 2, 3, 4, 5, 6, 7, 8, 9, 12, 13, 15, 17, 18, 24, 26, 30, 31, 34, 36, 38, 39, 40, 44, 57, 59, 61, 62	(20, 8, 8, 8, 19, 1)
163, 912, 33	1, 2, 3, 4, 6, 7, 9, 10, 11, 13, 14, 16, 18, 22, 23, 29, 30, 34, 37, 40, 43, 49, 54, 57, 60, 62, 63, 64	(18, 4, 20, 4, 17, 1)
164, 679, 30	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 17, 19, 22, 24, 28, 31, 34, 35, 39, 42, 46, 47, 55, 56, 59, 60, 63	(21, 4, 14, 4, 20, 1)
165, 1007, 44	1, 2, 3, 4, 5, 6, 7, 8, 13, 14, 15, 19, 23, 25, 30, 34, 37, 41, 43, 44, 45, 49, 51, 53, 54, 57, 59, 60	(20, 2, 20, 2, 19, 1)
166, 1852, 34	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 15, 24, 26, 28, 34, 35, 36, 40, 41, 48, 50, 51, 52, 53, 56, 57	(26, 2, 8, 2, 25, 1)
167, 119, 17	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 24, 25, 26, 29, 30, 32, 33, 35, 43, 45, 47, 48, 53, 54, 56, 57, 60, 64	(18, 8, 12, 8, 17, 1)
168, 630, 40	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 15, 16, 17, 21, 32, 33, 35, 39, 42, 43, 44, 45, 47, 50, 56, 57, 60	(18, 7, 14, 7, 17, 1)
169, 788, 32	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 15, 16, 17, 21, 32, 33, 35, 39, 42, 43, 44, 45, 47, 50, 56, 57, 60	(18, 7, 14, 7, 17, 1)
170, 788, 32	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 26, 27, 28, 29, 30, 31, 32, 33, 36, 39, 44, 47, 48, 54, 59, 63	(10, 17, 10, 17, 9, 1)
171, 156, 14	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 24, 25, 26, 28, 30, 31, 32, 36, 39, 44, 51, 53, 54, 55, 59, 62, 63	(13, 14, 10, 14, 12, 1)
173, 131, 13	1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 20, 23, 24, 25, 29, 31, 38, 39, 40, 41, 46, 49, 58, 59, 60, 62	(12, 12, 16, 12, 11, 1)

G, n_1, n_2	Elements of the Difference Set	Invariant Factors
174, 5, 1	1, 2, 3, 4, 5, 6, 8, 9, 12, 13, 14, 18, 20, 25, 26, 27, 29, 30, 33, 38, 40, 48, 50, 52, 55, 56, 57, 63	(16, 8, 16, 8, 15, 1)
176, 375, 23	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 17, 23, 24, 25, 29, 32, 36, 38, 39, 43, 44, 50, 51, 56, 57, 59, 61	(22, 4, 12, 4, 21, 1)
177, 136, 13	1, 2, 3, 4, 5, 6, 7, 8, 9, 12, 18, 20, 22, 23, 24, 25, 27, 29, 30, 33, 38, 50, 52, 59, 60, 61, 63, 64	(18, 8, 12, 8, 17, 1)
179, 245, 23	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 16, 18, 25, 30, 33, 35, 37, 41, 49, 52, 56, 57, 59, 63, 64	(18, 7, 14, 7, 17, 1)
180, 635, 27	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 18, 19, 21, 24, 25, 32, 33, 34, 37, 39, 42, 44, 48, 50, 53, 54, 64	(13, 14, 10, 14, 12, 1)
181, 153, 28	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 16, 17, 18, 21, 22, 24, 29, 35, 44, 46, 48, 52, 53, 54, 55, 57, 60	(12, 16, 8, 16, 11, 1)
182, 990, 35	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 18, 19, 21, 22, 25, 26, 33, 34, 37, 39, 46, 50, 52, 58, 63, 64	(15, 14, 6, 14, 14, 1)
183, 140, 10	1, 2, 3, 4, 5, 6, 8, 10, 11, 13, 17, 18, 22, 29, 32, 37, 41, 43, 44, 45, 47, 49, 50, 54, 58, 60, 62, 64	(13, 8, 22, 8, 12, 1)
184, 255, 16	1, 2, 3, 4, 5, 6, 7, 8, 9, 18, 24, 25, 27, 32, 34, 39, 41, 43, 45, 47, 49, 50, 51, 54, 56, 59, 60, 62	(20, 10, 4, 10, 19, 1)
187, 188, 14	1, 2, 3, 4, 5, 6, 8, 9, 10, 14, 18, 20, 21, 25, 31, 32, 33, 34, 35, 37, 38, 42, 45, 49, 51, 54, 60, 62	(12, 15, 10, 15, 11, 1)
188, 300, 21	1, 2, 3, 4, 5, 6, 8, 9, 10, 13, 16, 17, 26, 28, 30, 32, 35, 41, 42, 43, 44, 50, 57, 58, 59, 60, 62, 64	(16, 13, 6, 13, 15, 1)
188, 300, 21	1, 2, 3, 4, 5, 6, 8, 9, 10, 13, 15, 20, 21, 23, 24, 28, 31, 32, 41, 42, 43, 44, 47, 48, 49, 55, 59, 62	(15, 15, 4, 15, 14, 1)
188, 300, 21	1, 2, 3, 4, 5, 6, 8, 9, 10, 13, 15, 17, 21, 24, 25, 26, 28, 31, 32, 40, 41, 43, 48, 49, 55, 58, 61, 62	(17, 11, 8, 11, 16, 1)
188, 300, 21	1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 13, 15, 20, 21, 24, 27, 29, 30, 41, 42, 45, 47, 48, 55, 58, 60, 62, 64	(14, 17, 2, 17, 13, 1)
202, 374, 19	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 18, 31, 32, 35, 36, 41, 45, 47, 49, 52, 53, 55, 56, 57, 59	(14, 12, 12, 12, 13, 1)
246, 133, 17	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 21, 22, 24, 29, 35, 36, 37, 40, 42, 44, 51, 53, 57, 58, 60, 61	(18, 10, 8, 10, 17, 1)
250, 41, 9	1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 17, 18, 24, 25, 29, 31, 38, 42, 45, 49, 57, 58, 59, 60, 61, 62	(14, 8, 20, 8, 13, 1)
251, 337, 26	1, 2, 3, 4, 5, 6, 8, 9, 11, 14, 15, 16, 17, 26, 30, 31, 34, 37, 38, 39, 41, 42, 43, 50, 51, 55, 59, 62	(9, 15, 16, 15, 8, 1)
253, 732, 36	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 13, 19, 25, 26, 31, 32, 40, 41, 43, 47, 48, 49, 51, 56, 58, 61, 62, 63	(19, 10, 6, 10, 18, 1)
254, 472, 29	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 15, 16, 17, 28, 30, 36, 37, 38, 39, 40, 44, 49, 55, 59, 61, 62	(16, 14, 4, 14, 15, 1)

ACKNOWLEDGMENT
The author wishes to acknowledge the generous support by the Deanship of academic Research at the University of Jordan.

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