EMBEDDING OF SIGNED GRAPHS IN GRACEFUL SIGNED GRAPHS

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Abstract

In this paper, we generalize to the class of signed graphs the well known result that *every numbered graph can be embedded as an induced subgraph in a gracefully numbered graph* .

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1 Introduction

Unless mentioned otherwise, for standard terminology and notation in graph theory we follow F. Harary [11].

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A *signed graph* (for short, "sigraph") is a graph where some edges are *positive* and others are *negative*. A *graceful numbering* f of a sigraph $S = (G, \sigma)$ (where G is the underlying graph of S and σ is a function, called the *signature* of S, which assigns a plus sign or a minus sign to each of the edges of G), is an injective function from the vertex set $V(S)$ of S to the set of integers $\{0, 1, \ldots, q = m+n\}$ such that when each edge $uv \in E(S)$ is assigned $g_f(uv) = \sigma(uv)|f(u) - f(v)|$ the positive edges receive distinct labels from the set $\{1, 2, \ldots, m\}$ and the negative edges receive distinct labels from the set $\{-1, -2, \ldots, -n\}$. The sigraph S is called a *graceful sigraph* if it admits a graceful numbering and *nongraceful* otherwise (see [3–6]). We show that every sigraph can be embedded as an induced subsigraph of some gracefully numbered sigraph, and therefore, it is impossible to characterize the sigraphs that have graceful numbering by excluded induced subsigraphs. This generalizes to sigraphs work of B.D. Acharya [1] on graphs.

There is a rationale behind considering embedding of numbered sigraphs in graceful sigraphs. The most important question for utilizing a graceful addressing and identification system involves being better able to determine whether an arbitrary model of a communication network (in which some connections may exist and some may not and some connections are 'good' and some are 'bad') is in a graceful configuration. If it is, and somehow known *a priori*, then how should it be numbered? If it is not, since provision needs to be made for growth of any addressing scheme for the network anyway to accommodate additional communication links on demand, then is it possible to embed the existing one in a larger graceful configuration? A numbering scheme would enable one to not only distinguish the nodes of the network by their unique numerical addresses but also would automatically assign unique numerical addresses to the communication links in the network invoking the very numerical addresses of their termini (see [7–10, 12]). We shall show that such provisions can be made by using appropriate methods of embedding the given network into another with desired properties.

An injective assignment of nonnegative integers to the vertices of a sigraph S is called a *numbering* (or an 'indexer') of S. If f is a numbering of S we shall let $M(f)$ denote the maximum value of the numbers assigned by f to the vertices of S and $\theta(S)$ denote the minimum of such numbers taken over all the possible numberings of S; $\theta(S)$ is called the *index of gracefulness* of S such that a numbering should result in distinct labels on the positive edges and on the negative edges respectively, (*see* B.D. Acharya and M.K. Gill [2] in the case of graphs). A numbering f of S for which $M(f) = \theta(S)$ is an *optimal numbering* of S.

The following results of B.D. Acharya [1] are known for graphs:

Theorem 1. *If f is an optimal numbering of a graph G then $0 \in f(G)$.*

Theorem 2. *Every numbered graph can be embedded as an induced subgraph in a gracefully numbered graph.*

In this note, we extend the above results to sigraphs.

2 Definitions

We let $E^+(S)$ and $E^-(S)$ denote respectively the set of *positive* and the set of *negative* edges of the given sigraph S so that $E^+(S) \cup E^-(S) =: E(S)$ is the *edge set* of S. Further, if $|E^+(S)| = m$ and $|E^-(S)| = n$ so that $m + n = q$ then we shall refer to S as a (p, m, n)-sigraph. An *all-positive* sigraph S is one in which $E^+(S) = E(S)$ and an *all-negative* sigraph is one in which $E^-(S) = E(S)$. A sigraph is *homogeneous* if it is either all-positive or all-negative, and *heterogeneous* otherwise (e.g., see [3–6]).

Given a numbering f of a sigraph S, its *complement*, denoted f^c, is defined as the function f^c such that $f^c(u) = M(f) - f(u), \forall u \in V(S)$. Obviously, for any sigraph S and for any optimal numbering f of S, f^c is also an optimal numbering of S and it satisfies $g_f = g_{f^c}$.

For a sigraph S, $\eta(S)$ is a sigraph obtained from S by changing the sign of each edge of S to its opposite and is called the *negation* of S.

Observation 3. *If f is an optimal numbering of a sigraph S then it is also an optimal numbering of the sigraph $\eta(S)$.*

A graph G is said to be *embedded* in a graph G', written as $G \preceq G'$, if there exists an induced subgraph of G' which is isomorphic to G. If $G \preceq G'$ it is sometimes convenient to regard G itself as an induced subgraph of G' (see [1]), which we will do indeed; this convention enables us to consider extension of any function defined on a sigraph S on G to a function on a sigraph S' on G' in which S is a subsigraph. Specifically, then, for any numbering f of S, we shall say that f has an *extension* f_e on S_e if f_e is a numbering of S_e and f is equal to $f_e|_S$, the restriction of f_e to $V(S)$. Hence, the numbered sigraph (S, f) is said to be *embedded* in the numbered sigraph (S_e, f_e).

3 The Main Result

The following is a fundamental result on numberings.

Proposition 4. *If a numbering f of a sigraph S does not assign 0 to any vertex of S then there exists a numbering f_0 of S such that $0 \in f_0(S)$.*

Proof. Suppose there is a numbering f which does not assign a 0 to any vertex of S. Let $k := \min_{u \in V(S)} f(u)$ and $f_0(v) = f(v) - k$. Clearly, f_0 is also a numbering which assigns 0 to a vertex of S. □

The above proposition yields.

Lemma 5. *Every optimal numbering of a sigraph S assigns 0 to some vertex of S.*

For any optimally numbered sigraph (S, f), we have $u, v \in V(S)$ such that $f(u) = 0$ and $f(v) = M(f)$. We are now ready to state our main result, which generalizes Theorem 2.

Theorem 6. *Every numbered sigraph can be embedded as an induced subsigraph in a gracefully numbered sigraph.*

Proof. Let (S, f) be any numbered sigraph with numbering f. If S is graceful, then the theorem is trivial. If S is a nongraceful homogeneous sigraph then the proof follows from Theorem 2. Hence, we let S be a nongraceful heterogeneous sigraph with an optimal numbering f. We let

$$\mathcal{M}^+ = \{1, 2, \ldots, M(f)\} - g_f(E^+(S))$$

and

$$\mathcal{M}^- = \{-1, -2, \ldots, -M(f)\} - g_f(E^-(S))$$

for the sets of numbers which are missed by g_f on positive and negative edges respectively.

Suppose $\mathcal{M}^+$ is non empty. Let $i_1, i_2, \ldots, i_t$ be the missing numbers on positive edges which are not in $f(S)$ and let $j_1, j_2, \ldots, j_r$ be the missing numbers on positive edges which are in $f(S)$. For each $b, 1 \leq b \leq r$, we take a new vertex y_b with label $f(y_b) = M(f) + j_b$ and join it to v and u by positive edges. Let the resulting sigraph be S_e at this stage and f_e be the so extended numbering of S_e.

Now, $M(f_e) = M(f) + j_r$ and we have a set $\mathcal{M}_e^+$ of numbers in $\{1, 2, \ldots, M(f_e)\}$ that are missing on positive edges and also on vertices. For each number $i \in \mathcal{M}_e^+$, take a new vertex x_i with label $f_e(x_i) = i$ and join it to u by a positive edge. We have $M(f_e') = M(f_e)$. Thus by construction, we have a numbered sigraph (S_e', f_e') in which positive edges have all the numbers $1, 2, \ldots, M(f_e')$. The negative edges have exactly the same numbers as they did in (S, f). Hence, (S, f) is an induced subsigrph.

Thus, we may assume that in (S, f) the positive edges are numbered $1, 2, \ldots, M(f)$. By negating S while keeping the same numbering f, we get a numbered sigraph (S, f) in which the negative edges are numbered $-1, -2, \ldots, -M(f)$. Applying the same construction as we discussed before, we get a numbered sigraph (S_e', f_e') in which the negative edges are numbered by consecutive numbers from -1 to $-M(f)$ and the positive edges are numbered by consecutive numbers from 1 to $M(f_e')$. The vertex numbers are contained in the set $\{0, 1, 2, \ldots, |E_e'|\}$. Also, (S, f) is an induced subsigraph. Thus, (S_e', f_e') is a gracefully numbered sigraph and the original numbered sigraph is an induced subsigraph of it. □

Thus, the above result shows that there is no "forbidden subsigraph" characterization of graceful sigraphs, generalizing Theorem 2.

Finally, we declare the following natural question open for investigation: *Given a sigraph S what is the least number of vertices (edges) of a graceful sigraph that contains S as an induced subsigraph?*

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REFERENCES

1. B.D. Acharya, *Construction of certain infinite families of graceful graphs from a given graceful graph*, Def. Sci. J., **32**(3)(1982), 231-236.

2. B.D. Acharya and M.K. Gill, *On the index of gracefulness of a graph and the gracefulness of two dimensional square lattice graphs*, Indian J. Math. **23**(1981), 8-94.

3. M. Acharya and T. Singh, *Graceful signed graphs*, Czech. Math. J., **54**(129), 2004, 291-302.

4. M. Acharya and T. Singh, *Graceful signed graphs: II. The case of signed cycles with connected negative sections*, Czech. Math. J., **55**(130), 2005, 25-40.

5. M. Acharya and T. Singh, *Graceful signed graphs: III. The case of signed cycles in which negative sections form maximum matchings*, Graph Theory Notes of New York, **45**(2003), 11-15.

6. T. Singh, **Advances in the theory of signed graphs**, Ph.D. thesis, University of Delhi, Delhi, 2003.

7. J.C. Bermond, *Graceful graphs, radio antennae and French windmills*, Proc. One-day Combinatorics Conference. (Open Univ., Pitmann, London, May 1978) 1979, 18-37.

8. J.C. Bermond, A. Kotzig and J. Turgeon, *On a combinatorial problem of antennas in radio astronomy*, Proc. V. Hungarian Colloquium on Combinatorics (Keszthely), 1976.

9. G.S. Bloom and S.W. Golomb, *Applications of numbered undirected graphs*, Proc. IEEE, **65** (1977), 562-570.

10. S.W. Golomb, *How to number a graph*, In: **Graph Theory and Computing** (Ed. R.C. Read), Academic Press, New York, 1972, 23-37.

11. F. Harary, **Graph Theory**, Addison-Wesley Pub. Co., Reading, Massachusetts, 1969.

12. A. Rosa, *On certain valuations of the vertices of a graph*; In. **Theory of Graphs**, Internat. Symp., Rome, 1966, (Ed.: P. Rosentiehl), Gordon and Breach, New York and Dunod, Paris, 1967, 349-355.

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